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Quantifier elimination in the theory of $L_p(L_q)$-Banach lattices

C Ward Henson
Yves Raynaud

Abstract: We introduce the class of doubly atomless bands in $L_p(L_q)$-Banach lattices and show that this class is axiomatizable by positive bounded sentences in the language of Banach lattices. (Here $p \neq q$ are fixed and in the interval $1 \leq p, q < \infty$.) The theory of this class is complete (indeed, we show it is separably categorical) and model complete. Further, we show that it satisfies quantifier elimination if and only if the ratio p/q is not an integer. On the functional analytic side, the proof of the latter result uses a positive-coefficient version of the well known Rudin-Plotkin-Hardin extension theorems.

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Introduction

This paper is a continuation of our paper [HR] in which we began the study of the class of $L_p(L_q)$-Banach lattices from a model theoretic point of view. (Throughout this paper we assume $1 \leq p, q < \infty$ and $p \neq q$.) The logical framework we use for this study is the setting of positive bounded formulas and approximate satisfaction initiated by Henson in [He] and developed by Henson and Iovino in [HI]. In order to have a class that is axiomatizable in this framework one needs to expand the class of $L_p(L_q)$-spaces (since, for example, this class is not closed under ultraproducts). In [HR] we considered the class BL_pL_q of *bands* of $L_p(L_q)$-Banach lattices, which turns out to be natural in both functional analysis and model theory.

One of the main results of [HR] shows that the class of BL_pL_q-Banach lattices is axiomatizable by positive bounded sentences in the language of Banach lattices. (See [HR, Corollary 2.10].) An explicit set of axioms for this class can be derived from the results in Section 3 of [HR], which show that a Banach lattice is in the class

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exactly when it can be paved in an almost isometric sense by finite dimensional BL_pL_q-sublattices. (See [HR, Proposition 3.6]; to express the needed axioms as positive bounded sentences one needs bounds on the dimensions of the BL_pL_q-sublattices as given in [HR, Proposition 3.7].)

In this paper we identify and study a natural axiomatizable subclass of BL_pL_q whose theory is very well behaved from the model theoretic point of view. This is the class of *doubly atomless* BL_pL_q-Banach lattices; see the beginning of Section 2 for the definition. Every member of this class turns out to be elementarily equivalent to the Banach lattice $L_p([0,1];L_q([0,1]))$, which we denote by $\boldsymbol{L_p(L_q)}$; hence the theory of the class of doubly atomless BL_pL_q-Banach lattices is complete. Indeed, every separable member of this class is isomorphic to $\boldsymbol{L_p(L_q)}$ (*i.e.*, this theory is in fact separably categorical).

The main focus of this paper is to determine the values of p,q for which the theory of doubly atomless BL_pL_q-Banach lattices has quantifier elimination. (See [HI, Definition 13.13] and the material that follows it for a discussion of quantifier elimination in the positive bounded setting.) When $1 \leq p \neq q < \infty$, we show that this theory has quantifier elimination if and only if the ratio $\alpha := p/q$ is not an integer. (See Corollary 4.4 and Proposition 5.4.)

In Section 6 we indicate (without full details) that the theory of doubly atomless BL_pL_q-Banach lattices is model complete for all values of p,q, and indeed that it is the model companion of the theory of all BL_pL_q-Banach lattices.

In the background of this investigation are corresponding results for L_p-spaces that were already known. (For this reason we are excluding the case $p = q$ in this paper.) As shown in [HI, Example 13.4], the class of L_p-Banach lattices is axiomatizable by positive bounded sentences. Further, the class of atomless L_p-Banach lattices is axiomatizable and its theory is complete, separably categorical, and has quantifier elimination for all $1 \leq p < \infty$. (See Henson's [He, Theorem 2.2] for axiomatizability of the atomless condition and for completeness of the theory, and [HI, Example 13.18] for quantifier elimination.)

In certain ways, the model theoretic study of BL_pL_q-Banach lattices has more features in common with the model theory of L_p-Banach *spaces* than it does with the model theory of L_p-Banach *lattices*. As noted in the previous paragraph, for atomless L_p-Banach lattices one has quantifier elimination for all values of p. In that result there is nothing like the restriction (that p/q not be an integer) needed for atomless BL_pL_q-Banach lattices to have quantifier elimination. However, when L_p-spaces are considered without the lattice structure, a similar complexity arises. In [HI, Example 13.8] it is discussed that the full class of L_p-Banach spaces as well as the class of atomless L_p-spaces are both axiomatized by positive bounded sentences in the pure language of Banach spaces. As in the lattice case, and for the same reasons, the theory of atomless L_p-Banach spaces is complete and separably categorical. However, this theory has quantifier elimination if and only if $p \neq 4, 6, 8, \dots$, as is discussed in [HI, Example 13.18 at the end]. Further, the reasoning behind that result is similar to what we use in Sections 3, 4, and 5 of this paper to treat quantifier elimination for the theory of doubly atomless BL_pL_q-Banach lattices. Finally, we note that it can be shown for all values of $1 \leq p < \infty$ that the theory of atomless L_p-Banach spaces is the model companion of the theory of all L_p-Banach spaces, using the same argument we use in Section 6 to prove the analogous result for doubly atomless BL_pL_q-Banach lattices.

We close this Introduction by indicating the contents of this paper section by section. Section 1 contains preliminaries from analysis needed for the spaces we consider here. Section 2 defines the class of doubly atomless BL_pL_q-spaces and proves that it is axiomatizable, and that its theory is separably categorical and complete. Here we also show that the classes of BL_pL_q-spaces we study are closed under unions of increasing chains. Section 3 contains some results about extending isometries on the positive cone of L_α topological vector lattices for $0 < \alpha < \infty$; these results are central to our treatment of quantifier elimination for doubly atomless BL_pL_q-Banach lattices. In Sections 4 and 5 we prove our quantifier elimination results. In Section 6 we briefly discuss model completeness for the theory of doubly atomless BL_pL_q-Banach lattices. At the end of the paper we indicate some possible directions for further research in this area.

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1 Preliminaries

$L_p(L_q)$-spaces

If $1 \leq p < \infty$, Y is a Banach space, and (Ω, Σ, μ) is a measure space, then $L_p(\Omega, \Sigma, \mu; Y)$ is the space of (classes of) Bochner-measurable functions $f \colon \Omega \to Y$ such that (the class of) the function $N_Y(f) \colon \Omega \to \mathbb{R}_+$ defined by $\omega \mapsto \|f(\omega)\|_Y$ belongs to $L_p(\Omega, \Sigma, \mu)$. (Recall that Bochner-measurable functions are limits of sequences of measurable simple functions.) In this definition "class" means as usual "equivalence class with respect to equality almost everywhere". If Y itself is a space of the form $L_q(\Omega', \Sigma', \mu')$, $1 \leq q < \infty$, then $L_p(\Omega, \Sigma, \mu; Y)$ can be identified with a Banach lattice X of scalar-valued measurable functions on the product measure space $(\Omega \times \Omega', \Sigma \bar{\otimes} \Sigma', \mu \bar{\otimes} \mu')$.

The definition of this product measure space is standard when μ, μ' are both σ-finite. In that case the class of a $\mu\bar{\otimes}\mu'$-measurable function f belongs to X iff the (classes of the) partial functions $f(\omega,\cdot)$ (which are ν-measurable for μ-almost every $\omega \in \Omega$) belong to $L_q(\Omega',\Sigma',\nu')$ for μ-almost every $\omega \in \Omega$ and the (class of the) function $N_q(f)\colon \Omega \to \mathbb{R}_+$, defined by $\omega \mapsto (\int |\tilde{f}(\omega,\omega')|^q \, d\mu'(\omega'))^{1/q}$, belongs to $L_p(\Omega,\Sigma,\mu)$. When considering ultraproducts of such spaces, it is necessary to consider also the case of non σ-finite measure spaces. In fact we may restrict to *decomposable* measure spaces: a measure space (Ω,Σ,μ) is *decomposable* (or *strictly localizable*) if there exists a partition $\Omega = \bigcup_\alpha \Omega_\alpha$ of Ω into elements Ω_α of Σ that have finite nonzero μ-measure, such that

(i) a subset A of Ω belongs to Σ if and only if each $\Omega_\alpha \cap A$ belongs to Σ; and

(ii) in that case one also has $\mu(A) = \sum\limits_\alpha \mu(A \cap \Omega_\alpha)$.

Every L_p-space ($1 \le p < \infty$) can be represented as the L_p-space of a decomposable measure space.

Such a measure space (Ω,Σ,μ) is in particular *Maharam*; that is, it is *semi-finite* (*i.e.*, every set $A \in \Sigma$ with $\mu(A) > 0$ contains a subset $B \in \Sigma$ with $0 < \mu(B) < \infty$) and the space $L_\infty(\Omega,\Sigma,\mu)$ is *order-complete* (*i.e.*, every order bounded family of elements has a least upper bound; equivalently, $L_0(\Omega,\Sigma,\mu)$ is order-complete). Consequently, the dual of $L_1(\Omega,\Sigma,\mu)$ is identifiable with $L_\infty(\Omega,\Sigma,\mu)$ via the pairing given by the integral. The product of two decomposable measure spaces can be defined in such a way to also be decomposable, and the interpretation of $L_p(\Omega,\Sigma,\mu;L_q(\Omega',\Sigma',\mu'))$ as a space of measurable functions on the product measure space is very similar to the σ-finite case. (The condition for a $\mu\otimes\mu'$-measurable function to define an element of $L_p(\Omega,\Sigma,\mu;L_q(\Omega',\Sigma',\mu'))$ is the same as in the σ-finite case provided the function is supported by a product of σ-finite sets, and conversely, every element of $L_p(\Omega,\Sigma,\mu;L_q(\Omega',\Sigma',\mu'))$ has such a representing measurable function.)

Abstract $L_p(L_q)$-spaces

A description of ultraproducts of $L_p(L_q)$-Banach lattices has been known since the 1980s. This class is not closed under ultraproducts, but a somewhat larger class is, namely the class of *bands* of $L_p(L_q)$-Banach lattices ([LR1], [HLR]). A *band* in $L_p(\Omega,\Sigma,\mu;L_q(\Omega',\Sigma',\mu'))$ is the range R_S of a projection $f \mapsto \mathbf{1}_S.f$, where $S \subset \Omega\times\Omega'$ is some $\mu\otimes\mu'$-measurable subset. Equivalently R_S consists of all S-supported elements of $L_p(L_q)$ and S is called the support of R_S. Let us recall a definition of bands in a general Banach lattice, relying only on the Banach lattice structure: a band in a Banach lattice X is a subset of the form $A^\perp$, where $A \subset X$ and $A^\perp = \{g \in X \mid \forall f \in X, f \perp g\}$ is the set of the elements of X that are disjoint from all elements in A.

An "abstract" description of these bands as Banach lattices has been given. By an *abstract $L_p(L_q)$-space* (in short, an *AL_pL_q-space*) we mean a Banach lattice X which, for some measure space (Ω, Σ, μ), can be equipped with the structure of an $L_\infty(\Omega, \Sigma, \mu)$-module and with a map $N\colon X \to L_p(\Omega, \Sigma, \mu)_+$ such that

(i) For every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ and $x \in X$, if $\varphi \geq 0$ and $x \geq 0$, then $\varphi.x \geq 0$;

(ii) $N(x+y) \leq N(x) + N(y)$ for every $x, y \in X$;

(iii) $N(\varphi.x) = |\varphi| N(x)$ for every $\varphi \in L_\infty(\Omega, \Sigma, \mu)$ and $x \in X$;

(iv) if $0 \leq |x| \leq |y|$, then $N(x) \leq N(y)$, for every $x, y \in X$;

(v) $N(x+y)^q = N(x)^q + N(y)^q$ for every pair of disjoint $x, y \in X$;

(vi) $\|x\|_X = \|N(x)\|_{L_p}$ for every $x \in X$.

A map N satisfying the axioms (ii)–(vi) is called a *q-random norm*. The ordinary Bochner spaces $L_p(L_q)$ are clearly AL_pL_q-spaces (the random norm of which was denoted by N_q in the previous paragraph); in particular, L_p-spaces and L_q-spaces are AL_pL_q-spaces. More generally, any band B in a Bochner $L_p(\Omega, \Sigma, \mu; L_q)$-space inherits the structure of an AL_pL_q-space. Conversely, it was proved in [LR1] (see also [LR2] and [HLR]) that every AL_pL_q-space is isomorphic to a band of a Bochner $L_p(L_q)$-space. In fact the isomorphism preserves the action of L_∞ and the q-random norm. This provides a characterization of bands in $L_p(L_q)$-spaces as the Banach lattices arising from AL_pL_q-spaces by ignoring the action of L_∞ and the random norm N. Such spaces were called *BL_pL_q-Banach lattices* in [HR]. It was proved in [HR] that the class of BL_pL_q-Banach lattices is axiomatizable in the signature of Banach lattices.

Note that the class of AL_pL_q-spaces is closed under $\oplus_p$ (p-direct sum). In the separable case, every AL_pL_q-space X has a concrete representation as a p-direct sum of Bochner $L_p(L_q)$-spaces; namely, X is isomorphic to a Banach lattice of the form

$$\Big(\bigoplus_{n\geq 0} L_p(\Omega_n; \ell_q^n)\Big)_p \underset{p}{\oplus} L_p(\Omega_\infty; \ell_q) \underset{p}{\oplus} \Big(\bigoplus_{n\geq 0} L_p(\Omega'_n; L_q([0,1]) \underset{q}{\oplus} \ell_q^n)\Big)_p \underset{p}{\oplus} L_p(\Omega'_\infty; L_q([0,1]) \underset{q}{\oplus} \ell_q).$$

It was noted in [LR1] that (for fixed p, q) the class of AL_pL_q-spaces is closed under ultraproducts. Let $\mathcal{U}$ be an ultrafilter on the set I, $(X_i)_{i\in I}$ a family of AL_pL_q-spaces, and $N_i\colon X_i \mapsto L_p(\Omega_i, \Sigma_i, \mu_i)_+$ the corresponding q-random norms. Let $\mathcal{L}$ be the abstract L_p-space $\prod_\mathcal{U} L_p(\Omega_i, \Sigma_i, \mu_i)$. If $L_p(\tilde\Omega, \tilde\Sigma, \tilde\mu)$ is any representation of $\mathcal{L}$ as an L_p-space over a decomposable measure space, then $\prod_\mathcal{U} L_\infty(\Omega_i, \Sigma_i, \mu_i)$ may be identified with a subalgebra $\mathcal{Z}$ of $L_\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ acting naturally on $\prod_{\mathcal{U}} X_i$ by the ultraproduct action, which extends in a unique way to an action of $L_\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ on $\prod_{\mathcal{U}} X_i$. Then the ultraproduct map $N_{\mathcal{U}} \colon \prod_{\mathcal{U}} X_i \to \prod_{\mathcal{U}} L_p(\Omega_i, \Sigma_i, \mu_i)$ defines a q-random norm on the Banach lattice $\prod_{\mathcal{U}} X_i$, with respect to this action of $L_\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$.

Isomorphisms and embeddings

We specialize some terminology from [HI] to the setting of the structures considered here. Given two Banach lattices X and Y, a map $T \colon X \to Y$ that is linear, isometric and preserves the lattice operations will simply be called an *embedding*. If an embedding T is surjective, it will be called an *isomorphism*.

In a few places we consider certain Banach lattices only as ordered Banach spaces, and consider maps that are only linear, positive and isometric. We shall not give any particular name to such maps (to speak of ordered normed space embeddings would be appropriate, but needlessly pedantic).

For some purposes we consider convex cones in Banach spaces (*e.g.*, positive cones of Banach lattices). It is always understood that the vertex of such a cone is at the origin. Let Γ_1 and Γ_2 be two such cones and $T \colon \Gamma_1 \to \Gamma_2$ be a map. We call T a *normed cone morphism* if it is affine (equivalently, additive and positively homogeneous of degree one) and norm-preserving. Observe that this does not necessarily imply that T is distance-preserving (*e.g.*, consider the map $T \colon L_1^+ \to \mathbb{R}_+$ defined by $T(f) = \|f\|_1$ for all f).

Terminology

We use standard notation and terminology from functional analysis; for Banach lattice notions see [LT] [MN] [Sch]. In particular, when we refer to a *sublattice* of a lattice ordered vector space, we mean it to be a *linear sublattice*.

2 The class of doubly atomless BL_pL_q-Banach lattices

Let $1 \le p \neq q < \infty$ and X be a AL_pL_q-Banach lattice, and $N \colon X \to L_p(\Omega, \mathcal{A}, \mu)$ a q-random norm on X. We say that two elements $u, v \in X$ are *base-disjoint* if $N(x) \wedge N(y) = 0$.

Let $a \in X$ be a non-zero element. We say that a is a *base-atom* if every decomposition of a as a sum of base-disjoint vectors is trivial, *i.e.*, one of the vectors is zero. We say that a is a *fiber-atom* if in every decomposition of a as a sum of disjoint vectors, the vectors are in fact base-disjoint.

We say that X is *base-atomless* (resp. *fiber-atomless*) if there is no base-atom (resp. fiber-atom) in X; and that X is *doubly atomless* if it is both base-atomless and fiber-atomless.

The following Lemma shows that these concepts depend only on the Banach lattice structure of X (not on the random norm); therefore we may speak of base-atomless, fiber-atomless or doubly atomless BL_pL_q-Banach lattices.

2.1 Lemma *1) a is a base-atom in X iff for all $u, v \in X$ one has*
$\big(a = u + v, |u| \wedge |v| = 0$ *and* $\|u\|^p + \|v\|^p = \|u + v\|^p\big)$ *implies* $\big(u = 0$ *or* $v = 0\big)$;
2) a is a fiber-atom in X iff for all $u, v \in X$ one has
$$\big(a = u + v \text{ and } |u| \wedge |v| = 0\big) \text{ implies } \|u\|^p + \|v\|^p = \|u + v\|^p].$$

Proof It suffices to prove that two disjoint vectors $u, v \in X$ are base-disjoint iff $\|u\|^p + \|v\|^p = \|u+v\|^p$. This condition is clearly necessary; conversely, if it is satisfied, then
$$\int (N(u)^p + N(v)^p) = \int (N(u)^q + N(v)^q)^{p/q}$$
but we have $(N(u)^p + N(v)^p) - (N(u)^q + N(v)^q)^{p/q} \geq 0$ if $p \leq q$, resp. ≤ 0 if $p \geq q$. Hence the equality of integrals implies
$$N(u)^p + N(v)^p = (N(u)^q + N(v)^q)^{p/q}$$
which in turn implies that $N(u)$ and $N(v)$ are disjoint since $p \neq q$. □

2.2 Lemma *Consider X as a band in a space $L_p(\Omega, \mathcal{A}, \mu; L_q(\Omega', \mathcal{A}', \mu'))$. Let S_X be the support of this band (considered as a measurable subset of $\Omega \times \Omega'$). Let $R_X = \Omega_X \times \Omega'_X$ be the smallest rectangle containing S_X. Then X is base-atomless iff the measure space $(\Omega_X, \mathcal{A}\,|_{\Omega_X}, \mu|_{\Omega_X})$ is atomless; and X is fiber-atomless iff the measure space $(\Omega'_X, \mathcal{A}'\,|_{\Omega'_X}, \mu'|_{\Omega'_X})$ is atomless.*

Proof i) It is clear that if the measure space $(\Omega_X, \mathcal{A}|_{\Omega_X}, \mu|_{\Omega_X})$ is atomless, then X is base-atomless. Conversely, let X be base-atomless and A be a σ-finite set in $\mathcal{A}$ that is contained in Ω_X. Then there exists an element $x \in X$ with Ω-support A (that is, $\mathrm{Supp}(N(x)) = A$). If $x = u + v$ is a nontrivial decomposition of x with disjoint random norms $N(u), N(v)$, then the supports A_u, A_v of $N(u), N(v)$ are disjoint, have positive measure and $A = A_u \cup A_v$, so A is not an atom.

ii) Assume that the measure space $(\Omega'_X, \mathcal{A}'|_{\Omega'_X}, \mu'|_{\Omega'_X})$ is atomless, and let $x \in X$. We want to find a disjoint decomposition $x = u + v$ with $N(u) \wedge N(v) \neq 0$. Note that it is sufficient to do that for some non zero component of x. (Recall that a component of x is an element $u \in X$ such that $x-u$ is disjoint from u; equivalently $u = Px$ for some band projection P in X). So we may assume that x belongs to L_∞ and that the support of x is included in a rectangle $A_x \times A'_x$, where A_x (resp. A'_x) is a subset of positive finite μ-measure (resp. μ'-measure) included in Ω_X (resp. Ω'_X). If for some decomposition $A'_x = A'_1 \cup A'_2$ of A'_x into two disjoint sets of positive measure the vectors $u = \mathbf{1}_{A_x \times A'_1} x$ and $v = \mathbf{1}_{A_x \times A'_2} x$ are not base-disjoint, we are done. Otherwise, for every integer $N \geq 1$ we can find partitions $A'_x = A'_1 \cup \ldots \cup A'_N$ and $A_x = A_1 \cup \ldots \cup A_N$ with $\mu'(A'_i) = 1/N$, $i = 1, \ldots n$, such that $x = \sum_{i=1}^N \mathbf{1}_{A_i \times A'_i} x$. Then $\|x\|^p = \sum_{i=1}^N \|\mathbf{1}_{A_i \times A'_i} x\|^p \leq \|x\|_\infty^p \sum_{i=1}^N \mu(A_i) N^{-p/q} = \|x\|_\infty \mu(A_x) N^{-p/q} \to 0$ when $N \to \infty$, which is a contradiction.

Conversely, assume that the measure space $(\Omega'_X, \mathcal{A}'|_{\Omega'_X}, \mu'|_{\Omega'_X})$ has an atom A', which necessarily has finite measure. Then for some $A \in \mathcal{A}$, the function $x = \mathbf{1}_{A \times A'}$ is an element of X. Consider a decomposition $x = u + v$ as the sum of two disjoint nonzero elements u, v; then $u = \mathbf{1}_U$, $v = \mathbf{1}_V$, where U, V form a measurable partition of $A \times A'$. Since A' is an atom we necessarily have $U = B \times A'$ and $V = C \times A'$, where (B, C) is a measurable partition of A. Hence u, v are base-disjoint, and x is a fiber-atom. □

2.3 Lemma *a) X is base-atomless iff for every $x \in X$ there exist two disjoint elements $u, v \in X$ with $x = u + v$ and $\|u\| = \|v\| = 2^{-1/p}\|x\|$.*
b) X is fiber-atomless iff for every $x \in X$ there exist two disjoint elements $u, v \in X$ with $x = u + v$ and $\|u\| = \|v\| = 2^{-1/q}\|x\|$.

Proof a) Consider an x in X. If such a decomposition exists for x, then the two components of x are clearly base-disjoint (by the proof of Lemma 2.1), so x is not a base-atom; hence the condition is sufficient. Conversely, if X is base-atomless, then it can be represented as a band in some space $L_p(\Omega, \mathcal{A}, \mu; L_q(\Omega', \mathcal{A}', \mu'))$, where the measure space $(\Omega, \mathcal{A}, \mu)$ is atomless. Then for every $x \in X$ there exist disjoint sets $A, B \in \mathcal{A}$ such that $A \cup B$ is the support of $N(x)$ and $\|\mathbf{1}_A N(x)\|_p = \|\mathbf{1}_B N(x)\|_p = 2^{-1/p}\|N(x)\|_p$. Then set $u = \mathbf{1}_A x$ and $v = \mathbf{1}_B x$.

b) A disjoint decomposition $x = u + v$ with $\|u\| = \|v\| = 2^{-1/q}\|x\|$ cannot be base-disjoint (unless $x = 0$); hence, if such a decomposition exists for every $x \in X$, then X is fiber-atomless.

Conversely, assume that X is fiber-atomless, and represent it as a band in in some space $L_p(\Omega, \mathcal{A}, \mu; L_q(\Omega', \mathcal{A}', \mu'))$, where the measure space $(\Omega', \mathcal{A}', \mu')$ is atomless. Let $x \in X$ be a non-zero element. Then there exists a component x_1 of x such that $x_1 \neq 0$ and $N(x_1) \leq 2^{-1/q} N(x)$. For, by hypothesis there is a disjoint decomposition $x = u + v$ with

$N(u) \wedge N(v) \neq 0$; since $N(u)^q + N(v)^q = N(x)^q$ we have $N(u)^q \wedge N(v)^q \leq (1/2)N(x)^q$. Denoting the support of $(N(u) - N(v))_+$ by A, we set $x_1 = \mathbf{1}_{A^c}u + \mathbf{1}_A v$. Then x_1 is a non-zero component of x and $N(x_1) = N(u) \wedge N(v) \leq 2^{-1/q}N(x)$. Iterating this procedure we may find for every k a component x_k of x with $x_k \neq 0$ and $N(x_k) \leq 2^{-k/q}N(x)$.

The set $\mathcal{C}_x$ of components of x can be ordered by the relation $u \prec v$ iff $|u| \leq |v|$ (equivalently, the support of u is included in that of v). Then every linearly ordered family Γ of elements of $\mathcal{C}_x$ has a least upper bound u_Γ, which is still an element of $\mathcal{C}_x$. Thus by Zorn's Lemma the set $\{u \in \mathcal{C}_x \mid N(u) \leq 2^{-1/q}N(x)\}$ (which is non empty by the preceding discussion) has maximal elements. Let u_0 be such a maximal element. If $N(u_0) \neq 2^{-1/q}N(x)$, there is some positive real ε and some measurable subset A of the support of $N(x)$ such that $\mathbf{1}_A N(u_0) \leq (2^{-1/q} - \varepsilon)N(x)$. Let $w_0 = \mathbf{1}_A(x - u_0)$; we can find a component w_1 of w_0 such that $N(w_1) \leq \varepsilon N(w_0)$, hence $N(w_1) \leq \varepsilon \mathbf{1}_A N(x)$. Then $u_1 = u_0 + w_1$ is a component of x such that $N(u_1) \leq 2^{-1/q}N(x)$, $u_0 \prec u_1$ and $u_0 \neq u_1$, which contradicts the maximality of u_0. Hence $N(u_0) = 2^{-1/q}N(x)$; consequently if we set $v_0 = x - u_0$, we have also $N(w_0) = 2^{-1/q}N(x)$, and thus $\|u_0\| = \|v_0\| = 2^{-1/q}\|x\|$. □

2.4 Remark It results easily from the proof of part (b) of Lemma 2.3 that if X is a fiber atomless AL_pL_q-Banach lattice, equipped with a q-random norm N with values in $L_p(\Omega, \mathcal{A}, \mu)$, then for every element $x \in X$ and every $\varphi \in L_\infty(\Omega, \mathcal{A}, \mu)$ such that $0 \leq \varphi \leq 1$ there exists a disjoint decomposition $x = u + v$ in two components such that $N(u)^q = \varphi.N(x)^q$, $N(v)^q = (1 - \varphi).N(x)^q$.

2.5 Proposition *The classes of base-atomless, resp. fiber-atomless, resp. doubly atomless BL_pL_q-Banach lattices are axiomatizable (in the signature of Banach lattices).*

Proof By [HR] the class of BL_pL_q-Banach lattices is axiomatizable. Thus we need only to give positive bounded sentences in the language of Banach lattices, the approximate satisfaction of which is equivalent, in BL_pL_q-Banach lattices, to the property of being base-atomless, resp. fiber-atomless. From Lemma 2.3 we know that these properties are respectively characterized by the following statements (expressed in the usual formal language of mathematics):

(A) $\qquad \forall x\, \exists u \left[\| |u| \wedge |x - u| \| = 0 \text{ and } \|u\| = 2^{-1/p}\|x\| \text{ and } \|x - u\| = 2^{-1/p}\|x\| \right]$

respectively

(B) $\qquad \forall x\, \exists u \left[\| |u| \wedge |x - u| \| = 0 \text{ and } \|u\| = 2^{-1/q}\|x\| \text{ and } \|x - v\| = 2^{-1/q}\|x\| \right].$

Now consider for each real $\varepsilon \geq 0$ the following sentences (which are positive bounded sentences in the language of Banach lattices):

$$(A_\varepsilon) \qquad \forall_1 x\, \exists_1 u \left[\| |u| \wedge |x-u| \| \leq \varepsilon \text{ and } | \|u\| - 2^{-1/p}\|x\| | \leq \varepsilon \right. \\ \left. \text{and } | \|x-u\| - 2^{-1/p}\|x\| | \leq \varepsilon \right]$$

and

$$(B_\varepsilon) \qquad \forall_1 x\, \exists_1 u \left[\| |u| \wedge |x-u| \| \leq \varepsilon \text{ and } | \|u\| - 2^{-1/q}\|x\| | \leq \varepsilon \right. \\ \left. \text{and } | \|x-v\| - 2^{-1/q}\|x\| | \leq \varepsilon \right]$$

(Recall that in the bounded quantifiers $\forall_1 x$ and $\exists_1 x$ the variable x is taken to range over all elements of norm ≤ 1.)

It is easy to see that in all Banach lattices, (A_0) and (B_0) are equivalent to (A) and (B), respectively. Likewise, simple homogeneity and continuity arguments show that a Banach lattice satisfies all the (A_ε), resp. (B_ε), $\varepsilon > 0$, if and only if it approximately satisfies (in the sense of [HI]) the positive bounded sentence (A_0), resp. (B_0).

We will show that for ε sufficiently small, a BL_pL_q-Banach lattice X that satisfies (A_ε), resp. (B_ε) is in fact base-atomless, resp. fiber-atomless; that is, it verifies (A_0), resp. (B_0). It follows that for any Banach lattice X in the class BL_pL_q, approximate satisfaction of (A_0), resp. (B_0) by X is equivalent to X being base-atomless, resp. fiber-atomless.

First we prove this claim for the sentences (A_ε) and the property of being base-atomless. Assume that X is a BL_pL_q-Banach lattice satisfying (A_ε), but that X contains some non-zero base-atom a. Let $u, v \in X$ with $a = u + v$, $\| |u| \wedge |v| \| \leq \varepsilon$, and $| \|u\| - 2^{-1/p}\|a\| | \leq \varepsilon$, $| \|v\| - 2^{-1/p}\|a\| | \leq \varepsilon$. We can find disjoint elements u', v' in X such that $\|u - u'\| \leq \varepsilon$ and $\|v - v'\| \leq \varepsilon$ (*e.g.*, $u' = u - u_+ \wedge |v| + u_- \wedge |v|$, $v' = v - v_+ \wedge |u| + v_- \wedge |u|$). We have then:

$$\|a - (u' + v')\| \leq 2\varepsilon, \quad | \|u'\| - 2^{-1/p}\|a\| | \leq 2\varepsilon, \text{ and } | \|v'\| - 2^{-1/p}\|a\| | \leq 2\varepsilon$$

Let $P_{u'}$ and $P_{v'}$ be the projections in X onto the bands generated respectively by u' and v', and set:

$$b = P_{u'}a, \qquad c = P_{v'}a$$

We have:

$$\|b - u'\| = \|P_{u'}(a - (u' + v'))\| \leq \|a - (u' + v')\| \leq 2\varepsilon \text{ and similarly } \|c - v'\| \leq 2\varepsilon$$

Note that $P = P_{u'} + P_{v'}$ is also a band projection, and $P(u' + v') = u' + v'$. Thus

$$\|a - (b + c)\| = \|(I - P)a\| = \|(I - P)(a - (u' + v'))\| \leq \|a - (u' + v')\| \leq 2\varepsilon$$

and consequently:

(*) $\quad \|a-(b+c)\| \le 2\varepsilon,\ \big|\|b\| - 2^{-1/p}\|a\|\big| \le 4\varepsilon,$ and $\big|\|c\| - 2^{-1/p}\|a\|\big| \le 4\varepsilon.$

Observe that since b, c are both components of the base-atom a, their random norms $N(b)$ and $N(c)$ must be proportional to each other, and thus proportional to $\|b\|$ and $\|c\|$. Since b and c are disjoint we have $N(b+c) = (N(b)^q + N(c)^q)^{1/q}$, and thus:

(**) $$\|b+c\| = (\|b\|^q + \|c\|^q)^{1/q}$$

From (*) we obtain:

$$\big|\|a\| - \|b+c\|\big| \le 2\varepsilon \quad \text{and} \quad \big|(\|b\|^q + \|c\|^q)^{1/q} - 2^{-1/p+1/q}\|a\|\big| \le 8\varepsilon$$

Therefore conditions (*) are not compatible with (**) if $|1 - 2^{-1/p+1/q}| > 10\varepsilon$.

Finally, we we prove our claim for the sentences (B_ε) and the property of being base-atomless. Assume now that X is a BL_pL_q-Banach lattice satisfying (B_ε), but that X contains some non-zero fiber-atom a. Let $u, v \in X$ with $a = u + v$, $\big\||u| \wedge |v|\big\| \le \varepsilon$, and $\big|\|u\| - 2^{-1/q}\|a\|\big| \le \varepsilon$, $\big|\|v\| - 2^{-1/q}\|a\|\big| \le \varepsilon$. Consider, as in the preceding paragraph, disjoint elements u', v' in X such that $\|u-u'\| \le \varepsilon$ and $\|v - v'\| \le \varepsilon$, and set $b = P_{u'}a$ and $c = P_{v'}a$. We obtain now:

(†) $\quad \|a-(b+c)\| \le 2\varepsilon,\ \big|\|b\| - 2^{-1/q}\|a\|\big| \le 4\varepsilon,$ and $\big|\|c\| - 2^{-1/q}\|a\|\big| \le 4\varepsilon$

Since the elements b and c are two components of the same fiber-atom a, they must be base-disjoint. Thus

(††) $$\|b+c\| = (\|b\|^p + \|c\|^p)^{1/p}$$

Conditions (†) and (††) are incompatible if $|1 - 2^{-1/q+1/p}| > 10\varepsilon$. □

2.6 Proposition *Every separable doubly atomless BL_pL_q-Banach space is isomorphic to $L_p([0,1]; L_q([0,1])$ (which we denote by $\boldsymbol{L_p(L_q)}$). Hence the theory of doubly atomless BL_pL_q-Banach lattices is separably categorical (i.e., it has only one separable model up to isomorphism).*

Proof Let X be a separable doubly atomless BL_pL_q-Banach lattice. It can be represented as a band in a space $L_p(\Omega_1, \mathcal{A}_1, \mu_1; L_q(\Omega_2, \mathcal{A}_2, \mu_2))$, where $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are atomless measure spaces. Moreover one may assume that the least rectangle containing the support of this band is $\Omega_1 \times \Omega_2$.

First we will show that the spaces $L_1(\Omega_1, \mathcal{A}_1, \mu_1)$ and $L_1(\Omega_2, \mathcal{A}_2, \mu_2)$ are both separable. Indeed, each of these L_1-spaces is isomorphic to a direct sum of spaces of the form $L_1([0,1]^\kappa)$, where $[0,1]$ is equipped with Lebesgue measure and κ is some cardinal number. If for example $L_1(\Omega_2, \mathcal{A}_2, \mu_2)$ contains a component $L_1(B) \simeq L_1([0,1]^\kappa)$ with $\kappa > \aleph_0$, then since the support S_X intersects $\Omega_1 \times B$ (*i.e.*, the intersection has positive measure) there is some component $L_1(A) \simeq L_1([0,1]^\alpha)$ of $L_1(\Omega_1, \mathcal{A}_1, \mu_1)$ such that S_X intersects $A \times B$. That is, $T\colon = S_X \cap (A \times B)$ has positive measure. Since the measure space $[0,1]^\alpha \times [0,1]^\kappa = [0,1]^{\alpha+\kappa}$ is homogeneous, $L_1(A \times B)$ is isomorphic to its band $L_1(T, (\mathcal{A}_1 \otimes \mathcal{A}_2)|_T, (\mu_1 \otimes \mu_2)|_T)$; hence this last space is not separable. Then the band generated by T in X is not separable either (since the norm of X dominates the L_1-norm up to a constant factor on the set T, which is of finite measure). So in fact every component in the decomposition of both L_1-spaces is separable. On the other hand both of the measure spaces are σ-finite: if for example $(\Omega_2, \mathcal{A}_2, \mu_2)$ were not σ-finite, then Ω_2 would contain an uncountable family (B_γ) of disjoint measurable sets of positive measure. Each of the bands generated by $\Omega_1 \times B_\gamma$ in X would in turn contain a non-zero element x_γ, and X would contain an uncountable family(x_γ) of pairwise disjoint, non zero elements and would not be separable, a contradiction. This completes the proof that $L_1(\Omega_1, \mathcal{A}_1, \mu_1)$ and $L_1(\Omega_2, \mathcal{A}_2, \mu_2)$ are both separable.

Since both measure spaces $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are also atomless, it follows that the corresponding L_p-spaces are isomorphic to the usual Lebesgue spaces $L_p([0,1])$. Thus we may suppose that X is a band in $\boldsymbol{L_p(L_q)}$.

Let $f_0 \in X$ be a positive element of maximal support. We may suppose that $N(f_0)$ is an indicator function; its support is then the base-support Ω_X of X. Let $w = |f_0|^q$; denoting also by w a non-negative function measurable in the two variables representing w, set

$$W(s,t) = \int_0^t w(s,u)\,du.$$

Then W is a measurable function of the two variables (s,t), increasing and continuous with respect to the second variable, and $W(s,1) = 1$. The change of variable formula for the Lebesgue integral [Ran, Corollary 6.3.17] yields that

$$\int_0^1 h(W(s,u))\,w(s,u)\,du = \int_0^1 h(W(s,u))\,dW_s(u) = \int_0^1 h(t)\,dt$$

for every $h \in L_1([0,1])$. Let $Y = L_p(\Omega_X; L_q([0,1])$. Then we define a linear map $T\colon L_0(\Omega_X \times [0,1])$ into $L_0([0,1]^2)$ by

$$Tf(s,t) = f(s, W(s,t))f_0(s,t)$$

Then clearly

$$N(Tf) = N(f).$$

In particular T defines an embedding from Y into $L_p([0,1]; L_q([0,1]))$, in fact into the band generated by f_0, which is X. Note that T preserves random norms and is modular for the action of $L_\infty(\Omega_X)$, that is

$$T(\phi f) = \phi f$$

for all $f \in Y$ and $\phi \in L_\infty(\Omega_X)$. Let us show that T is surjective (from Y onto X). If $0 \le a \le 1$, $E_a = \{(s,u) \in [0,1]^2 \mid u \le W(s,a)\}$ and $F_a = \Omega_X \times [0,a]$, we have

$$W^{-1}(E_a) = \{(s,t) \mid W(s,t) \le W(s,a)\} \supset \Omega_X \times [0,a] = F_a$$

thus

$$T\mathbf{1}_{E_a} = (\mathbf{1}_{E_a} \circ W) f_0 \ge \mathbf{1}_{F_a} f_0$$

On the other hand $N(T\mathbf{1}_{E_a}) = N(\mathbf{1}_{E_a}) = W(\cdot,a)^{1/q} = N(\mathbf{1}_{F_a} f_0)$ and thus $\|T\mathbf{1}_{E_a}\| = \|\mathbf{1}_{F_a} f_0\|$. Since the lattice norm on $L_p(L_q)$ is strictly monotone, we have thus

$$T\mathbf{1}_{E_a} = \mathbf{1}_{F_a} f_0.$$

By modularity of T we have that $\mathbf{1}_{A\times F_a} f_0$ belongs to the range of T for every measurable subset A of Ω_X and $a \in [0,1]$. By linearity and density, the range of T (which is closed since T is isometric) contains X. So $T\colon Y \to X$ is an isomorphism.

Finally we may find an isomorphism U from $L_p(\Omega_X)$ onto $L_p([0,1])$, *e.g.*, mapping $\mathbf{1}_{\Omega_X}$ onto $|\Omega_X|^{1/p}\mathbf{1}_{[0,1]}$. Then $U \otimes I$ maps Y onto $L_p([0,1]; L_q([0,1])$ isometrically. □

2.7 Corollary *For any pair X_1, X_2 of doubly atomless BL_pL_q-Banach lattices there is an ultrafilter $\mathcal{U}$ such that the corresponding ultrapowers $(X_1)_\mathcal{U}, (X_2)_\mathcal{U}$ are isomorphic.*

Proof By the Downward Löwenheim-Skolem Theorem (see [HI, Proposition 9.13]) every BL_pL_q-Banach lattice X contains a separable one X_0 that is an elementary substructure of X. In particular X and X_0 are elementary equivalent. If X is doubly-atomless, then by Proposition 2.5 so is X_0, which is thus isomorphic to $\boldsymbol{L_p(L_q)}$, by Proposition 2.6. Hence all the doubly-atomless BL_pL_q-spaces are elementary equivalent and the Ultrapower Theorem ([HI, Theorem 10.7]) shows that any two doubly atomless BL_pL_q-Banach lattices have isomorphic ultrapowers. □

2.8 Proposition *The following classes of Banach lattices are closed under unions of increasing chains:*
i) the class of BL_pL_q-Banach lattices;
ii) the class of doubly atomless BL_pL_q-Banach lattices.

Proof i) Let $(X_i)_{i\in I}$ be an increasing chain of BL_pL_q-Banach lattices and $X = \overline{\cup_i X_i}$ be the completion of their union. By [HR, Proposition 3.6], it suffices to check that X is a $(\mathcal{L}_p\mathcal{L}_q)_1$-Banach lattice, that is: for every finite family $x_1, \ldots, x_n$ of disjoint elements of X and every $\varepsilon > 0$, there exists a finite dimensional sublattice F of X and a vector lattice isomorphism T from F onto a finite-dimensional BL_pL_q-space such that $\|T\|\,\|T^{-1}\| < 1+\varepsilon$ and $\mathrm{dist}(x_k, F) < \varepsilon$, $k = 1, \dots n$.

In fact, decomposing x_k, $k = 1, \dots, n$ into their positive and negative parts, we may reduce to the case where $x_k \geq 0$, $k = 1, \dots, n$. Then find $i \in I$ and $y_1, \dots y_n$ in X_i^+ such that $\|x_k - y_k\| \leq \varepsilon' := \varepsilon/(2n+1)$, $k = 1, \dots n$. For $j \neq k$ we have then $\|\,|y_j| \wedge |y_k|\,\| \leq 2\varepsilon'$ by the triangle inequality. Setting $z_j = \big(y_j - \bigvee_{k\neq j} y_k\big)_+$, we obtain positive, disjoint elements $z_1, \dots z_n \in X_i$ with $\|x_j - z_j\| \leq 2n\varepsilon'$, $j = 1, \dots n$. Since X_i is a $(\mathcal{L}_p\mathcal{L}_q)_1$-Banach lattice, it contains a finite-dimensional sublattice F such that $\mathrm{dist}\,(z_j, F) < \varepsilon'$, $j = 1, \dots n$ and there is a vector lattice isomorphism T from F onto a finite-dimensional BL_pL_q-space with $\|T\|\,\|T^{-1}\| < 1+\varepsilon$. Then $\mathrm{dist}\,(x_j, F) < (2n+1)\varepsilon' < \varepsilon$, $j = 1, \dots n$. Since $F \subset X$ we are done.

ii) We now have to prove that if, moreover, the BL_pL_q-Banach lattices in the chain are doubly atomless, then so is X. This is a trivial consequence of the axiomatization of both base- and fiber-atomless properties by means of a $\forall\exists$ axiom (see the proof of Proposition 2.5). □

3 Continuation of positive isometries on subcones of L_α^+

The following equimeasurability result on the positive axis is known (see [Li, Lemma 2]); for the sake of completeness we give a short proof, following [Ray].

3.1 Lemma *Let α be a positive real number that is not an integer. Let μ, ν be two probability measures on $[0, +\infty)$, each having a moment of order α. If*

$$\int_0^{+\infty} (t+a)^\alpha \, d\mu(t) = \int_0^{+\infty} (t+a)^\alpha \, d\nu(t) \tag{1}$$

for every $a \geq 0$, then $\mu = \nu$.

Proof Assume first $0 < \alpha < 1$. We make use of the formula

$$\forall s > 0, \quad s^\alpha = C_\alpha \int_0^{+\infty} (1 - e^{-us}) \frac{du}{u^{\alpha+1}} \,.$$

Then by Fubini's theorem

$$\int_0^{+\infty} (t+a)^\alpha \, d\mu(t) = C_\alpha \int_0^{+\infty} \int_0^{+\infty} (1 - e^{-u(t+a)}) \frac{du}{u^{\alpha+1}} \, d\mu(t) =$$

$$= C_\alpha \int_0^{+\infty} \left[1 - \int_0^{+\infty} e^{-u(t+a)} \, d\mu(t)\right] \frac{du}{u^{\alpha+1}}$$

$$= C_\alpha \int_0^{+\infty} \left[1 - e^{-ua} \mathcal{L}\mu(u)\right] \frac{du}{u^{\alpha+1}}$$

where $\mathcal{L}\mu$ denotes the Laplace transform of μ. For all $u > 0$, set

$$\Phi_\mu(u) = \frac{1 - \mathcal{L}\mu(u)}{u^{\alpha+1}} \, .$$

Observe that $\mathcal{L}_\mu(u) \leq 1$, hence $\Phi_\mu(u) \geq 0$ for all $u > 0$. If we put $a = 0$ in the above formula, we get

$$C_\alpha \int_0^{+\infty} \Phi_\mu(u) \, du = \int_0^{+\infty} t^\alpha \, d\mu(t) < +\infty$$

so

$$\int_0^{+\infty} (t+a)^\alpha \, d\mu(t) = C_\alpha \int_0^{+\infty} \left(\frac{1 - e^{-ua}}{u^{\alpha+1}} + e^{-ua} \Phi_\mu(u) \right) du =$$

$$= a^\alpha + \mathcal{L}_{\Phi_\mu}(a)$$

Hence equation (1) implies that $\mathcal{L}_{\Phi_\mu} = \mathcal{L}_{\Phi_\nu}$; then by injectivity of the Laplace transform [Ka, p. 63] we have $\Phi_\mu = \Phi_\nu$, so $\mathcal{L}\mu = \mathcal{L}\nu$, and thus $\mu = \nu$.

This proves the Lemma in the case $0 < \alpha < 1$. Now if $\alpha > 1$, we may differentiate both sides of equation (1) with respect to a and obtain

$$\alpha \int_0^{+\infty} (t+a)^{\alpha-1} \, d\mu(t) = \alpha \int_0^{+\infty} (t+a)^{\alpha-1} \, d\nu(t)$$

which is exactly the result of replacing α by $\alpha - 1$ in equation (1). Iterating, we descend to the exponent $\beta = \alpha - [\alpha]$ (the fractional part of α) which satisfies $0 < \beta < 1$ unless $\alpha \in \mathbb{N}$, and we conclude the proof using the above reasoning. □

3.2 Proposition *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\alpha > 0$, $\alpha \notin \mathbb{N}$. Assume that $f_1, \ldots, f_n, g_1, \ldots, g_n$ are elements of the positive cone of $L_\alpha(\Omega, \mathcal{A}, P)$ satisfying*

$$\|\mathbf{1} + \sum_{j=1}^n \lambda_j f_j\|_\alpha = \|\mathbf{1} + \sum_{j=1}^n \lambda_j g_j\|_\alpha \tag{2}$$

for every system $(\lambda_1, \ldots, \lambda_n)$ of positive coefficients ($\mathbf{1}$ denotes the constant unit function). Then the random vectors $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ have the same joint probability distribution.

Proof From Lemma 3.1 we see that for any $\lambda_1, ..., \lambda_n \geq 0$

$$\operatorname{dist}\big(\sum_{j=1}^{n} \lambda_j f_j\big) = \operatorname{dist}\big(\sum_{j=1}^{n} \lambda_j g_j\big)$$

Let ν_f, resp. ν_g be the probability distribution of the random vector $f = (f_1, .., f_n)$, resp. $g = (g_1, ..., g_n)$. We have for every $\lambda_1, ... \lambda_n \in \mathbb{R}_+$

$$\int_{\mathbb{R}^n_+} \exp\big[-\sum_{j=1}^{n} \lambda_j t_j\big]\, d\nu_f(t) = \int_{\mathbb{R}^n_+} \exp\big[-\sum_{j=1}^{n} \lambda_j t\big]\, d\nu_g(t)$$

and therefore

$$\mathcal{L}\nu_f = \mathcal{L}\nu_g;$$

i.e., ν_f and ν_g have the same multivariate Laplace transform. By injectivity of the multivariate Laplace transformation (*e.g.*, see [Ka, Theorem 4.3]) it follows that $\nu_f = \nu_g$. □

3.3 Proposition *Let $0 < \alpha \notin \mathbb{N}$ and $\Gamma \subseteq L^+_\alpha(\mu)$ be a subcone of the positive cone of some $L_\alpha(\mu)$-space. Let $T \colon \Gamma \to L^+_\alpha(\nu)$ be a normed cone morphism into the positive cone of another L_α-space. Then T extends uniquely to an isomorphism from the closed sublattice generated by Γ onto the closed sublattice generated by $T(\Gamma)$.*

Proof The argument follows familiar lines (see Hardin's [Ha, proof of Theorem 2.2]). The uniqueness of this extension, if it exists, is clear. We show its existence.

a) We assume first that μ, ν are probability measures, Γ contains the positive constants, and that T maps constants to constants ($T\mathbf{1} = \mathbf{1}$).

Then by Proposition 3.2 we have $\operatorname{dist}(Tf_1, ..., Tf_n) = \operatorname{dist}(f_1, ..., f_n)$ for every $f_1, ..., f_n \in \Gamma$. In particular for all Borel subsets $B_1, \ldots, B_n$ of $\mathbb{R}$ we have $\mu(\{f_i \in B_i, i = 1, ...n\}) = \nu(Tf_i \in B_i, i = 1, ..n)$. Set $\rho(\{f_i \in B_i, i = 1, ...n\}) = \{Tf_i \in B_i, i = 1, ...n\}$ for every finite system $(f_1, ..., f_n, B_1, ..., B_n)$; this map is well defined up to sets of measure zero and preserves measure. It extends first to a Boolean measure-preserving isomorphism from the ring of measurable sets generated by the $f \in \Gamma$ onto that generated by the $Tf \in T(\Gamma)$, and then between the σ algebras $\sigma(\Gamma)$ and $\sigma(T(\Gamma))$ generated by Γ, resp. $T(\Gamma)$. This isomorphism in turn induces an isometry U_ρ from $L_\alpha(\Omega, \sigma(\Gamma), \mu)$ onto $L_\alpha(\Omega', \sigma(T(\Gamma)), \nu)$. Approximating every $f \in \Gamma$ by step functions on sets $a_i \leq f < b_i$, one sees that U_ρ restricts to T on Γ. Moreover the sublattice E generated by Γ is clearly included in $L_\alpha(\Omega, \sigma(\Gamma), \mu)$, and since U_ρ is an embedding, its restriction to E gives the desired extension of T (in fact since $\mathbf{1} \in \Gamma$ it turns out that $E = L_\alpha(\Omega, \sigma(\Gamma), \mu)$).

b) We treat now the general case. If $\Phi = \{f_1, ..., f_n\}$ is a finite subset in Γ, the cone Γ_Φ generated by Φ has an element of maximal support $f = f_1 + ... + f_n$. This is also the maximal support of the closed sublattice generated by Γ_Φ. Let S_Φ be the support of f and S'_Φ that of Tf. After a change of density (reducing to the case where $f = \mathbf{1}_{S_\Phi}$, $Tf = \mathbf{1}_{S'_\Phi}$) limited to the bands generated by Γ_Φ, resp. $T(\Gamma_\Phi)$, the argument in part (*a*) gives an isomorphism U_Φ from the closed sublattice generated by Γ_Φ onto that generated by Γ_{T_Φ}, which extends $T_{|\Gamma_\Phi}$. If $\Phi_1 \subset \Phi_2$, then $U_{\Phi_2|\Gamma_{\Phi_1}} = U_{\Phi_1}$ by uniqueness, so there exists a common extension U of all U_Φ's to a map from the sublattice generated by Γ onto that generated by $T(\Gamma)$. Since U, U^{-1} are isometric they extend to the closures. □

4 Quantifier elimination for the theory of doubly atomless BL_pL_q

Let us first recall what it means for a normed structure $\mathcal{M}$ to be ω_1-*saturated*. For $r > 0$ we let B_r denote the closed ball of radius r in $\mathcal{M}$. Then $\mathcal{M}$ is ω_1-saturated if for every countable subset C of $\mathcal{M}$, every set $\Gamma(x_1, \dots, x_n)$ of positive bounded formulas in the signature of $\mathcal{M}$ in which elements of C are allowed as parameters, and every $r > 0$, if each finite subset of $\Gamma(x_1, \dots, x_n)$ can be satisfied in $\mathcal{M}$ by elements $x_1, \dots, x_n$ of B_r, then the entire set $\Gamma(x_1, \dots, x_n)$ can be satisfied in $\mathcal{M}$ by elements of B_r. (This definition is equivalent to [HI, Definition 9.17] in view of [HI, Proposition 9.20].)

In this section we sometimes need that certain normed structures are ω_1-saturated. For this reason we consider ultrapowers with respect to a given countably incomplete ultrafilter $\mathcal{U}$. Indeed, when the signature of a normed structure $\mathcal{M}$ is countable (as is the case for Banach lattices) and $\mathcal{U}$ is countably incomplete, the ultrapower $\mathcal{M}_\mathcal{U}$ is ω_1-saturated. (See [HI, Proposition 9.18].)

The definition of ω_1-saturation as well as the preceding observation extend without difficulty to the case of p-normed structures, $0 < p \leq 1$. (See [JL, p. 1102] for the definition of p-norms.) We shall use the following two facts about ω_1-saturated structures:

– An ω_1-saturated and order-continuous Banach lattice L is not σ-finite; that is, for every sequence $(c_k \mid k \geq 1)$ of elements of L there exists a non-zero element x of L that is disjoint from every c_k. (Proof: for every $n \geq 1$ there exists a normalized element x_n in L with $\||x_n| \wedge |c_k|\| < \frac{1}{n}$ for all $k = 1, \dots, n$. Applying ω_1-saturation of L using the countable set of parameters $C = \{c_k \mid k \geq 1\}$ and the set of positive bounded formulas $\Gamma(x) = \{\||x| \wedge |c_k|\| < \frac{1}{n} \mid n, k \geq 1\} \cup \{\|x\| = 1\}$ yields a normalized element x satisfying $\||x| \wedge |c_k|\| = 0$ for all k.)

– If $0 < \alpha < \infty$ and $L_\alpha(\Omega, \mathcal{A}, \mu)$ is atomless and ω_1-saturated, then every embedding T from a sublattice of $L_\alpha[0,1]$ into $L_\alpha(\Omega, \mathcal{A}, \mu)$ can be extended to an embedding of $L_\alpha[0,1]$ into $L_\alpha(\Omega, \mathcal{A}, \mu)$. This is proved in [HI, Example 13.18] in the case $1 \leq \alpha < \infty$, and the proof extends easily to the non convex case $0 < \alpha < 1$ (where L_α is only α-normed).

4.1 Lemma *Let $1 \leq p, q < \infty$ and let $\mathcal{Y}$ be a fiber-atomless AL_pL_q-space.*
a) For every $g \in L_p^+$, $y \in \mathcal{Y}$ and $\varepsilon > 0$ there exists $x \in \mathcal{Y}$ such that $N(x) = g$ and $\||x| \wedge |y|\| \leq \varepsilon(\|g\| + \|y\|)$.
b) If $\mathcal{Y} = Y_\mathcal{U}$ for some AL_pL_q-Banach lattice Y and some countably incomplete ultrafilter $\mathcal{U}$, the preceding statement is true with $\varepsilon = 0$.

Proof a) We may assume $\varepsilon \leq 1$. Since the random norm N maps $\mathcal{Y}$ onto the positive cone of the associated L_p-space, there is $x_0 \in \mathcal{Y}$ with $N(x_0) = g$. Because $\mathcal{Y}$ is fiber-atomless, we may find a component z of $|x_0| + |y|$ with $N(z) = \varepsilon N(|x_0| + |y|)$. Setting $x = \varepsilon^{-1} \frac{g}{N(|x_0|+|y|)} z$, we have

$$|x| \wedge |y| \leq |x| \wedge (|x_0| + |y|)) \leq (\varepsilon^{-1}|z|) \wedge (|x_0| + |y|) = |z|$$

and hence:

$$N(|x| \wedge |y|) \leq N(z) = \varepsilon N(|x_0| + |y|) \text{ and } \||x| \wedge |y|\| \leq \varepsilon \|N(|x_0| + |y|\|) \leq \varepsilon(\|g\| + \|y\|).$$

b) Assume that Y is represented as an AL_pL_q-space, with random norm $N\colon Y \to L$ (for some abstract L_p-space L). Then $\mathcal{Y}_U$ is represented as an AL_pL_q-space, with random norm $N_\mathcal{U}\colon Y_\mathcal{U} \to L_\mathcal{U}$. The pair (Y, L) of Banach lattices, together with the additional function $N\colon A \to L$ (which is 1-Lipschitz) form a 2-sorted normed space structure in the sense of [HI]. Its ultrapower $(Y, L, N)_\mathcal{U} = (Y_\mathcal{U}, L_\mathcal{U}, N_\mathcal{U})$ is ω_1-saturated, so statement b) follows immediately from a). $\square$

4.2 Proposition *Let $1 \leq p, q < \infty$ with $\alpha := p/q \notin \mathbb{N}$ and $\mathcal{U}$ a countably incomplete ultrafilter. Let X, Y be BL_pL_q-Banach lattices with X separable, Y doubly atomless. For every proper closed sublattice E of X and every embedding T from E into $Y_\mathcal{U}$ there is a proper extension $\widetilde{T}$ of T that is still an embedding.*

Proof We may represent X as a band in some separable $L_p(\Omega, \mathcal{A}, \mu; L_q(\Omega', \mathcal{A}', \mu'))$ and assume that Y is equipped with a q-random norm N with values in some atomless L_p-space $L_p(S, \Sigma, \nu)$. Then $Y_{\mathcal{U}}$ is equipped with a random norm with values in the abstract L_p-space $(L_p(S, \Sigma, \nu))_{\mathcal{U}}$ that we may represent as a concrete L_p-space $L_p(\tilde{S}, \tilde{\Sigma}, \tilde{\nu})$. Fix $g \in X \setminus E$. It is sufficient to prove that for every finite-dimensional sublattice F of E the restriction T_F of T to F has an extension $\widetilde{T_F}$ to the sublattice generated by F and g that is an embedding. Indeed this is equivalent to the existence of an element h in $Y_{\mathcal{U}}$ satisfying

$$\|t(Tf_1, ..., Tf_n, \lambda_1, ..., \lambda_m, h)\| = \|t(f_1, .., f_n, \lambda_1, ..., \lambda_m, g)\|$$

for every lattice term $t(x_1, ..., x_n, t_1, ..., t_m, y)$ and every choice of parameters $f_1, .., f_n \in F$, $\lambda_1, ..., \lambda_m \in \mathbb{R}$. One can make this set of conditions Γ_F countable by taking a countable dense subset of parameters. Consider a sequence (F_n) of finite-dimensional sublattices of E that generates E as closed sublattice of X. The existence of such a sequence results, for example, from the representation of the order-continuous Banach lattice E as a Köthe function space and a standard argument based on approximation by simple functions. By ω_1-saturation of $Y_{\mathcal{U}}$ there is an element $h \in Y_{\mathcal{U}}$ that satisfies all the conditions in the union of the sets Γ_{F_n}. Then there is a isomorphism $\widetilde{T}$ from the closed sublattice generated by E and g into $Y_{\mathcal{U}}$ extending T and such that $\widetilde{T}g = h$, and such an extension is clearly unique.

So in the following we may suppose that E is finite-dimensional. Fix disjoint positive elements $f_1, ..f_n$ generating E; set $u = \sum_{i=1}^n f_i$. Let g_0 be the component of g in the complementary band $E^\perp$ (the subspace of elements of X that are disjoint from E). We shall assume also that $g = g_0 + \mathbf{1}_A u$ for some measurable set $A \subset \Omega \times \Omega'$. Then iterating the above procedure we shall be able to treat the case where $g = g_0 + \gamma.u$ where γ is a simple function, and finally, by a density argument, the case of a general g. Let $\varphi_i = N(f_i)^q$ and $\psi_i = N(Tf_i)^q$. Let $g'_i = \mathbf{1}_A f_i$ and $g''_i = \mathbf{1}_{A^c} f_i$; note that the closed sublattice generated by E and g is simply the linear span of $g_0, g'_1, g''_1, ..., g'_n, g''_n$. Set $\varphi'_i = N(g'_i)^q$, $\varphi''_i = N(g''_i)^q$ and $\varphi_0 = N(g_0)^q$. Note that $\varphi_i = \varphi'_i + \varphi''_i$ for all $i = 1, ..., n$. For every $\lambda_1, ..., \lambda_n \in \mathbb{R}$ we have

$$\|\sum_j \lambda_j f_j\|_X^q = \|N(\sum_j \lambda_j f_j)\|_p^q = \|\big(\sum_j |\lambda_j|^q N(f_j)^q\big)^{1/q}\|_p^q = \|\sum_j |\lambda_j|^q \varphi_j\|_\alpha$$

and similarly

$$\|\sum_j \lambda_j T(f_j)\|_{Y_{\mathcal{U}}}^q = \|\sum_j |\lambda_j|^q \psi_j\|_\alpha .$$

Hence for any positive coefficients $\lambda_1, .., \lambda_n \geq 0$, we have

$$\|\sum_j \lambda_j \varphi_j\|_\alpha = \|\sum_j \lambda_j \psi_j\|_\alpha.$$

By Proposition 3.3 there is an unique isomorphism V from the closed sublattice generated by $\varphi_1, .., \varphi_n$ onto the closed sublattice generated by $\psi_1, .., \psi_n$ such that $V\varphi_i = \psi_i$, $i = 1, ..., n$.

Let $\mathcal{A}_0 = \sigma(\varphi_0, \varphi_1', \varphi_1'' ..., \varphi_n', \varphi_n'')$ be the generated σ-algebra. Since $(L_\alpha(S, \Sigma, \nu))_{\mathcal{U}}$ is an ω_1-saturated and atomless L_α-space (recall that $Y_{\mathcal{U}}$ is base-atomless like Y) there exists an embedding U, extending V, from $L_\alpha(\mathcal{A}_0)$ into this L_α-space. We have

$$\psi_i = U\varphi_i = U\varphi_i' + U\varphi_i''$$

for every $i = 1, ..., n$, with $U\varphi_i', U\varphi_i'' \geq 0$. We need to find a disjoint decomposition $Tf_i = h_i' + h_i'' \in Y_{\mathcal{U}}$ such that

$$N(h_i')^q = U\varphi_i', N(h_i'')^q = U\varphi_i''$$

Since $Y_{\mathcal{U}}$ is fiber-atomless, for every $\chi_i \in L_\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\nu})$ with $0 \leq \chi_i \leq 1$ we can find disjoint elements h_i' and h_i'' in $Y_{\mathcal{U}}$ such that $Tf_i = h_i' + h_i''$ and $N(h_i')^q = \chi_i N(Tf_i)^q, N(h_i'')^q = (1 - \chi_i)N(Tf_i)^q$ (see Remark 2.4). In the present case, choose $\chi_i = (U\varphi_i')/\psi_i$. Also take $h_0 \in (Y_{\mathcal{U}})_+$ disjoint from $h_1, ..., h_n$ such that $N(h_0)^q = U\varphi_0$. Such an element exists by Lemma 4.1(b), since the ultrafilter $\mathcal{U}$ is countably incomplete and the ultrapower $Y_{\mathcal{U}}$ is fiber-atomless. We have then for every $\lambda_0, \lambda_1', \lambda_1'', ...\lambda_n', \lambda_n'' \in \mathbb{R}$

$$\begin{aligned} N\big(\lambda_0 h_0 + \sum_{j=2}^{n}(\lambda_j' h_j' + \lambda_j'' h_j'')\big)^q &= |\lambda_0|^q N(h_0)^q + \sum_{j=2}^{n}(|\lambda_j'|^q N(h_j')^q + |\lambda_j''|^q N(h_j'')^q) \\ &= |\lambda_0|^q U\varphi_0 + \sum_{j=2}^{n}(|\lambda_j'|^q U\varphi_j' + |\lambda_j''|^q U\varphi_j'') \\ &= U\big(|\lambda_0|^q \varphi_0 + \sum_{j=2}^{n}(|\lambda_j'|^q \varphi_j' + |\lambda_j''|^q \varphi_j'')\big) \end{aligned}$$

while similarly

$$N\big(\lambda_0 g_0 + \sum_{j=2}^{n}(\lambda_j' g_j' + \lambda_j'' g_j'')\big)^q = |\lambda_0|^q \varphi_0 + \sum_{j=2}^{n}(|\lambda_j'|^q \varphi_j' + |\lambda_j''|^q \varphi_j'').$$

Since U is an isometry, this implies

$$
\begin{aligned}
\|\lambda_0 h_0 + \sum_{j=2}^{n}(\lambda_j' h_j' + \lambda_j'' h_j'')\|_X^q &= \|N\big(\lambda_0 h_0 + \sum_{j=2}^{n}(\lambda_j' h_j' + \lambda_j'' h_j'')\big)^q\|_\alpha \\
&= \|N\big(\lambda_0 g_0 + \sum_{j=2}^{n}(\lambda_j' g_j' + \lambda_j'' g_j'')\big)^q\|_\alpha \\
&= \|\lambda_0 g_0 + \sum_{j=2}^{n}(\lambda_j' g_j' + \lambda_j'' g_j'')\|_{Y_{\mathcal{U}}}^q .
\end{aligned}
$$

Therefore, we may define an embedding $\widetilde{T}$ extending T by setting $\widetilde{T}g_0 = h_0$, $\widetilde{T}g_i' = h_i'$, and $\widetilde{T}g_i'' = h_i''$, for $i = 1, \dots n$. □

4.3 Corollary *Let $1 \le p, q < \infty$ with $\alpha := p/q \notin \mathbb{N}$ and $\mathcal{U}$ a countably incomplete ultrafilter. Let X, Y be BL_pL_q-Banach lattices with X separable, Y doubly atomless. For every closed sublattice E of X and every embedding from E into $Y_{\mathcal{U}}$, there is an extension $\widetilde{T}$ of T to the whole of X that is still an embedding.*

Proof Using Zorn's Lemma one finds a maximal extension $\widetilde{T}$ of the given embedding to some closed sublattice $\widetilde{E}$ of X. By Proposition 4.2 we necessarily have $\widetilde{E} = X$. □

4.4 Corollary *For every $1 \le p, q < \infty$ with $\alpha := p/q \notin \mathbb{N}$, the theory of doubly atomless BL_pL_q-Banach lattices has quantifier elimination.*

Proof By [HI, Proposition 13.17] it is sufficient to prove that every embedding T from a sublattice E of a separable doubly atomless BL_pL_q-space X into another doubly atomless BL_pL_q-space Y can be extended to an embedding from X into a suitable ultrapower of Y. This is an immediate consequence of Corollary 4.3 if we take an ultrapower of Y defined by a countably incomplete ultrafilter. □

5 A counterexample to QE for integer values of p/q $(\neq 1)$

As in Lusky's paper [Lu] we consider the interval $I = (0, \infty)$ equipped with the measures μ_1 and μ_2 defined by

$$d\mu_1(t) = t^3 e^{-t}(1 + \sin t)\, dt \text{ and } d\mu_2(t) = t^3 e^{-t}(1 - \sin t)\, dt.$$

In view of the identity

$$\int_0^\infty t^{s-1} e^{-t} \sin t \, dt = \Gamma(s) 2^{-s/2} \sin\frac{s\pi}{4} \text{ for all } s > 0$$

we have

$$\int_0^\infty t^k \, d\mu_1 = \int_0^\infty t^k \, d\mu_2$$

for all $k \in 4\mathbb{N}$. Hence

$$\int_0^\infty f^m \, d\mu_1 = \int_0^\infty f^m \, d\mu_2$$

for every integer m and every f in the cone generated by the functions $t^{4\ell}$, $\ell \in \mathbb{N}$. In particular if we consider the cone C generated by the functions $\mathbf{1}$ and $f_1 : t \mapsto t^4$, the identity operator $T = \mathrm{id}_{|C} \colon C \to C$ is norm-preserving for the norms of $L_m(\mu_1)$, resp. $L_m(\mu_2)$ (for any integer $m \geq 1$).

5.1 Lemma *Let m be an integer ≥ 1. There is no positively linear extension $\widetilde{T} \colon L_m(\mu_1)_+ \to L_m(\mu_2)_+$ of T with $\|\widetilde{T}\| \leq 1$ (that is, $\|\widetilde{T}f\|_m \leq \|f\|_m$ for every $f \in L_m(\mu)_+$).*

Proof If such an extension $\widetilde{T}$ exists, then $\|\widetilde{T}f\|_\infty \leq \|f\|_\infty$ for every $f \in L_\infty(\mu)_+$ since $0 \leq f \leq \mathbf{1}$ implies $0 \leq \widetilde{T}f \leq \widetilde{T}\mathbf{1} = \mathbf{1}$. By a standard lattice interpolation argument we obtain $\|\widetilde{T}f\|_q \leq \|f\|_q$ for every $f \in L_q(\mu)_+$, for every $m \leq q \leq \infty$. This is indeed a simple consequence of the identity

$$u^\theta = \inf\{au + b \mid a, b > 0, a^\theta b^{1-\theta} = C_\theta\}$$

which is valid for every $u \geq 0$ and $0 < \theta < 1$ with $C_\theta = \theta^\theta (1-\theta)^{1-\theta}$. If $m < q < \infty$, set $\theta = m/q$. For every $f \in L_q(\mu)_+$ we have $g := f^{1/\theta} \in L_m(\mu)_+$. Then using the fact that $\widetilde{T}$ is non-decreasing, positively linear and $\widetilde{T}\mathbf{1} = T\mathbf{1} = \mathbf{1}$

$$\widetilde{T}f = \widetilde{T}(g^\theta) = \widetilde{T}(\inf_{a^\theta b^{1-\theta} = C_\theta} ag + b\mathbf{1}) \leq \inf_{a^\theta b^{1-\theta} = C_\theta} a\widetilde{T}g + bT\mathbf{1} = (\widetilde{T}g)^\theta$$

hence

$$\|\widetilde{T}f\|_q \leq \|(\widetilde{T}g)^\theta\|_q = \|\widetilde{T}g\|_m^\theta \leq \|g\|_m^\theta = \|f\|_q$$

Now we compare the L_q-norms of f_1 and $\widetilde{T}f_1 = Tf_1$

$$\|Tf_1\|_q^q - \|f_1\|_q^q = -2 \int_0^{+\infty} t^{4q+3} e^{-t} \sin t \, dt = -2^{-2q-1} \Gamma(4(q+1)) \sin[(q+1)\pi]$$

For every m we can find $q > m$ such that the last expression is strictly positive, *i.e.* $\|Tf_1\|_q > \|f_1\|_q$, which is a contradiction. □

The following result is a consequence of some of the results in [GR] in the case where $E = F$. In the separable case it is implied by the results of Koldobskii [Ko]. Since the results of [GR] are somewhat dispersed and are stated in greater generality, we give here a self-contained proof.

5.2 Lemma *If T is an embedding of a space $E = L_p(\mu; L_q(\nu))$ into a space $F = L_p(\mu'; L_q(\nu'))$ there exists a positive and isometric linear map S from $L_p(\mu)$ into $L_p(\mu')$ such that*

$$\forall f \in E \ \ [N(Tf) = SN(f))]$$

Proof We first remark that T maps base-disjoint vectors onto base-disjoint vectors. For, T maps disjoint vectors onto disjoint vectors, and if f, g are base-disjoint

$$\|Tf + Tg\|^p = \|f + g\|^p = \|f\|^p + \|g\|^p = \|Tf\|^p + \|Tg\|^p$$

hence Tf, Tg are base-disjoint (see the proof of Lemma 2.1).

We now show that $N(Tf)$ depends only on $N(f)$. If the space $L_q(\nu)$ has dimension 1 this is trivial (then $N(f) = |f|$ and $N(Tf) = N(|Tf|) = N(T|f|)$ since T is an embedding). If $L_q(\nu)$ has dimension at least 2, we can decompose it into two complementary bands B_1, B_2 (hence $L_q(\nu) = B_1 \oplus_q B_2$). Let $E_i = L_p(\mu; B_i)$, $i = 1, 2$. If $x_i \in B_i$, $i = 1, 2$ are two non-zero vectors and $\varphi \in L_p(\mu)$, then setting $f_i = \varphi \otimes x_i$

$$\begin{aligned}\int (N(Tf_1)^q + N(Tf_2)^q)^{p/q} \, d\mu' &= \int N(Tf_1 + Tf_2)^p \, d\mu' = \|T(f_1 + f_2)\|^p \\ &= \|f_1 + f_2\|^p = (\|x_1\|_q^q + \|x_2\|_q^q)^{p/q} \|\varphi\|_p^p \\ &= (\|f_1\|^q + \|f_2\|^q)^{p/q} = (\|Tf_1\|^q + \|Tf_2\|^q)^{p/q} \\ &= (\|N(Tf_1)\|_p^q + \|N(Tf_2)\|_p^q)^{p/q}.\end{aligned}$$

In other words

$$\|N(Tf_1)^q + N(Tf_2)^q\|_{p/q} = \|N(Tf_1)^q\|_{p/q} + \|N(Tf_2)^q\|_{p/q};$$

i.e., the functions $N(Tf_1)^q$ and $N(Tf_2)^q$ satisfy the equality case in Minkowski's inequality if $p > q$, resp. the reverse Minkowski inequality if $p < q$. This is possible only if they are proportional, hence $N(Tf_2) = \lambda N(Tf_1)$ with $\lambda = \|N(Tf_2)\|_p / \|N(Tf_1)\|_p = \|f_2\| / \|f_1\|$, and since

$$\frac{\|Tf_i\|}{\|x_i\|} = \frac{\|f_i\|}{\|x_i\|} = \|\varphi\| \, , \ i = 1, 2$$

we obtain

$$\frac{N(Tf_1)}{\|x_1\|} = \frac{N(Tf_2)}{\|x_2\|} := \alpha$$

for some element $\alpha \in L_p^+(\mu')$ satisfying $\|\alpha\|_p = \|\varphi\|_p$. It is clear that α depends only on φ (not on the particular choice of the $x_i \in B_i$); accordingly, we shall write $\alpha = A(\varphi)$. Moreover if $x \in L_q(\nu)$, denoting by x_i its component in B_i

$$N(T(\varphi \otimes x)) = (N(T(\varphi \otimes x_1))^q + N(T(\varphi \otimes x_2))^q)^{1/q} = (\|x_1\|^q + \|x_2\|^q)^{1/q} A(\varphi) = \|x\| A(\varphi).$$

Since the equality of the first and last term was also clear in the case where $L_q(\nu)$ is 1-dimensional, from now on we do not make any assumption about the dimension of $L_q(\nu)$.

The map $A\colon L_p^+(\mu) \to L_p^+(\mu')$ is clearly homogeneous of degree one and preserves disjointness (since T preserve base-disjointness). It follows immediately that A is additive on simple functions in $L_p^+(\mu)$, and since it is continuous (by continuity of T) it is additive on $L_p^+(\mu)$. It is extended to a map $S\colon L_p(\mu) \to L_p(\nu)$ by setting

$$\varphi = \psi - \theta \text{ with } \psi, \theta \in L_p^+(\mu) \Longrightarrow S(\varphi) = A(\psi) - A(\theta).$$

Then S is a positive linear isometry since (denoting by φ_+ and φ_- the positive and negative part of φ)

$$\|S\varphi\|^p = \|A(\varphi_+) - A(\varphi_-)\|^p = \|A(\varphi_+)\|^p + \|A(\varphi_-)\|^p = \|\varphi_+\|^p + \|\varphi_-\|^p = \|\varphi\|^p.$$

Note that if $\varphi_1, .., \varphi_n$ are disjoint elements in $L_p^+(\mu)$ and $x_1, .., x_n$ are arbitrary vectors (not necessarily disjoint) in $L_q(\nu)$ we have

$$\begin{aligned} N\big(T\big(\sum_i \varphi_i \otimes x_i\big)\big) &= \sum_i N\big(\big(T\varphi_i \otimes x_i\big)\big) = \sum_i \|x_i\| S(\varphi_i) \\ &= S\big(\sum_i \|x_i\|\varphi_i\big) = S\big(N\big(\sum_i \varphi_i \otimes x_i\big)\big). \end{aligned}$$

By density of the simple functions in $L_p(\mu; L_q(\nu))$ we obtain

$$N(Tf) = S(N(f))$$

for every $f \in E$. $\square$

5.3 Lemma *Let $n \geq 1$ be an integer, $1 \leq p \neq q < \infty$, $\alpha = p/q$, $L_p = L_p(\Omega, \mathcal{A}, \mu)$, $L_q = L_q(\Omega', \mathcal{A}', \mu')$ and $E = L_p(L_q)$. Assume that L_q is at least n-dimensional and denote by L_α the Lebesgue space $L_\alpha(\Omega, \mathcal{A}, \mu)$. If the theory of the Banach lattice E satisfies quantifier elimination, then for all n-tuples $\underline{\varphi} = (\varphi_1, .., \varphi_n)$, $\underline{\psi} = (\psi_1, .., \psi_n)$ in L_α^+ such that*

$$\forall \lambda_1, ..., \lambda_n \in \mathbb{R}_+ \ \big(\|\sum_{i=1}^n \lambda_i \varphi_i\|_\alpha = \|\sum_{i=1}^n \lambda_i \psi_i\|_\alpha\big)$$

there is a normed cone morphism S_α from L_α^+ into the positive cone $\mathcal{L}_+$ of some ultrapower $\mathcal{L}$ of L_α such that $S_\alpha \varphi_i = \psi_i$, $i = 1, ...n$.

Proof We can find $\underline{f} = (f_1, ..., f_n)$ and $\underline{g} = (g_1, ...g_n)$ in E such that the f_i (resp the g_i) are pairwise disjoint and $N(f_i)^q = \varphi_i$, $N(g_i)^q = \psi_i$, $i = 1, ..., n$. Then for every

$\lambda_1, ..., \lambda_n \in \mathbb{R}$ we have

$$N\big(\sum_{i=1}^{n} \lambda_i f_i\big) = \sum_{i=1}^{n} |\lambda_i|^q \varphi_i$$

$$N\big(\sum_{i=1}^{n} \lambda_i g_i\big) = \sum_{i=1}^{n} |\lambda_i|^q \psi_i$$

hence

$$\|\sum_{i=1}^{n} \lambda_i f_i\|_E = \|\sum_{i=1}^{n} \lambda_i g_i\|_E.$$

Thus the linear map $T_0 \colon \text{span}(\text{f}_\text{i}) \to \text{E}$ such that $T_0 f_i = g_i$, $i = 1, \ldots, n$ is an isometry; in fact $F := \text{span}(\text{f}_\text{i})$ is a sublattice of E and T_0 is an embedding. By quantifier elimination for E there exists an embedding T from E into some ultrapower $E_{\mathcal{U}}$ of E such that $Tf_i = g_i$, $i = 1, ..n$ (see [HI, Proposition 13.17]). $E_{\mathcal{U}}$ is an AL_pL_q-space with random norm taking values in $(L_p)_{\mathcal{U}}$ and can be viewed as a band in some $(L_p)_{\mathcal{U}}(L_q(\nu))$-space (up to an isomorphism preserving the $(L_p)_{\mathcal{U}}$-valued random norm). That is, we can replace T by an embedding into $(L_p)_{\mathcal{U}}(L_q(\nu))$. By Lemma 5.2 there exists a positive isometric linear map S of L_p into $(L_p)_{\mathcal{U}}$ such that

$$N(Tf) = SN(f)$$

for every $f \in L_p$. Since every isometric linear and positive map between L_p-spaces automatically preserves the lattice operations, S induces embeddings S_α of the whole scale of corresponding L_α-spaces, defined by

$$S_\alpha f = (\operatorname{sgn} Sf)(S|f|^{\alpha/p})^{p/\alpha}$$

In particular for $\alpha = p/q$ this embedding $L_\alpha \to (S_\alpha)_{\mathcal{U}}$ satisfies

$$S_\alpha \varphi_i = (S\varphi_i^{1/q})^q = (SN(f_i))^q = N(g_i)^q = \psi_i$$

for all $i = 1, ...n$. $\square$

5.4 Proposition *Let $\boldsymbol{L_p} = L_p([0,1])$ and L_q be any at least 2-dimensional L_q-space. Then for $p/q \in \mathbb{N}$, $p/q > 1$ the theory of the Banach lattice $\boldsymbol{L_p}(L_q)$ does not satisfy quantifier elimination.*

Proof Let μ_1, μ_2 be the measures on $(0, \infty)$ defined at the beginning of the section, and $f_0 = \mathbf{1}$, $f_1 \colon t \to t^4$ be the functions considered in Lemma 5.1. Let $\alpha = p/q$ and U, resp. V be isomorphisms from $L_\alpha(\mu_1)$, resp. $L_\alpha(\mu_2)$ onto $\boldsymbol{L}_\alpha$; set $\varphi_i = Uf_i, \psi_i = Vf_i$, $i = 1, 2$. By construction of f_1, f_2 we have

$$\|\lambda_1\varphi_1 + \lambda_2\varphi_2\|_\alpha = \|\lambda_1\psi_1 + \lambda_2\psi_2\|_\alpha$$

for all scalars λ_1, λ_2. If $\boldsymbol{L_p}(L_q)$ has quantifier elimination, then by Lemma 5.3 there are an ultrapower $(\boldsymbol{L}_\alpha)_\mathcal{U}$ and an embedding $S\colon \boldsymbol{L}_\alpha \to (\boldsymbol{L}_\alpha)_\mathcal{U}$ such that $S\varphi_i = \psi_i$, $i = 0, 1$. Then $T = V_\mathcal{U}^{-1}SU$ is an embedding from $L_\alpha(\mu_1)$ into $L_\alpha(\mu_2)_\mathcal{U}$ preserving the functions f_0, f_1. Since $\alpha > 1$ there is a positive contractive projection P from $L_\alpha(\mu_2)_\mathcal{U}$ onto $L_\alpha(\mu_2)$. [In the present case it can be presented as the weak limit projection, but its existence also results from the fact that every sublattice of L_α, $\alpha \geq 1$ is contractively complemented.] Then PT restricts to a positively linear norm-decreasing map from $L_\alpha^+(\mu_1)$ to $L_\alpha^+(\mu_2)$ fixing $\mathbf{1}$ and f_1, which contradicts Lemma 5.1. $\square$

In particular, it follows from Propositions 2.6 and 5.4 that the theory of doubly atomless BL_pL_q-Banach lattices does not have quantifier elimination when $1 \leq p \neq q < \infty$ and $p/q \in \mathbb{N}$.

6 Model completeness for integer values of p/q ($\neq 1$)

In this final section we discuss (without proofs) the theory of doubly atomless BL_pL_q-Banach lattices in the cases where it does not have quantifier elimination. Namely, we consider $1 \leq p \neq q < \infty$ such that p/q is an integer. Although this theory does not have quantifier elimination (as was shown in the previous section), it does have the slightly weaker property of being *model complete*.

A theory T is said to be *model complete* if whenever X, Y are models of T and X is a substructure of Y, it is always the case that X is an elementary substructure of Y. This is obviously true when T admits quantifier elimination. In classical model theory, T is model complete iff every formula is equivalent in T to an *existential* formula, which is one written in prenex normal form as

$$\exists x_1 \ldots \exists x_m \varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

in which φ contains no quantifiers. In the setting of Banach lattices, in which one uses an approximate logic such as the logic of positive bounded formulas [HI], model completeness is expressed equivalently by the condition that an arbitrary formula can be approximated arbitrarily closely by a sequence of existential formulas (uniformly on bounded balls of each fixed radius in all models of the theory).

A "soft" way to prove model completeness in certain cases is provided by the following special case of a theorem that is due to Lindström in the setting of classical model theory:

6.1 Proposition *Let T be a theory of Banach lattices that has no finite dimensional models. If T is separably categorical and the class of models of T is closed under unions of increasing chains, then T is model complete.*

Proof See Theorem 7.3.4 in [Ho] for a proof of the analogous result in classical model theory. This proof can be easily adapted to the current setting, using tools from [HI]. We omit the details. $\square$

6.2 Corollary *For every $1 \leq p \neq q < \infty$, the theory of doubly atomless BL_pL_q-Banach lattices is model complete.*

Proof The theory of doubly atomless BL_pL_q-Banach lattices is axiomatizable by Proposition 2.5, so it is the class of all models of its theory. Clearly every doubly atomless BL_pL_q-Banach lattice is infinite dimensional. The theory of this class is separably categorical by Proposition 2.6 and it is closed under unions of increasing chains by Proposition 2.8. The Corollary follows using Proposition 6.1. $\square$

We remark that Corollary 6.2 allows us to identify the theory of doubly atomless BL_pL_q-Banach lattices as the *model companion* of the theory of all BL_pL_q-Banach lattices. First we discuss the concept in general. Suppose we are given theories S, T in the same language. We say S is a model companion of T if (a) every model of T embeds in a model of S; (b) every model of S embeds in a model of T; and (c) S is model complete. In that situation it can be shown that S is uniquely determined by T. Indeed, letting T_0 be the theory of the class of substructures of models of T, the model companion S of T will (when it exists) have exactly the existentially closed models of T_0 as its models. (A discussion of model companions in the setting of classical model theory is in [Ho, pp. 198–200]; this material can be easily carried over to the positive bounded setting.) In model theory, the passage from a given theory T to its model companion S (when it exists) often yields a theory with very good model theoretic properties that can also be used to study (substructures of) models of T.

6.3 Corollary *The theory of doubly atomless BL_pL_q-Banach lattices is the model companion of the theory of all BL_pL_q-Banach lattices.*

Proof Each of the classes mentioned is axiomatizable (by Proposition 2.5 and [HR, Corollary 2.10]); thus each is the class of all models of its theory. The theory of doubly atomless BL_pL_q-Banach lattices is model complete by Corollary 6.2, and it extends the theory of all BL_pL_q-Banach lattices. Finally, every BL_pL_q-Banach lattice is contained (as a band) in an L_pL_q-Banach lattice, which can be extended to a doubly atomless L_pL_q-Banach lattice (by extending its underlying measure spaces to atomless measure spaces). □

We conclude by indicating directions for further research:

The paper [BBH] contains an extensive study of model theoretic properties of atomless L_p-Banach lattices. In particular, it is shown that the theory of this class is ω-stable, and the model theoretic independence relation of this theory is characterized using familiar concepts from analysis. Further, it is shown that canonical bases exist as tuples from the model. The corresponding properties should be studied for the theory of doubly atomless BL_pL_q-Banach lattices.

In particular:

(1) Suppose X, Y are doubly atomless BL_pL_q-Banach lattices and $a \in X^n$ and $b \in Y^n$ are finite tuples; what does it mean for (X, a) and (Y, b) to be elementarily equivalent? That is, what do model theoretic types express in doubly atomless BL_pL_q-Banach lattices? (For the meaning of types in atomless L_p-Banach lattices see Proposition 3.7 in [BBH] and the discussion following it, as well as Proposition 5.4 in [BBH].)

(2) Is the theory of doubly atomless BL_pL_q-Banach lattices stable? If so, (a) for which cardinals $\kappa \geq \omega$ is this theory κ-stable? (b) What is the model theoretic independence relation for this theory? (c) Do types have canonical bases that are sets of ordinary elements, or must one add imaginaries? (For the answers to these questions for atomless L_p-Banach lattices, see the following results in [BBH]: Theorem 3.15 for (a); the theory is κ-stable for every κ. Theorem 4.12 and Lemma 5.11 for (b). Theorem 6.2 for (c); canonical bases that are sets of ordinary elements do exist.)

References

[BBH] I Ben Yaacov, A Berenstein and C W Henson, Model theoretic independence in the Banach lattices $L_p(\mu)$, *Israel J. Math.* **183** (2011), 285–320; doi: 10.1007/s11856-011-0050-4.

[GR] S Guerre and Y Raynaud, Sur les isométries de $L_p(X)$ et le théorème ergodique vectoriel, *Canadian J. Math.* **40** (1986), 360–391.

[Ha] C J Hardin, Isometries on subspaces of L_p, *Indiana Univ. Math. J.* **30** (1981), 449–465; doi: 10.1512/iumj.1981.30.30036.

[HLR] R Haydon, M Levy and Y Raynaud, *Randomly Normed Spaces*, Collection “Travaux en Cours", Hermann ed. 1992.

[He] C W Henson, Nonstandard hulls of Banach spaces, *Israel J. Math.* **25** (1976), 108–144.

[HI] C W Henson and J Iovino, Ultraproducts in analysis, in *Analysis and Logic* (C Finet, C Michaux eds.), Cambridge Univ. Press (2003), 1–113.

[HR] C W Henson and Y Raynaud, On the theory of $L_p(L_q)$-Banach lattices, *Positivity* **11** (2007), 201–230; doi: 10.1007/s11117-006-2023-0.

[Ho] W Hodges, *A Shorter Model Theory*, Cambridge Univ. Press, 1997.

[JL] W B Johnson and J Lindenstrauss, *Handbook of the Geometry of Banach Spaces*, vol. 2, North-Holland, 2003.

[Ka] O Kallenberg, *Foundations of Modern Probability*, Springer-Verlag, 1997.

[Ko] A Koldobskii, Isometries of $L_p(X;L_q)$ and equimeasurability, *Indiana Univ. Math. J.* **40** (1991), 677–705; doi: 10.1512/iumj.1991.40.40031.

[LR1] M Levy and Y Raynaud, Ultrapuissances de $L_p(L_q)$, *C. R. Acad. Sci. Paris* **299** no. 3 (1984), 81–84.

[LR2] M Levy and Y Raynaud, Ultrapuissances de $L_p(L_q)$, *Séminaire d'Analyse Fonctionnelle Paris VII–VI, année 1983–84*, Pub. Math. Univ. Paris VII **20**, 69–79.

[Li] W Linde, Uniqueness theorems for measures in L_r and $C_0(\Omega)$, *Math. Ann.* **274** (1986), 617–626; doi: 10.1007/BF01458597.

[LT] J Lindenstrauss and L Tzafriri, *Classical Banach Spaces, Vol. II: Function Spaces*, Springer-Verlag, 1979.

[Lu] W Lusky, Some consequences of Rudin's paper: "L_p-isometries and Equimeasurability", *Indiana Univ. Math. J.* **27** (1978), 859–866; doi: 10.1512/iumj.1978.27.27057.

[MN] P Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.

[Ran] I K Rana, *An Introduction to Measure and Integration*, 2nd ed., Amer. Math. Soc. 2002.

[Ray] Y Raynaud, Sur les sous-espaces de $L_p(L_q)$, *Séminaire d'Analyse Fonctionnelle Paris VII–VI, année 1984–85*, Publ. Math. Univ. Paris VII **26** (1986), 49–71.

[Sch] H H Schaefer, *Banach Lattices and Positive Operators*, Grundlehren der math. Wissenschaften, 21, Springer-Verlag, 1974.

Mathematics Department, University of Illinois at Urbana-Champaign
1409 W. Green Street, Urbana, Illinois 61801, USA.

Institut de Mathématiques de Jussieu (CNRS and UPMC–Univ. Paris 6)
Projet Analyse Fonctionnelle, Case 247, 4 place Jussieu, 75005 Paris, France.

henson(at)math(dot)uiuc(dot)edu, yves.raynaud(at)upmc(dot)fr

http://www.math.uiuc.edu/~henson

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