Journal of Logic & Analysis 17:5 (2025) 1–8
ISSN 1759-9008

On finite index subfactors of super McDuff II_1 factors

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Abstract: A II_1 factor M has the super McDuff property if the central sequence algebra $M' \cap M^{\mathcal{U}}$ is a II_1 factor. Suppose that $N \subset M$ be an inclusion of II_1 factors with finite Jones index. In this note we prove that N has the super McDuff property if and only if M has the super McDuff property. We prove also that the same permanence result holds in the case of the *uniform* super McDuff property introduced recently in [16]. This answers a question posed by I. Goldbring.

2020 Mathematics Subject Classification 46L10 (primary); 03C66 (secondary)

Keywords: II_1 factor, subfactor, Jones index, super McDuff property

To fuffle with affection.

1 Introduction

The study of von Neumann algebras has been a central theme of study in functional analysis over the last century. These operator algebras have a very robust structure and often require interdisciplinary approaches of study, as evidenced by connections with probability theory and entropy [30], ergodic theory and dynamical systems [29], knot theory and quantum algebra [20]. This article will address an aspect of the continuous model theory of von Neumann algebras, especially II_1 factors (see [15, 13]), which are centerless continuous von Neumann algebras admitting a trace. The foundational work of Murray and von Neumann [24], and then crucial works of McDuff [22, 23] and Connes [11] provided striking reasons for why the study of approximately central sequences of elements in II_1 factors is important for the subject. Since then there have been several important insights in this thread, see for instance [12, 6, 28, 25, 29, 4, 18].

There is a natural central sequence algebra associated to a II_1 factor M, namely the von Neumann algebra $M' \cap M^{\mathcal{U}}$, where $M^{\mathcal{U}}$ is the tracial ultrapower over a countably incomplete ultrafilter $\mathcal{U}$. It can be that $M' \cap M^{\mathcal{U}} = \mathbb{C}$, in which case M is said to have *property Gamma*. $M' \cap M^{\mathcal{U}}$ can also be nontrivial but abelian. If $M' \cap M^{\mathcal{U}}$ is non-abelian, M is said to have *McDuff's property* [23], in which case, per [23], the

Published: June 2025 https://doi.org/10.4115/jla.2025.17.5

central sequence algebra either has center or is a II_1 factor. The property of $M' \cap M^{\mathcal{U}}$ being II_1 factor has been called the *super McDuff property* and has been studied in recent works [14, 2, 9]. A recent paper [16] introduced the notion of a *uniformly super McDuff* II_1 factor which is a quantitative strengthening of the super McDuff property. This property asks if the central sequence algebra $N' \cap N^{\mathcal{U}}$ is a II_1 factor for all $N \equiv M$. It is also equivalent to $M^{\mathcal{U}}$ being super McDuff. This property has applications to the elementary equivalence classification program for II_1 factors which has seen activity in recent years [14, 7, 10, 21, 17, 19]. In [16], several natural examples of these factors were shown including $N \overline{\otimes} R$ for any non Gamma factor N, and the family of infinitely generic II_1 factors.

Since the emergence of modern subfactor theory [20], studying finite index subfactors has been crucial due to connections with various other disciplines including quantum algebra and tensor categories. We still don't understand fully what II_1 factors can occur as subfactors of finite index, even in basic examples like free group factors. It is also interesting to know what what axiomatizable properties are also stable under finite index inclusions. This note shows that super McDuff and uniformly super McDuff II_1 factors are preserved by taking subfactors and superfactors of finite Jones index [20], a heretofore open basic problem concerning these factors. In the case of uniformly super McDuff II_1 factors, this answers a question posed by I. Goldbring in a private correspondence. We remark that subfactors of finite index are a central object of study in the subfactor theory. They also naturally correspond to finite index subgroups.

Theorem A *Suppose that $N \subset M$ be an inclusion of separable II_1 factors with finite Jones index. Then N has the (uniform) super McDuff property if and only if M has the (uniform) super McDuff property.*

The proof of this theorem is of course inspired by a result of Pimsner and Popa [26, Proposition 1.11] which proves this permanence for the McDuff property. However, to control the center of the central sequence algebra, there are added subtleties that need to be addressed. Due to the short nature of this note, we do not spell out preliminaries and we assume the reader is familiar with the standard terminology in II_1 factors, the definition of ultraproducts, and basic facts on subfactors.

Acknowledgements

We thank Dietmar Bisch for kindly pointing to us the reference [3]. We also thank Adrian Ioana for helpful comments. We further thank the anonymous referee for their helpful comments.

2 Proof of the Main Theorem

Let (M, τ) be a finite von Neumann algebra and $Q \subset M$ be a von Neumann subalgebra. The basic construction $\langle M, e_Q \rangle$ is defined as the von Neumann subalgebra of $\mathbb{B}(L^2M)$ generated by M and the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q)$. There is a semifinite faithful normal trace on $\langle M, e_Q \rangle$ satisfying $\mathrm{Tr}(xe_Qy) = \tau(xy)$, for every $x, y \in M$. Recall also the notion of *finite index* for subfactors ([1, Definition 9.4.9]): $Q \subset M$ is finite index if $L^2(M)$ as a right Q module is finite dimensional. Equivalently, the conditional expectation E_Q satisfies $E_Q \geq \lambda I$ for some scalar λ.

We will need some lemmas for the main result. The following is extremely elementary:

Lemma 2.1 *Let $N, M \subset B(H)$ be two von Neumann algebras s.t. $N' \subset M$. Let $e \in N'$ be a projection. Then $(N \cap M)e = Ne \cap eMe$.*

Proof First note that $(N \cap M)e \subset Ne \cap eMe$ is clear. Conversely, let $p \in Ne \cap eMe$ be a projection. Let q be the projection onto $\overline{N'pH}$. Then q is the minimal projection in N majorizing p, so $qe = p$. We also have $p \in M$, so for any $x \in M' \subset N$,

$$x\overline{N'pH} \subset \overline{N'pxH} \subset \overline{N'pH}$$

Thus, $q \in N \cap M$ and $p = qe \in (N \cap M)e$. Since projections densely span any von Neumann algebra, this shows $Ne \cap eMe \subset (N \cap M)e$. □

We thank Adrian Ioana for pointing out to us that the following lemma is already obtained by Sorin Popa, see Lemma 3.1 in [27] (see also [8, page 4]). Nevertheless we include a proof for convenience of the reader.

Lemma 2.2 *Let $N \subset M$ be an inclusion II_1 factors with finite Jones index and $M \subset \widetilde{M}$ be an inclusion of II_1 factors. Then $M' \cap \widetilde{M} \subset N' \cap \widetilde{M}$ is of finite index.*

Proof By [20, Corollary 3.1.9], there exists $M_0 \subset N$, a finite index subfactor, s.t. $N \subset M$ is obtained via the basic construction to $M_0 \subset N$, i.e., $M = \langle N, e_{M_0} \rangle''$. Thus, there is a natural representation $\psi : M \to B(L^2(N))$ where N acts via the standard left action of N on $L^2(N)$ and e_{M_0} acts as the orthogonal projection of $L^2(N)$ onto $L^2(M_0)$. Now, let $\pi : \widetilde{M} \to B(K)$ be an infinite index representation. Then $\pi|_M$ is an infinite index representation of M, so we may write,

$$K = L^2(N) \otimes H$$

where H is an infinite-dimensional Hilbert space and $\pi|_M = \psi \otimes 1_H$. Let

$$J : L^2(N) \to L^2(N)$$

be the canonical anti-linear involution. Then,

$$M' = JM_0J \mathbin{\overline{\otimes}} B(H) \text{ and } N' = JNJ \mathbin{\overline{\otimes}} B(H)$$

Since $JM_0J \subset JNJ$ is a finite index inclusion, the expectation $E : N' \to M'$, defined by $E = E_{JM_0J} \otimes 1_{B(H)}$ where $E_{JM_0J} : JNJ \to JM_0J$ is the trace-preserving expectation, is a finite index expectation. Now, recall that $e_{M_0} = Je_{M_0}J \in (JM_0J)'$, so,

$$\iota : JM_0J \to JM_0Je_{M_0}, \iota(x) = xe_{M_0}$$

is an isomorphism. Furthermore, the expectation E_{JM_0J} is given by,

$$E_{JM_0J}(x) = \iota^{-1}(e_{M_0}xe_{M_0})$$

where we always have, whenever $x \in JNJ$, $e_{M_0}xe_{M_0} \in JM_0Je_{M_0}$. So, E is given by,

$$E(x) = (\iota \otimes 1_{B(H)})^{-1}(e_{M_0}xe_{M_0})$$

Since $e_{M_0} \in M \subset \widetilde{M}$, we have, if $x \in N' \cap \widetilde{M} = (JNJ \mathbin{\overline{\otimes}} B(H)) \cap \widetilde{M}$, then

$$e_{M_0}xe_{M_0} \in (JM_0Je_{M_0} \mathbin{\overline{\otimes}} B(H)) \cap e_{M_0}\widetilde{M}e_{M_0}.$$

Note that $(JM_0J \mathbin{\overline{\otimes}} B(H))' = M \subset \widetilde{M}$, so by Lemma 2.1,

$$(JM_0Je_{M_0} \mathbin{\overline{\otimes}} B(H)) \cap e_{M_0}\widetilde{M}e_{M_0} = [(JM_0J \mathbin{\overline{\otimes}} B(H)) \cap \widetilde{M}]e_{M_0}$$

Therefore, whenever $x \in N' \cap \widetilde{M} = (JNJ \mathbin{\overline{\otimes}} B(H)) \cap \widetilde{M}$,

$$E(x) = (\iota \otimes 1_{B(H)})^{-1}(e_{M_0}xe_{M_0}) \in (JM_0J \mathbin{\overline{\otimes}} B(H)) \cap \widetilde{M} = M' \cap \widetilde{M}$$

That is, there is a normal conditional expectation of finite index from $N' \cap \widetilde{M}$ to $M' \cap \widetilde{M}$, namely the restriction of E to $N' \cap \widetilde{M}$. By [**?**, Corollary 3.20], this implies $M' \cap \widetilde{M} \subset N' \cap \widetilde{M}$ is of finite index. □

The following lemma is probably well known to experts.

Lemma 2.3 *If $N \subset M$ is a finite index inclusion of tracial von Neumann algebras, then $Z(N)$ is finite-dimensional if and only if $Z(M)$ is finite-dimensional.*

Proof Using the basic construction, we see that it suffices to show if $Z(N)$ is finite-dimensional, then so is $Z(M)$. Assume to the contrary that $Z(M)$ is infinite-dimensional. Then there exists a sequence of nonzero projections $p_n \in Z(M)$ s.t. $p_n \to 0$ strongly. Let $E_N : M \to N$ be the trace-preserving conditional-expectation. It is easy to see that $E_N(Z(M)) \subset Z(N)$, so as $p_n \to 0$ strongly, $E_N(p_n) \to 0$ strongly in $Z(N)$. As $Z(N)$ is finite-dimensional, $\|E_N(p_n)\| \to 0$. However, as $N \subset M$ is of finite index, there exists $\lambda > 0$ s.t. $E_N \geq \lambda I$, so, $E_N(p_n) \geq \lambda p_n$ for all n, which is a contradiction. □

Proof of Theorem A, super McDuff case By the basic construction, if $N \subset M$ is of finite index, then M is a finite index subfactor of N^t for some t. Since being super McDuff is clearly preserved under amplifications, it suffices to show that, if $N \subset M$ is of finite index and M is super McDuff, then so is N.

Now, by Lemma 2.2, we have $M' \cap M^{\mathcal{U}} \subset N' \cap M^{\mathcal{U}}$ is of finite index. So, as M is super McDuff, we have $M' \cap M^{\mathcal{U}} = \mathbb{C}$ and thus by Lemma 2.3, $Z(N' \cap M^{\mathcal{U}})$ is finite-dimensional. We note that the trace-preserving conditional expectation $E_{N^{\mathcal{U}}} : M^{\mathcal{U}} \to N^{\mathcal{U}}$ is given by,

$$E_{N^{\mathcal{U}}}((x_n)_{\mathcal{U}}) = (E_N(x_n))_{\mathcal{U}}$$

from which it is easy to see that $N \subset M$ being of finite index implies $N^{\mathcal{U}} \subset M^{\mathcal{U}}$ is of finite index as well. It is also easy to see that $E_{N^{\mathcal{U}}}(N' \cap M^{\mathcal{U}}) = N' \cap N^{\mathcal{U}}$, i.e., $E_{N^{\mathcal{U}}}$ restricts to the trace-preserving conditional expectation from $N' \cap M^{\mathcal{U}}$ to $N' \cap N^{\mathcal{U}}$. Thus, $N' \cap N^{\mathcal{U}} \subset N' \cap M^{\mathcal{U}}$ is also of finite index. Applying Lemma 2.3 again, we see that $Z(N' \cap N^{\mathcal{U}})$ is finite-dimensional. By [23, Theorem 5], we have $Z(N' \cap N^{\mathcal{U}}) = \mathbb{C}$ and N is super McDuff. □

For the proof about the uniformly super McDuff case, we need the downward Löwenheim-Skolem theorem. For the convenience of readers, we include the version we specifically need here (cf [5, Proposition 7.3]):

Theorem 2.4 (Downward Löwenheim-Skolem theorem, as applied to inclusions of II_1 factors) *Let $N \subset M$ be an inclusion of II_1 factors, $A \subset N$ be a subset separable under the 2-norm. Then there exists separable II_1 factors $N_0 \subset N$, $M_0 \subset M$ s.t. $N_0 \subset M_0$, the structure $(N_0 \subset M_0, E_{N_0})$ is an elementary substructure of $(N \subset M, E_N)$, and $A \subset N_0$.*

We will also need the following lemma:

Lemma 2.5 *Let $N_1 \subset M_1$ and $N_2 \subset M_2$ be two inclusions of II_1 factors. If the structures $(N_1 \subset M_1, E_{N_1})$ and $(N_2 \subset M_2, E_{N_2})$ are elementarily equivalent, then one inclusion is of finite index if and only if the other is of finite index.*

Proof $N \subset M$ being of finite index is equivalent to $E_N - \lambda I \geq 0$ for some $\lambda > 0$, which in turn can be characterized as the existence of $\lambda > 0$ such that

$$\tau(y^* y(E_N(x^* x) - \lambda x^* x)) \geq 0$$

for all $x, y \in B_1(M)$ where $B_1(M)$ denote the operator norm closed unit ball of M. Equivalently, there exists $\lambda > 0$ such that

$$\inf_{x,y \in B_1(M)} \tau(y^* y(E_N(x^* x) - \lambda x^* x)) \geq 0$$

Since the left-hand side is a formula in the language of inclusions of II_1 factors with conditional expectations, the above property of inclusions is preserved by elementary equivalence, whence the lemma follows. □

Proof of Theorem A, uniformly super McDuff case In [16, Theorem 3.5], the implication ($6 \Rightarrow 4$) shows that if every separable $N \equiv M$ is super McDuff, then M is uniformly super McDuff. In the proof, it is shown that, to prove M is uniformly super McDuff, it suffices to prove for any finitely many $x_1, \cdots, x_n \in M^{\mathcal{U}}$, there is a separable elementary substructure M_0 of $M^{\mathcal{U}}$ containing $x_1, \cdots, x_n$ s.t. M_0 is super McDuff. Now, assume $N \subset M$ is of finite index and M is super McDuff. Let $x_1, \cdots, x_n \in N^{\mathcal{U}}$. Since $\{x_1, \cdots, x_n\} \subset N^{\mathcal{U}} \subset M^{\mathcal{U}}$, by downward Löwenheim-Skolem, there exist separable II_1 factors $N_0 \subset N^{\mathcal{U}}$ and $M_0 \subset M^{\mathcal{U}}$ s.t. $N_0 \subset M_0$, the structure $(N_0 \subset M_0, E_{N_0})$ is an elementary substructure of $(N^{\mathcal{U}} \subset M^{\mathcal{U}}, E_{N^{\mathcal{U}}})$, and $x_1, \cdots, x_n \in N_0$. The inclusion $N^{\mathcal{U}} \subset M^{\mathcal{U}}$ is of finite index, so by Lemma 2.5, $N_0 \subset M_0$ is of finite index as well. Since M is uniformly super McDuff, M_0 is super McDuff. Hence, N_0 is super McDuff, which suffices to show N is uniformly super McDuff. The proof where N is a finite index extension of M can be carried out similarly. Alternatively, it follows from taking the basic construction and then applying the previous case for finite index subfactors. □

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Received: 3 February 2025 Revised: 7 May 2025