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Locating subsets of $\mathcal{B}(H)$ relative to seminorms inducing the strong-operator topology

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Abstract: Let H be a Hilbert space, and $\mathcal{A}$ an inhabited, bounded, convex subset of $\mathcal{B}(H)$. We give a constructive proof that $\mathcal{A}$ is weak-operator totally bounded if and only if it is located relative to a certain family of seminorms that induces the strong-operator topology on $\mathcal{B}(H)$.

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This paper is a contribution to the programme of research in constructive functional analysis and operator theory. It lies entirely within a Bishop-style constructive framework; in other words, the logic is intuitionistic, and we use an underlying set theory, such as that presented by Aczel and Rathjen [1, 2], which avoid axioms that would imply essentially nonconstructive principles such as the law of excluded middle.[1]

Although carried out by strictly constructive means, our work is not insignificant within classical-logic-based computational functional analysis: each of our results and proofs is, *a fortiori,* classical. But constructive proofs, by their very nature, embody algorithms, and hence estimates,[2] that can be extracted—sometimes with surprising ease—and then implemented; such program-extraction and implementation can be found in Constable [8], Hayashi [9], and Schwichtenberg [13]. For example, consider our main result, Theorem 1, which deals with an inhabited,[3] bounded, convex set $\mathcal{A}$ of operators on an infinite-dimensional Hilbert space H. The first half of its proof is, essentially, an algorithm for converting

[1] A popular alternative foundation for constructive mathematics is Martin-Löf's type theory [12].

[2] A very different approach to the extraction of estimates (often optimal ones) is adopted by Kohlenbach: working with classical logic, he uses *proof-mining* to extract computational information from classical proofs; see Kohlenbach [11].

[3] To say that a set is **inhabited** means that we can construct an element of it. This is a constructively stronger notion that *nonempty* (although, confusingly, some earlier work on constructive analysis uses *nonempty* in the sense of *inhabited*).

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– finite ε-approximations to $\mathcal{A}$ relative to the seminorms defining the weak-operator topology on $\mathcal{B}(H)$
– into a computation of distances from $\mathcal{A}$ relative to a certain family of seminorms that induces the strong-operator topology on $\mathcal{B}(H)$.

The second half is an algorithm for carrying out this conversion in reverse. Of course, the practical extraction and implementation of these algorithms would be a nontrivial business; but it could be done.

We begin by recalling some definitions from the constructive theory of locally convex spaces. A subset S of a locally convex space $\left(X, (p_i)_{i\in I}\right)$, where the p_i are the seminorms defining the topology on X, is said to be **located** in X if

$$\inf\left\{\sum_{i\in F} p_i(x-s) : s \in S\right\}$$

exists for each $x \in X$ and each finitely enumerable[4] subset F of I. On the other hand, S is said to be **totally bounded** if for each finitely enumerable subset F of I and each $\varepsilon > 0$, there exists a finitely enumerable subset T of S with the property that for each $x \in S$ there exists $y \in T$ with $\sum_{i\in F} p_i(x-y) < \varepsilon$; such a set T is then called a **finitely enumerable ε-approximation** to S relative to the seminorm $\sum_{i\in F} p_i$.

We note these facts about total boundedness and locatedness:

– The image of a totally bounded set under a uniformly continuous mapping between locally convex spaces is totally bounded ([6], Proposition 5.4.2).
– Every totally bounded subset of X is located, and every located subset of a totally bounded set is totally bounded ([6], Propositions 5.4.4 and 5.4.5).

The following two locally convex topologies play a fundamental role in the classical theory of subalgebras of the space $\mathcal{B}(H)$ of bounded operators on a Hilbert space H:

▷ The **strong operator topology** τ_s: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow Tx$ is continuous for each $x \in H$; sets of the form

$$\{T \in \mathcal{B}(H) : \|Tx\| < \varepsilon\},$$

with $x \in H$ and $\varepsilon > 0$, form a sub-base of strong-operator neighbourhoods of the zero operator.

[4]A set is **finitely enumerable** if it is the range of a mapping from $\{1, \ldots, n\}$ for some natural number n; the set is **finite** if the mapping can be chosen one-one.

▷ The **weak operator topology** τ_w: the weakest topology on $\mathcal{B}(H)$ with respect to which the mapping $T \rightsquigarrow \langle Tx, y\rangle$ is continuous for all $x, y \in H$; sets of the form

$$\{T \in \mathcal{B}(H) : |\langle Tx, y\rangle| < \varepsilon\},$$

with $x, y \in H$ and $\varepsilon > 0$, form a sub-base of weak-operator neighbourhoods of the zero operator.

These topologies are induced, respectively, by the seminorms of the form $T \rightsquigarrow \|Tx\|$ with $x \in H$, and those of the form $T \rightsquigarrow |\langle Tx, y\rangle|$ with $x, y \in H$.

For each integer $N \geqslant 2$ we denote, for example, by $\mathbf{x}$ the N-tuple $(x_1, \dots, x_N)$ of elements of H, and we define H_N to be the Hilbert direct sum of N copies of H. Although one frequently describes the strong-operator topology by means of the **L_1-like seminorms**

$$\| \ \|_{1,\mathbf{x}} : T \rightsquigarrow \sum_{n=1}^{N} \|Tx_n\|,$$

where $\mathbf{x} \in H_N$, in this paper we focus our attention on an alternative family of seminorms inducing τ_s: namely, the family of **L_2-like seminorms**

$$\| \ \|_{2,\mathbf{x}} : T \rightsquigarrow \left(\sum_{n=1}^{N} \|Tx_n\|^2\right)^{1/2},$$

where $\mathbf{x} \in H_N$. We say that a subset $\mathcal{A}$ of $\mathcal{B}(H)$ is **k-located** if it is located relative to the family of L_k-like seminorms ($k = 1, 2$). Note that although each of the two L_k-families induces the strong-operator topology on $\mathcal{B}(H)$, it is not *a priori* the case that the metric-dependent notions of **1**-locatedness and **2**-locatedness coincide on a given subset $\mathcal{A}$ of $\mathcal{B}(H)$. It will be a consequence of our main result, which we now state, that these two notions of locatedness do coincide when $\mathcal{A}$ is inhabited, bounded, and convex.

Theorem 1 *Let H be an infinite-dimensional Hilbert space, and $\mathcal{A}$ an inhabited, bounded, convex subset of $\mathcal{B}(H)$. Then $\mathcal{A}$ is **2**-located if and only if it is weak-operator totally bounded.*

In the case where H is separable, the equivalence of **1**-locatedness and weak-operator total boundedness for inhabited, bounded, convex subsets of $\mathcal{B}(H)$ was proved by Spitters ([14], Corollary 10), who took a non-elementary route through trace-class operators and normal states. In the non-separable case, the implication from weak-operator total boundedness to **1**-locatedness is proved by Bridges, Ishihara and Vîţă [7]

(Theorem 3.8), using general results about infima of real-valued continuous functions on convex sets in normed spaces (a counterpart of which plays a role in our work below).

We shall prove Theorem 1 without separability and by relatively elementary methods. Before doing so, we remind ourselves of a common construction and deal with some preliminary results. The complicated proof of the first of these, due to Ishihara, can be found in [10] (Corollary 5) or Bridges and Vîţă [6] (Corollary 6.2.9).

Proposition 2 *Let C be an inhabited, bounded, convex subset of an inner product space H. Then C is located if and only if*

$$\sup \{\mathrm{Re}\,\langle x, y\rangle : y \in C\}$$

exists for each $x \in H$.

Our second preliminary result is a version of a classically trivial result about Banach spaces ([6], Proposition 5.3.4), whose known constructive proof is not trivial as it uses the Hahn-Banach theorem. However, in the case where X is a Hilbert space, there is a natural, more elementary proof, for which we need two items of information about dimensionality in a normed space X. First, we note that every finite-dimensional subspace of X is located ([6], Lemma 4.1.2). Secondly, we say that X is **infinite-dimensional** if for each finite-dimensional subspace V of X, there exists $x \in X$ with $\rho(x, V) > 0$ (in which case the orthogonal complement of V contains a unit vector). For additional material on finite- and infinite-dimensionality in normed spaces, see Chapter 4 of Bridges and Vîţă [6].

Lemma 3 *Let H be an infinite-dimensional Hilbert space, and let $x_1, \ldots, x_N$ be vectors in H. Then for each $t > 0$, there exist pairwise orthogonal unit vectors $e_1, \ldots, e_N$ in H such that the vectors $x_n' \equiv x_n + te_n$ $(1 \leqslant n \leqslant N)$ are linearly independent.*

Proof To begin with, pick a unit vector e_1 such that $x_1' \equiv x_1 + te_1 \neq 0$. Suppose that for some $n < N$ we have found the desired vectors $e_1, \ldots, e_n$, and let V be the n-dimensional subspace of H generated by the vectors $x_k' \equiv x_k + te_k$ $(1 \leqslant k \leqslant n)$. Either $\rho\left(x_{n+1}, V\right) > 0$ or $\rho\left(x_{n+1}, V\right) < t$. In the first case, $V \cup \{x_{n+1}\}$ generates an $(n+1)$-dimensional subspace W of H, and we can pick a unit vector e orthogonal to W. Then for each $v \in V$,

$$\|x_{n+1} + te - v\| = t\left\|e - t^{-1}\left(v - x_{n+1}\right)\right\| \geqslant t\rho(e, W) = t.$$

Hence $\rho\left(x_{n+1} + te, V\right) \geqslant t > 0$, so $x_{n+1} + te$ is linearly independent of V, and we can take $e_{n+1} \equiv e$.

In the case where $\rho\left(x_{n+1}, V\right) < t$, we pick a unit vector e orthogonal to V. With P the projection of H on V, and I the identity operator on H, we have

$$\begin{aligned} \left\|(I-P)\left(x_{n+1}+te\right)\right\| &\geqslant \|t\,(I-P)\,e\| - \|(I-P)\,x_{n+1}\| \\ &= t-\rho\left(x_{n+1}, V\right) > 0. \end{aligned}$$

Hence $\rho\left(x_{n+1}+te, V\right) > 0$, so $x_{n+1}+te$ is linearly independent of V, and we can take $e_{n+1} \equiv e$. □

Returning to the set-up of Theorem 1, for each $T \in \mathcal{B}(H)$ define

$$\widetilde{T}\mathbf{x} \equiv (Tx_1, \ldots, Tx_N)\,,$$

and for any subset $\mathcal{A}$ of $\mathcal{B}(H)$ define

$$\widetilde{\mathcal{A}} \equiv \left\{\widetilde{T} : T \in \mathcal{A}\right\}.$$

Lemma 4 *If $\mathcal{A}$ is an inhabited, bounded, $\mathbf{2}$-located subset of $\mathcal{B}(H)$, and $\mathbf{x} \in H_N$, then*

$$\widetilde{\mathcal{A}}\mathbf{x} \equiv \left\{\widetilde{T}\mathbf{x} : T \in \mathcal{A}\right\}$$

is located in H_N.

Proof We may assume that $\mathcal{A} \subset \mathcal{B}_1(H)$. Let $0 < \alpha < \beta$, and set $\varepsilon \equiv \frac{1}{3}(\beta-\alpha)$. By Lemma 3, since H is infinite-dimensional, there exist pairwise orthogonal unit vectors $e_1, \ldots, e_N$ such that the vectors

$$x_n' \equiv x_n + \frac{\varepsilon}{\sqrt{N}} e_n$$

are linearly independent. Given $\mathbf{y} \in H_N$, construct $S \in \mathcal{B}(H)$ such that $Sx_n' = y_n$ for each n. (This is possible since the locatedness of the n-dimensional span V of $\{x_1', \ldots, x_N'\}$ implies the existence of the projection P of H onto V, and hence enables us to set $Sx = 0$ if x is in the orthogonal complement of V.) Since $\mathcal{A}$ is $\mathbf{2}$-located in $\mathcal{B}(H)$,

$$\lambda \equiv \left\{\inf\left(\sum_{n=1}^{N}\left\|(S-T)x_n'\right\|^2\right)^{1/2} : T \in \mathcal{A}\right\}$$

exists. Either $\lambda > \alpha + \varepsilon$ or $\lambda < \beta - \varepsilon$. In the former case, for each $T \in \mathcal{A}$ we have

$$\begin{aligned}
&\left(\sum_{n=1}^{N} \|y_n - Tx_n\|^2\right)^{1/2} \\
\geqslant\ & \left(\sum_{n=1}^{N} \left\|(S-T)\,x_n'\right\|^2\right)^{1/2} - \left(\sum_{n=1}^{N} \left\|T(x_n - x_n')\right\|^2\right)^{1/2} \\
\geqslant\ & \lambda - \left(\sum_{n=1}^{N} \left\|x_n - x_n'\right\|^2\right)^{1/2} \\
>\ & \alpha + \varepsilon - \left(\sum_{n=1}^{N} \frac{\varepsilon^2}{N}\right)^{1/2} = \alpha.
\end{aligned}$$

In the case $\lambda < \beta - \varepsilon$, there exists $T \in \mathcal{A}$ such that

$$\left(\sum_{n=1}^{N} \left\|y_n - Tx_n'\right\|^2\right)^{1/2} < \beta - \varepsilon$$

and therefore

$$\begin{aligned}
\left(\sum_{n=1}^{N} \|y_n - Tx_n\|^2\right)^{1/2} &\leqslant \left(\sum_{n=1}^{N} \left\|y_n - Tx_n'\right\|^2\right)^{1/2} + \left(\sum_{n=1}^{N} \left\|T(x_n - x_n')\right\|^2\right)^{1/2} \\
&< \beta - \varepsilon + \left(\sum_{n=1}^{N} \frac{\varepsilon^2}{N}\right)^{1/2} = \beta.
\end{aligned}$$

It now follows from the constructive greatest-lower-bound principle ([6], Theorem 2.1.19) that

$$\rho\left(\mathbf{y}, \widetilde{\mathcal{A}}\mathbf{x}\right) = \inf\left\{\left(\sum_{n=1}^{N} \|y_n - Tx_n\|^2\right)^{1/2} : T \in \mathcal{A}\right\}$$

exists. □

The following lemma is similar to Lemma 3.2 of Bridges and Vîţă [7], and is needed to remove a preliminary restriction in part of the proof of Theorem 1.

Lemma 5 *Let $f_1, \ldots, f_N$ be bounded, nonnegative functions on a set S such that for each $\delta > 0$,*

$$m_\delta \equiv \inf\left\{\left(\sum_{n=1}^{N} (f_n(x) + \delta)^2\right)^{1/2} : x \in S\right\}$$

exists. Then $\inf\left\{\left(\sum_{n=1}^N f_n(x)^2\right)^{1/2} : x \in S\right\}$ *exists.*

Proof Compute $c > 0$ such that $\sum_{n=1}^N f_n(x) \leqslant c$ for each $x \in S$. Given $\varepsilon > 0$, pick $\delta > 0$ such that

$$2c\delta + N\delta^2 < \frac{\varepsilon}{2}.$$

Since m_δ exists, we can find $x_0 \in S$ such that

$$\sum_{n=1}^N (f_n(x_0) + \delta)^2 < \sum_{n=1}^N (f_n(x) + \delta)^2 + \frac{\varepsilon}{2}$$

for each $x \in S$. Then

$$\begin{aligned}\sum_{n=1}^N f_n(x_0)^2 &\leqslant \sum_{n=1}^N (f_n(x_0) + \delta)^2 < \sum_{n=1}^N (f_n(x) + \delta)^2 + \frac{\varepsilon}{2} \\ &= \sum_{n=1}^N f_n(x)^2 + 2\delta\sum_{n=1}^N f_n(x) + N\delta^2 + \frac{\varepsilon}{2} \\ &\leqslant \sum_{n=1}^N f_n(x)^2 + 2c\delta + N\delta^2 + \frac{\varepsilon}{2} < \sum_{n=1}^N f_n(x)^2 + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\inf\left\{\sum_{n=1}^N f_n(x)^2 : x \in X\right\}$$

exists; whence the desired infimum also exists. □

We now give the **proof of Theorem 1.**

Proof Assume that $\mathcal{A}$ is **2**-located in $\mathcal{B}(H)$. Let N be any positive integer, and define $H_N, \widetilde{T}, \widetilde{\mathcal{A}}$ as above. Then $\widetilde{\mathcal{A}}$ is an inhabited, bounded, convex subset of $\mathcal{B}(H_N)$. By Lemma 4, for each $\mathbf{x} \in H_N$ the inhabited, bounded, convex set

$$\widetilde{\mathcal{A}}\mathbf{x} \equiv \left\{\widetilde{T}\mathbf{x} : T \in \mathcal{A}\right\}$$

is located in H_N. It follows from Proposition 2 that for all $\mathbf{x}, \mathbf{y}$ in H_N,

$$\sigma_{\mathbf{x},\mathbf{y}} \equiv \sup\left\{\operatorname{Re}\left\langle \widetilde{T}\mathbf{x}, \mathbf{y}\right\rangle : T \in \mathcal{A}\right\} = \sup\left\{\operatorname{Re}\left\langle \mathbf{y}, \widetilde{T}\mathbf{x}\right\rangle : T \in \mathcal{A}\right\}$$

exists. Now,

$$S_{\mathbf{x},\mathbf{y}} \equiv \left\{\left(\langle Tx_1, y_1\rangle, \ldots, \langle Tx_N, y_N\rangle\right) : T \in \mathcal{A}\right\}$$

is an inhabited, bounded, and convex subset of the Hilbert space $\mathbf{C}^N$, taken with the usual inner product. Moreover, for each $\eta \in \mathbf{C}^N$,

$$\begin{aligned}\sup\{\mathrm{Re}\,\langle \eta, \zeta\rangle : \zeta \in S_{\mathbf{x},\mathbf{y}}\} &= \sup\left\{\mathrm{Re}\sum_{k=1}^{N}\eta_k\overline{\zeta_k} : \zeta \in S_{\mathbf{x},\mathbf{y}}\right\} \\ &= \sup\left\{\mathrm{Re}\sum_{k=1}^{N}\langle \eta_k y_k, Tx_k\rangle : T \in \mathcal{A}\right\} = \sigma_{\mathbf{x},\mathbf{z}}\end{aligned}$$

exists, where

$$\mathbf{z} \equiv (\eta_1 y_1, \ldots, \eta_N y_N) \in H_N.$$

Again applying Proposition 2, we see that $S_{\mathbf{x},\mathbf{y}}$ is located in $\mathbf{C}^N$, regarded as a Hilbert space over $\mathbf{C}$; being also bounded, $S_{\mathbf{x},\mathbf{y}}$ is therefore totally bounded. Since all norms on $\mathbf{C}^N$ are equivalent, it follows that for each $\varepsilon > 0$, there exists a finitely enumerable subset $\{T_1, \ldots, T_m\}$ of $\mathcal{A}$ such that the elements

$$\left(\langle T_k x_1, y_1\rangle, \ldots, \langle T_k x_N, y_N\rangle\right) \quad (k = 1, \ldots, m)$$

form a finitely enumerable ε-approximation to $S_{\mathbf{x},\mathbf{y}}$ relative to the norm

$$(\zeta_1, \ldots, \zeta_N) \rightsquigarrow \sum_{n=1}^{N}|\zeta_n|$$

on $\mathbf{C}^N$. Hence for each $T \in \mathcal{A}$ there exists $k \leqslant m$ such that

$$\sum_{n=1}^{N}|\langle (T - T_k)\, x_n, y_n\rangle| < \varepsilon.$$

Thus $\{T_k : 1 \leqslant k \leqslant m\}$ is a finitely enumerable ε-approximation to $\mathcal{A}$ relative to the seminorm $T \rightsquigarrow \sum_{n=1}^{N}|\langle Tx_k, y_k\rangle|$. It follows that $\mathcal{A}$ is weak-operator totally bounded.

To prove the converse, assume that $\mathcal{A}$ is weak-operator totally bounded. Let $S \in \mathcal{B}(H)$ and $\mathbf{x} \in H_N$. We need to prove that

$$\inf\left\{\left(\sum_{n=1}^{N}\|(S - T)\, x_n\|^2\right)^{1/2} : T \in \mathcal{A}\right\} \tag{1}$$

exists. For each $n \leqslant N$ and each $y \in H$, since the mapping $T \rightsquigarrow \mathrm{Re}\,\langle y, Tx_n\rangle$ is weak-operator uniformly continuous on the weak-operator totally bounded set $\mathcal{A}$,

$$\sup\{\mathrm{Re}\,\langle y, Tx_n\rangle : T \in \mathcal{A}\}$$

exists, by Corollary 2.2.7 of Bridges and Vîţă [6]; whence $\mathcal{A}x_n$ is located, by Proposition 2. Suppose for the moment that

$$\rho\,(Sx_n, \mathcal{A}x_n) > 0 \quad (1 \leqslant n \leqslant N)\,. \tag{2}$$

Note that $\widetilde{\mathcal{A}}$ is bounded, convex, and weak-operator totally bounded in $\mathcal{B}(H_N)$. It follows that

$$C \equiv \left\{ \left(\widetilde{S} - \widetilde{T} \right) \mathbf{x} : T \in \mathcal{A} \right\}$$

is a bounded, weakly totally bounded, convex subset of the Hilbert space H_N. Define $f : C \to \mathbf{R}$ by

$$f\left(\left(\widetilde{S} - \widetilde{T} \right) \mathbf{x} \right) \equiv \left(\sum_{n=1}^{N} \|(S-T)\, x_n\|^2 \right)^{1/2}.$$

Then f is a convex function. In view of (2) and Lemma 3.6 of Bridges, Ishihara and Vîţă [7], we see that the mappings $\left(\widetilde{S} - \widetilde{T} \right) \mathbf{x} \rightsquigarrow \|(S-T)\, x_n\|$ are uniformly differentiable on C, and hence (again note (2)) that f is also. It follows from Theorem 2.2 of the same reference that the infimum in (1) exists.

We now remove the condition (2). Let H' denote the direct sum $H \oplus H$ of two copies of H, let $\delta > 0$, and let $\mathcal{A}' \equiv \mathcal{A} \oplus \left\{ \delta^{1/2} I \right\}$, where I is the identity operator on H and

$$\left(T \oplus \delta^{1/2} I \right)(x, y) \equiv \left(Tx, \delta^{1/2} y \right) \qquad (T \in \mathcal{B}(H);\ x, y \in H).$$

Define $S \in \mathcal{B}(H')$ by $S'(x, y) \equiv (Sx, 0)$. Fix a unit vector $e \in H$, and let $x'_n \equiv (x_n, e)$ $(1 \leqslant n \leqslant N)$. Then for each $n \leqslant N$ and each $T \in \mathcal{A}$,

$$\left\| S' x'_n - \left(T, \delta^{1/2} \right) x_n \right\|^2 = \|Sx_n - Tx_n\|^2 + \delta \geqslant \delta,$$

so $\rho\left(S' x'_n, \mathcal{A}' x'_n \right) > 0$. It is easy to verify that $\mathcal{A}'$ is weak-operator totally bounded. Applying the first part of the proof to $\mathcal{A}', S'$, and $\mathbf{x}'$, we see that

$$\begin{aligned} m_\delta &\equiv \inf \left\{ \left(\sum_{n=1}^{N} \left\| S' x'_n - \left(T, \delta^{1/2} \right) x'_n \right\|^2 \right)^{1/2} : T \in \mathcal{A} \right\} \\ &= \inf \left\{ \left(\sum_{n=1}^{N} \|(S-T)\, x_n\|^2 + \delta \right)^{1/2} : T \in \mathcal{A} \right\} \end{aligned}$$

exists. Since $\delta > 0$ is arbitrary, it follows from Lemma 5 that the infimum at (1) exists in the general case. Since S and $\mathbf{x}$ are arbitrary, we conclude that $\mathcal{A}$ is **2**-located. □

Referring to Spitters [14] (Corollary 10) and Bridges, Ishihara and Vîţă [7] (Theorem 8), we immediately obtain

Corollary 6 *Let H be an infinite-dimensional Hilbert space, and $\mathcal{A}$ an inhabited, bounded, convex subset of $\mathcal{B}(H)$. Then $\mathcal{A}$ is $\mathbf{1}$-located if and only if it is $\mathbf{2}$-located.*

Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(H)$ with weak-operator totally bounded unit ball $\mathcal{A}_1$. Taken with the case $N = 1$ of Lemma 4, Theorem 1 tells us, in particular, that $\mathcal{A}_1 x$ is located in H for each $x \in H$. A major open question in constructive operator theory is this: under what conditions on the linear subspace $\mathcal{A}$ and the element x is the linear space $\mathcal{A}x$ located (in which case the projection on its closure exists)? The case of real interest is when $\mathcal{A}$ is a **von Neumann algebra**: a strong-operator closed subalgebra that contains the identity operator, has weak-operator totally bounded unit ball, and is closed under adjoints (in the sense that if $T \in \mathcal{A}$ and the adjoint T^* exists,[5] then $T^* \in \mathcal{A}$). Spitters has shown that if $\mathcal{A}$ is an abelian von Neumann algebra, then the space $\mathcal{A}x$ is located for each x in a dense subset of H ([14], Proposition 17). It is conjectured that the same conclusion holds when the word *abelian* is dropped from the antecedent.

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