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A Predicative Approach to the Constructive Integration Theory of Locally Compact Metric Spaces

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Abstract: Based on the inherently impredicative approach of Bishop to constructive integration theory, we present a predicative version of the integration theory of locally compact metric spaces. For that, we first introduce locally compact metric spaces with a modulus of local compactness. This notion of local compactness is incompatible to Mandelkern's but equivalent to both Bishop's and Chan's corresponding notions. Using our definition, we reconstruct the integration theory of continuous functions with compact support using set-indexed families of subsets, avoiding the impredicativity of the original constructive theory of Bishop and Cheng. We work within Bishop Set Theory, which provides an expressive framework for Bishop-style constructive mathematics and constitutes a minimal extension of Bishop's original theory of sets.

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1 Introduction

In general, the most popular approach to classical measure theory is to define integration through the concept of measure, see eg Halmos [13]. In contrast, the Daniell approach [10] proceeds the opposite direction and thus uses integration in order to define measure. The basic structure of the Daniell approach is a *Daniell space* $(X, L, \int)$, where L is a Riesz space of real-valued functions on X and $\int\colon L \to \mathbb{R}$ is a positive, linear functional that satisfies a certain continuity condition. A subset A of X is integrable, if its characteristic function χ_A is in L^1, an extension of L, that is defined through the non-constructive Bolzano–Weierstrass Theorem and the non-constructive axiom of completeness for $\mathbb{R}$. The measure $\mu(A)$ of A is then defined as the integral $\int \chi_A$. The Daniell approach has been incorporated in Bourbaki [5].

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Later, Bishop developed a constructive version of the Daniell approach, which we call *Bishop Measure Theory* (BMT) [2], and the approach was also adopted into *Bishop–Cheng Measure Theory* (BCMT), which was introduced in Bishop and Cheng [4] and substantially extended in Bishop and Bridges [3]. The basic structure of BCMT is an *integration space* $(X, L, \int)$, where $(X, =_X, \neq_X)$ is a Bishop set $(X, =_X)$ equipped with an inequality $x \neq_X x'$, a strong form of the classical (weak) negation $\neg(x =_X x')$. The category of sets for both constructive measure theories BMT and BCMT is **SetIneq**, the subcategory of **Set** of Bishop sets with an inequality. The arrows in **SetIneq** are *strongly extensional* functions $f\colon (X, =_X, \neq_X) \to (Y, =_Y, \neq_Y)$, ie functions $f\colon X \to Y$, such that $f(x) \neq_Y f(x') \Rightarrow x \neq_X x'$, for every $x, x' \in X$. Moreover, L is a set of strongly extensional, real-valued *partial* functions on X that has a structure similar to that of a Riesz space, and lastly $\int\colon L \to \mathbb{R}$ is a positive, linear functional that satisfies a constructive version of Daniell's continuity condition. The use of partial functions in L is crucial to the constructive realisation of the Daniell approach.

In order to avoid the use of the principle of the excluded middle (PEM) in the definition of the characteristic function χ_A of a subset A of X, Bishop employed *complemented subsets* $\boldsymbol{A} := (A^1, A^0)$ of X, where A^1, A^0 are disjoint subsets of X in a strong sense: $a^1 \neq a^0$ for every $a^1 \in A^1$ and $a^0 \in A^0$. The characteristic function $\chi_{\boldsymbol{A}}\colon A^1 \cup A^0 \to \{0,1\}$ of a complemented subset $\boldsymbol{A}$, defined by $\chi_{\boldsymbol{A}}(a) := 1$ if $a \in A^1$, and $\chi_{\boldsymbol{A}}(a) := 0$ if $a \in A^0$, is a partial, Boolean-valued function[1] on X, as it is defined on the subset $A^1 \cup A^0$ of X, without though the use of PEM! Similarly to the Daniell approach, Bishop and Cheng extended L to L^1 using the scheme of separation and the extensional property of integrable functions, ie $L^1 := \{f \in \mathfrak{F}(X)\colon f \text{ is integrable}\}$, where $\mathfrak{F}(X)$ is the totality of strongly extensional, real valued, partial functions on X. As the membership condition in the definition of $\mathfrak{F}(X)$ involves quantification over the universe $\mathbb{V}_0$ of predicative sets,[2] $\mathfrak{F}(X)$ is a proper class. Consequently, the use of the separation scheme over the proper class $\mathfrak{F}(X)$ in the definition of L^1 determines a proper class, and not a set.

This impredicativity involved in the definition of L^1 by Bishop and Cheng pervades BCMT and hinders the extraction of its computational content. It was noticed by

[1]The totality of complemented subsets of a set with an inequality X is a swap algebra, a generalisation of a Boolean algebra, while the totality of Boolean-valued, partial functions on X is a swap ring, a generalisation of a Boolean ring (see Petrakis and Zeuner [24] and Misselbeck-Wessel and Petrakis [15]).

[2]To define an element of $\mathfrak{F}(X)$, we need to define a partial function on X ie we need to construct a set A and an embedding of A into X. Clearly, this categorical notion of a subset of X involves quantification over $\mathbb{V}_0$.

Spitters in [28], and it is behind the subsequent abstract approach to the constructive treatment of measure and integration by Coquand, Palmgren and Spitters [8, 9, 28]. In [19, 30, 25] Petrakis and Zeuner provided a predicative treatment of Bishop–Cheng's L^1 by considering only the canonically integrable functions of a given pre-integration space. Using the indexisation-method, which is introduced by Bishop in [2] and elaborated much later in [19], the set-indexed family of canonically integrable functions is shown in [25, Theorem 10.9], to be an appropriate completion of the original pre-integration space. The predicative definition of L^1 in [25] ensures that all concepts defined through quantification over L^1 in BCMT, such as the notion of a full set, are also predicative. Here we continue the predicative reformulation of BCMT focusing on the integration theory of locally compact metric spaces.

Definition 1.1 (Bishop local compactness [2]) An inhabited metric space X is *Bishop locally compact*, if for every bounded $A \subseteq X$ there is a compact $K \subseteq X$ with $A \subseteq K$.

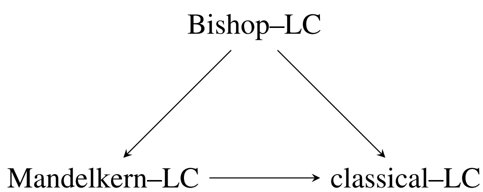
Clearly, Bishop local compactness (Bishop–LC) has an impredicative formulation, as it requires quantification over the proper class $\mathscr{P}(X)$ twice. First, we replace Bishop's impredicative definition of a locally compact metric space, given in [2, page 102], by a predicative and constructively equivalent notion of local compactness (see Definition 3.3 and Proposition 3.8). Notice that Bishop–LC is not equivalent to the classical definition of local compactness (*classical*-LC) according to which every point has a compact neighborhood. However, it holds that, if X is Bishop–LC, then it is also classical–LC: if X is inhabited by some x_0, then for every $x \in X$ we have that x is contained in the open ball $[d_{x_0} < n] := \{x \in X; d(x_0, x) < n\}$ and by hypothesis there is some compact subset K of X with $[d_{x_0} < n] \subseteq K$, hence K is a compact neighborhood of x. In [14, page 1111], Mandelkern, remarks the following:

> In the constructive theory of metric spaces as developed by Errett Bishop, the concept of locally compact space is unique. Virtually all other metric space concepts were successfully constructivized by a judicious choice of definition from among a variety of classically equivalent conditions. In this case, however, the definition differed from the classical definition. Bishop's locally compact spaces are those in which every bounded subset is contained in a compact subset. This allows the construction of a one-point compactification, and includes many traditional locally compact spaces, such as the Euclidean spaces. However, other spaces, such as open spheres in Euclidean space, are included only if given a new metric.

Classical–LC does not work constructively, as the one-point compactification of such a locally compact metric space need not be metrisable. In [14], Mandelkern provides a predicative definition of a locally compact metric space. A metric space is *Mandelkern locally compact* (Mandelkern–LC) if there is an ascending sequence $(H_n)_{n\in\mathbb{N}}$ of compact subsets of X, such that X is the countable union of H_n's and, for every $n \in \mathbb{N}$, H_{n+1} is a *uniform neighborhood* of H_n, ie there is some $r_n > 0$ such that:

$$\bigcup_{x\in H_n} [d_x < r_n] \subseteq H_{n+1}$$

Mandelkern showed that his notion is more general than Bishop–LC, ie if X is Bishop–LC, then X is Mandelkern–LC. Clearly, if X is Mandelkern–LC, then X is also classical–LC, ie we have the following implications:



Note that Mandelkern does not require a locally compact space to be inhabited. A predicative formulation of local compactness seems not to be Mandelkern's concern. Mandelkern wanted to generalise Bishop's notion, in order to include more examples. In [14], Mandelkern showed that his notion is strictly more general than Bishop–LC, as open spheres in $\mathbb{R}^n$ and the inhabited, metric complement of a located set in a Mandelkern–LC space are Mandelkern–LC spaces, while the latter is not in general a Bishop–LC space (see Bishop and Bridges [3, page 112]). Mandelkern's main result in [14] is the proof of the existence of the one-point compactification for his notion of locally compact metric space within BISH. However, an advantage of working with Bishop–LC is that BMT and BCMT generalise the integration theory of Bishop–LC spaces. A 'replacement' of Bishop–LC by Mandelkern–LC would be justified, from the integration theory point of view, if an integration theory of Mandelkern–LC metric spaces could be developed. This is still an open question.

Here we give a predicative reformulation of Bishop's notion of local compactness, which is close to Mandelkern's notion, however, avoiding his strong monotonicity condition for the sequence of compact sets $(H_n)_{n\in\mathbb{N}}$. The relation between our introduced notion of local compactness to Mandelkern–LC is discussed in the end of Section 3. In Definition 3.3 we equip an inhabited metric space with a modulus of local compactness, a 'witness' for Bishop–LC. Moduli of boundedness (Proposition 3.1), uniform continuity, continuity (Definition 3.5) as well as total boundedness (Definition 3.2) are concepts that witness the corresponding constructive properties of subsets, functions, and spaces, and allow the avoidance of (countable, or dependent) choice in constructive proofs. In type-theoretic words, the various moduli that are introduced in the constructive theory of metric spaces are instances of a *proof-relevance* that can be added to it (see also Petrakis [21]). As in the completely proof-relevant type theory of Martin-Löf the (type-theoretic) axiom of choice, ie the distributivity of the Π–type over the Σ–type, is provable (see eg The Univalent Foundations Program [29, Section 1.6]), the addition of moduli in the constructive theory of metric spaces makes the initial use of choice in some proofs unnecessary.

The paper is structured as follows:

- In Section 2 we give a brief account of Bishop Set Theory, in order to be self-contained. Especially, we define families and sets of subsets as well as families and sets of partial functions indexed by some given set. These concepts are crucial to the predicative reformulation of BCMT.
- In Section 3 we introduce locally compact metric spaces with a modulus of local compactness, a proof-relevant reformulation of Bishop–LC spaces. This notion of local compactness is shown to be equivalent to Bishop–LC and Chan–LC (Proposition 3.8). Its relation to Mandelkern–LC is also discussed.
- In Section 4 we define metric integration spaces with a modulus of continuity and unity (Definition 4.5). Our main result is Theorem 4.10, a predicative and proof-relevant version of [3, Theorem 1.10, page 220], the most fundamental result in the integration theory of locally compact metric spaces. According to it, if X is a locally compact metric space with modulus of local compactness and μ a positive measure on X with modulus of unity u, then there is a function $c\colon\ I \times \mathcal{F}(\mathbb{N}, C^{\mathrm{supp}}(X)) \to X$, such that $(X, C^{\mathrm{supp}}(X), \mathrm{Supp}(X), \mu)$ is a metric integration space with modulus of continuity and unity (c, u).

We work within the framework of *Bishop Set Theory*, a minimal extension of Bishop's theory of sets[3] that behaves like a high-level programming language. The type-theoretic interpretation of Bishop sets as setoids is developed mainly by Palmgren (see eg , [17, 18]). Other formal systems for Bishop-style constructive mathematics (BISH) are Myhill's system CST [16] and Aczel's system CZF [1]. For all notions and results of BST that we use without explanation or proof we refer to Petrakis [19, 20, 21, 22]. For all notions and facts from constructive analysis that we use without explanation or proof, we refer to Bishop [2], Bishop and Bridges [3] as well as Bishop and Richman [6].

[3]The relation of BST to Bishop's original theory of sets is discussed in Petrakis [19, Section 1.2]

2 Some prerequisites from Bishop Set Theory

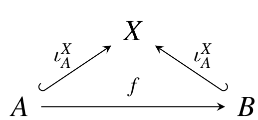
Bishop Set Theory (BST) is an informal, constructive theory of *totalities* and *assignment routines* between totalities that accommodates BISH and serves as an intermediate step between Bishop's original theory of sets and an adequate and faithful formalisation of BISH in Feferman's sense [11]. Totalities, apart from the basic, undefined set of natural numbers $\mathbb{N}$, are defined through a membership-condition. The *universe* $\mathbb{V}_0$ of predicative sets is an open-ended totality, which is not considered a set itself, and every totality the membership-condition of which involves quantification over the universe is not considered a set, but a proper class. *Sets* are totalities the membership-condition of which does not involve quantification over $\mathbb{V}_0$, and are equipped with an equality relation ie an equivalence relation. Assignment routines are of two kinds: *non-dependent* ones and *dependent* ones. A *function* is a non-dependent assignment routine between sets that respects equality. Two sets are equal in $\mathbb{V}_0$ if there is a bijection between them.[4] A function $f\colon X \to Y$ is an *embedding* if for all x, x' in X we have that $f(x) =_Y f(x')$ implies $x =_X x'$, and we write $f\colon X \hookrightarrow Y$.

For a set I and a non-dependent assignment routine $\lambda_0\colon I \rightsquigarrow \mathbb{V}_0$, a *dependent operation* Φ over λ_0 assigns for every $i \in I$ some $\Phi_i := \Phi(i) \in \lambda_0(i)$. In this case, we also write:

$$\Phi\colon \bigwedge_{i\in I} \lambda_0(i)$$

If $\Phi, \Psi\colon \bigwedge_{i\in I} \lambda_0(i)$, then they are equal if $\Phi(i) =_{\lambda_0(i)} \Psi(i)$, for every $i \in I$.

Let $(X, =_X)$ and $(A, =_A)$ be sets. If $\iota_A^X : A \hookrightarrow X$ is an embedding, the pair (A, ι_A^X) is called a *subset*. If (B, ι_B^X) is another subset of X, then we say that (A, ι_A^X) is a *subset* of (B, ι_B^X), in symbols $(A, \iota_A^X) \subseteq (B, \iota_B^X)$, if there is a modulus of the subset property, ie there exists a function $f : A \to B$ such that the following triangle commutes:



In this case we write $f\colon A \subseteq B$. We denote by $\mathscr{P}(X)$ the totality of all subsets of X. Its equality is given by the condition $(A, \iota_A^X) =_{\mathscr{P}(X)} (B, \iota_B^X) :\iff A \subseteq B \wedge B \subseteq A$. If $f : A \subseteq B$ and $g : B \subseteq A$, then we write $(f, g) : A =_{\mathscr{P}(X)} B$. The powerset $\mathscr{P}(X)$ is not considered to be a set, as its membership-condition involves quantification over $\mathbb{V}_0$.

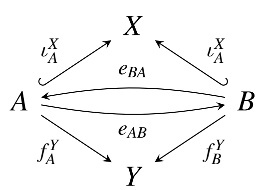
[4] With this equality the universe in BISH can be called *univalent* in the sense of Homotopy Type Theory [29], as, by definition, an equivalence between sets is an equality. Similarly, the type-theoretic function-extensionality axiom is incorporated in BST as the canonical equality of the function space.

If $(X, =_X)$ is a set, a formula $P(x)$ is an *extensional property* on X if for every $x, y \in X$ it holds that $(x =_X y \wedge P(x)) \Rightarrow P(y)$. By the separation scheme, $P(x)$ induces then the so-called *extensional* subset X_P of X. We also write $\{x \in X; P(x)\}$ instead of X_P.

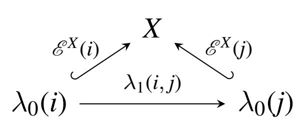
Eg the *diagonal* of a set X is the following extensional subset of $X \times X$:

$$D(X) := \{(x, y) \in X \times X; x =_X y\}$$

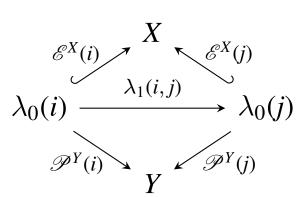
Let X, Y be sets, (A, ι_A^X) a subset of X and $f_A^Y : A \to Y$. Then we call (A, ι_A^X, f_A^Y) a *partial function* from X to Y and write $f_A^Y : X \rightharpoonup Y$. The totality $\mathfrak{F}(X, Y)$ of partial functions from X to Y is not a set as its membership-condition requires the quantification over the universe $\mathbb{V}_0$. Moreover, $(A, \iota_A^X, f_A^Y) =_{\mathfrak{F}(X,Y)} (B, \iota_B^X, f_B^Y)$, if there are moduli $e_{AB} : A \to B$ and $e_{BA} : B \to A$ such that the following upper and lower triangles commute:



Let $(X, =_X)$ and $(I, =_I)$ be sets, $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$ a non-dependent assignment routine, and $\mathscr{E}^X : \bigcurlywedge_{i \in I} \mathcal{F}(\lambda_0(i), X)$ a dependent operation, where for every $i \in I$ we have that $\mathscr{E}^X(i)$ is an embedding and $\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathcal{F}(\lambda_0(i), \lambda_0(j))$ a dependent operation where $\lambda_1(i, i) := \mathrm{id}_{\lambda_0(i)}$ for every $i \in I$. We call $(\lambda_0, \mathscr{E}^X, \lambda_1)$ an *I–family of subsets* of X if for every $(i, j) \in D(I)$ the following triangle commutes:



As $(j, i) \in D(I)$, if $(i, j) \in D(I)$, a similar commutativity holds for $\lambda_1(j, i)$, which together with $\lambda_1(i, j)$ witness the equality of $\lambda_0(i)$ and $\lambda_0(j)$ in $\mathbb{V}_0$. If $(Y, =_Y)$ is also a set, let further $(\lambda_0, \mathscr{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$ and $\mathscr{P}^Y : \bigcurlywedge_{i \in I} \mathcal{F}(\lambda_0(i), Y)$. We call $(\lambda_0, \mathscr{E}^X, \lambda_1, \mathscr{P}^Y)$ an *I-family of partial functions* from X to Y, if the following triangles commute:



Let $\mathrm{Fam}(I, X)$ and $\mathrm{Fam}(I, X, Y)$ be the totality of all families of subsets of X and the totality of all families of partial functions from X to Y, respectively. If $(\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$, such that for all $i, j \in I$ we have that $\lambda_0(i) =_{\mathscr{P}(X)} \lambda_0(j) \Rightarrow i =_I j$, we call $(\lambda_0, \mathcal{E}^X, \lambda_1)$ an *I-set of subsets* of X. Additionally, we define $\mathrm{Set}(I, X)$ to be the totality of I–sets of subsets of X.

Similarly, if $(\lambda_0, \mathcal{E}^X, \lambda_1, \mathscr{P}^Y) \in \mathrm{Fam}(I, X, Y)$, we call $(\lambda_0, \mathcal{E}^X, \lambda_1, \mathscr{P}^Y)$ an *I–set of partial functions* from X to Y, if the equality

$$(\lambda_0(i), \mathcal{E}^X(i), \mathscr{P}^Y(i)) =_{\mathfrak{F}(X,Y)} (\lambda_0(j), \mathcal{E}^X(j), \mathscr{P}^Y(j))$$

implies that $i =_I j$. Let $\mathrm{Set}(I, X, Y)$ be the totality of I–sets of partial functions from X to Y.

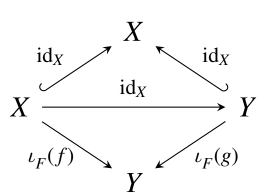
Proposition 2.1 *Let $(X, =_X)$ and $(Y, =_Y)$ be sets and $(F, \iota_F) \subseteq \mathcal{F}(X, Y)$. We define:*

- *$\lambda_0 \colon F \rightsquigarrow \mathbb{V}_0$ is the constant non-dependent assignment routine $\lambda_0(f) := X$, for all $f \in F$.*
- *$\mathcal{E}^X \colon \curlywedge_{f \in F} \mathcal{F}(X, X)$ is the constant dependent operation $\mathcal{E}^X(f) := \mathrm{id}_X$, for all $f \in F$.*
- *$\lambda_1 \colon \curlywedge_{(f,g) \in D(F)} \mathcal{F}(X, X)$ is the constant dependent operation $\lambda_1(f, g) := \mathrm{id}_X$, for all $f, g \in F$ such that $f =_F g$.*
- *$\mathscr{P}^Y \colon \curlywedge_{f \in F} \mathcal{F}(X, Y)$ is the dependent operation $\mathscr{P}^Y(f) := \iota_F(f)$, for every $f \in F$.*

Then $\bar{F} := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathscr{P}^Y)$ is a well-defined F–set of partial functions.

Proof Clearly, our definitions are well-defined and match the signatures. The signatures in turn fit the requirements for $\bar{F}$ to be an F–family of partial functions from X to Y, and we show that $\bar{F}$ respects the equality of F.

Let $f, g \in F$ be such that $f =_F g$. Then due to the extensionality of ι_F, ie $\iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(g)$, the following diagrams commute.



Together we have that $\bar{F} \in \mathrm{Fam}(F, X, Y)$. To show that $\bar{F}$ is in fact a *set* of partial functions, let $f, g \in F$ and $e_1, e_2 \colon X \to X$ such that $(e_1, e_2) : (X, \mathrm{id}_X, \iota_F(f)) =_{\mathfrak{F}(X,Y)}$

$(X, \mathrm{id}_X, \iota_F(g))$. By definition this implies that $\mathrm{id}_X =_{\mathcal{F}(X,X)} \mathrm{id}_X \circ e_1$ and $\mathrm{id}_X =_{\mathcal{F}(X,X)} \mathrm{id}_X \circ e_2$, ie $e_1 =_{\mathcal{F}(X,X)} e_2 =_{\mathcal{F}(X,X)} \mathrm{id}_X$. With the other part of the definition we see that $\iota_F(f) =_{\mathcal{F}(X,Y)} \iota_F(g) \circ \mathrm{id}_X =_{\mathcal{F}(X,Y)} \iota_F(g)$. As ι_F is an embedding, we see that $f =_F g$, ie $\bar{F} \in \mathrm{Set}(F, X, Y)$. □

If $(X, =_X)$ and $(I, =_I)$ are sets and $(\lambda_0, \mathscr{E}^X, \lambda_1) \in \mathrm{Fam}(I, X)$ is an I–family of subsets of X, then the *intersection* and *union* of $(\lambda_0, \mathscr{E}^X, \lambda_1)$ are denoted by $\bigcap_{i\in I} \lambda_0(i)$ and $\bigcup_{i\in I} \lambda_0(i)$, respectively, and are defined in Petrakis [19, pages 91, 97].

3 Locally Compact Metric Spaces with a Modulus of Local Compactness

A metric space is a triplet $(X, =_X, d_X)$, where $(X, =_X)$ is a set and d is a metric on X. If $(X, =_X, d_X)$ and $(Y, =_Y, d_Y)$ are metric spaces, a function $f\colon X \to Y$ is *uniformly continuous* with *modulus of uniform continuity* $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$, if

$$d_X(x, y) \le \omega(\varepsilon) \Rightarrow d_Y(f(x), f(y)) \le \varepsilon$$

for all $x, y \in X$ and $\varepsilon \in \mathbb{R}^+$.

A bounded subset of a metric space has the special property, that the distance between two elements is bounded by some real number. We model this by requiring the subset to be inhabited by some element, and by giving a bound on the distance an arbitrary element is allowed to have from this element.

Specifically, if $(X, =_X, d)$ is a metric space inhabited by x_0 and $M \in \mathbb{R}^+$, X is *bounded* with *modulus of boundedness* M, if for all $x \in X$ we have that $d(x_0, x) \le M$. If $(A, \iota,) \subseteq X$ is inhabited by a_0, then A is a *bounded subset* of X, if there is some $M_A \in \mathbb{R}^+$ such that $(A, =_A, d_A)$, where d_A is the relative metric on A, is bounded with *modulus of boundedness* M_A. The next fact is straightforward to show.

Proposition 3.1 *Let $(X, =_X, d)$ be a metric space and (A, ι) a bounded subset of X with modulus of boundedness $M \in \mathbb{R}^+$ and inhabited by a_0. Then A is included in a closed ball around $\iota(a_0)$, ie there exists $n \in \mathbb{N}^+$ such that $(A, \iota) \subseteq [d_{\iota(a_0)} \le n]$. If (Y, d_Y) is a metric space, then for any uniformly continuous function $f\colon [d_{\iota(a_0)} \le n] \to Y$ with modulus of uniform continuity ω, the restriction function $f \upharpoonright A : A \to Y$ defined by $f \upharpoonright A(x) := f(\iota(x))$ is also uniformly continuous with modulus of uniform continuity ω.*

In order to define total boundedness, we explain our notions of finiteness an subfiniteness. If $(X, =_X)$ is a set, let its set of *subfinite* subsets and set of *finite* subsets be defined by

$$\mathscr{P}^{\text{subfin}}(X) := \bigcup_{n\in\mathbb{N}} \mathcal{F}(\mathbb{N}^{<n}, X)$$
$$\mathscr{P}^{\text{fin}}(X) := \bigcup_{n\in\mathbb{N}} \mathrm{Emb}(\mathbb{N}^{<n}, X)$$

respectively, where $\mathrm{Emb}(Y, X)$ denotes the embeddings from Y to X, and $\mathbb{N}^{<n}$ is defined as the set $\{m \in \mathbb{N};\, m < n\}$. Notice, that subfinite and finite subsets are not actually sets, but functions. Intuitively, we identify each (sub-)finite set with its image. Then the subfiniteness property is equivalent to the more intuitive statement that a subfinite set is a subset of a finite set. In classical set theory there is no difference between finite and subfinite sets, but constructively a subset of a finite set need not be finite itself.

In a totally bounded metric space X there is a subfinite set that ε–approximates the whole metric space, for every for every $\varepsilon \in \mathbb{R}^+$, ie every element of X has a distance less than ε from some element of the subfinite set. Clearly, total boundedness is stronger than boundedness. The basic theory of totally bounded (and compact) metric spaces in Bishop and Bridges [3] can be reconstructed in a computationally more informative way using the notion of modulus of total boundedness (see Grubmüller [12, Sections 2.4, 2.5]), which is defined next.

Definition 3.2 (Totally bounded metric spaces with a modulus of total boundedness) Let $(X, =_X, d)$ be a metric space and let $n \in \mathbb{N}$ such that $A\colon\ \mathbb{N}^{<n} \to X$ is a subfinite subset of X. Let further $\varepsilon > 0$ and let $f\colon\ X \to \mathbb{N}^{<n}$. Then we call (A, h) a *subfinite ε–approximation* of X, if for all $x \in X$ we have that $d\,(x, A(h(x))) < \varepsilon$.

If $\alpha\colon\ \mathbb{R}^+ \to \mathscr{P}^{\text{subfin}}(X) \times \bigcup_{n\in\mathbb{N}} \mathcal{F}(X, \mathbb{N}^{<n})$ is a function such that for all $\varepsilon \in \mathbb{R}^+$ we have that $\alpha_\varepsilon := (A_\varepsilon, h_\varepsilon)$ is a subfinite ε–approximation for X, then we call $(X, =_X, d)$ a *totally bounded metric space* with the *modulus of total boundedness* α.

Definition 3.3 (Locally compact metric spaces with a modulus of local compactness) Let $(X, =_X, x_0, d)$ be a metric space inhabited by x_0. Let further $(K_n, \iota^X_{K_n})_{n\in\mathbb{N}}$ be a sequence of compact subsets of X and $\kappa\colon\ \mathbb{N} \to \mathbb{N}$ a function. We say that $(X, =_X, x_0, d)$ is *locally compact* with *modulus of local compactness* $\big((K_n, \iota^X_{K_n})_{n\in\mathbb{N}}, \kappa\big)$ for $(X, =_X, x_0, d)$, if for every $n \in \mathbb{N}$ we have that $[d_{x_0} \leq n] \subseteq K_{\kappa(n)}$.

If $(\mathbb{R}, =_\mathbb{R})$ is equipped with the standard Euclidean metric d_ε, then $(\mathbb{R}, =_\mathbb{R}, 0, d_\varepsilon)$ is locally compact with modulus of local compactness $\big(([d_0 \leq n])_{n\in\mathbb{N}}, \mathrm{id}_\mathbb{N}\big)$.

Proposition 3.4 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $\big((K_n, \iota_n)_{n\in\mathbb{N}}, \kappa\big)$. If $x_1 \in X$, then there is some $\kappa' \colon \mathbb{N} \to \mathbb{N}$ such that $\big((K_n, \iota_n)_{n\in\mathbb{N}}, \kappa'\big)$ is a modulus of local compactness for $(X, =_X, x_1, d)$.*

Proof Let the function $\kappa' \colon \mathbb{N} \to \mathbb{N}$ be defined by $\kappa'(n) := \kappa(\mathcal{K}_{n+d(x_0,x_1)})$, where $\mathcal{K}_{n+d(x_0,x_1)}$ denotes the canonical bound[5] of $n + d(x_0, x_1)$. Now let $n \in \mathbb{N}$. The basic idea is that the ball $[d_{x_1} \leq n]$ is included in the ball $[d_{x_0} \leq n + d(x_0, x_1)]$ which itself is included in the compact set $K_{\kappa(\mathcal{K}_{n+d(x_0,x_1)})}$. To prove this, let $x \in [d_{x_1} \leq n]$, and note that $d(x, x_0) \leq d(x, x_1) + d(x_1, x_0) \leq n + d(x_0, x_1) < \mathcal{K}_{n+d(x_0,x_1)}$ ie $[d_{x_1} \leq n] \subseteq K_{\kappa'(n)}$. Hence, $\big((K_n, \iota_n)_{n\in\mathbb{N}}, \kappa'\big)$ is a modulus of local compactness for $(X, =_X, x_1, d)$. □

Definition 3.5 Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $\big((K_n, \iota^X_{K_n})_{n\in\mathbb{N}}, k\big)$ and $(Y, =_Y, d_Y)$ a metric space. A function $f \colon X \to Y$ is *continuous* with *modulus of continuity* $(\omega_n)_{n\in\mathbb{N}}$, where $\omega_n \colon \mathbb{R}^+ \to \mathbb{R}^+$, for every $n \in \mathbb{N}$, if, for every $n \in \mathbb{N}$, the restriction function $f_n := f \upharpoonright K_n$ is uniformly continuous with modulus of uniform continuity ω_n. Let $C(X, Y)$ be the set of all continuous functions from X to Y, and let $C(X) := C(X, \mathbb{R})$.

The basic theory of locally compact metric spaces in [3, Section 4.6] can be reformulated in a computationally more informative way through the use of moduli of local compactness (see [12, Section 2.6]). Eg if $(X, =_X, x_0, d)$ is a locally compact metric space with modulus of local compactness $((K_n, \iota_n)_{n\in\mathbb{N}}, \tilde{\kappa})$ and (A, ι^X_A) is a closed and located subset of X (hence it is inhabited by some element a_0), then A is locally compact with a modulus of local compactness $((K'_n, \iota'_n)_{n\in\mathbb{N}}, \kappa')$, which can explicitly be defined from the constructions within the proof and the original modulus of local compactness for X.

Next we compare our definition of local compactness to Bishop and Chan local compactness. In the end, we discuss the relation of our definition to Mandelkern local compactness. The prima facie impredicativity of Bishop local compactness is easy to amend. Definition 3.3 as well as the definition of local compactness by Chan [7] are both equivalent predicative reformulations. While Chan first defines local compactness according to [3], later he introduces an equivalent definition using countable dense subsets. The specific countably dense subsets that are used are binary approximations.

Definition 3.6 (Binary approximation, Chan [7]) Let $(X, =_X, x_0, d)$ be an inhabited metric space. If $A_0 := \{x_0\}$ and $(A_n)_{n\in\mathbb{N}}$ is an ascending sequence of metrically

[5]For a real number $x := (x_n)_{n\in\mathbb{N}}$ its canonical bound $\mathcal{K}_x$ is the least positive natural number, such that $|x_1| + 2 < \mathcal{K}_x$. In this case, for all $y := (y_n)_{n\in\mathbb{N}} \in \mathbb{R}$ with $y =_{\mathbb{R}} x$, we have that $|y_n| < \mathcal{K}_x$, for every $n \in \mathbb{N}^+$.

discrete and finite subsets of X, then $(A_n)_{n\in\mathbb{N}}$ is called a *binary approximation* of X relative to x_0, if for all $n\in\mathbb{N}, n\geq 1$ the following hold:

$$[d_{x_0}\leq 2^n]\subseteq \bigcup_{x\in A_n}[d_x\leq 2^{-n}] \tag{1}$$

$$\bigcup_{x\in A_n}[d_x\leq 2^{-n+1}]\subseteq [d_{x_0}\leq 2^{n+1}] \tag{2}$$

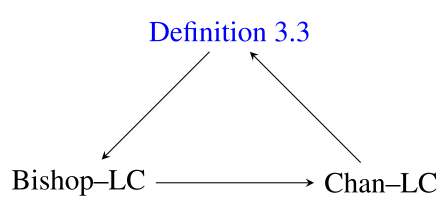
The existence of binary approximations defines Chan local compactness.

Definition 3.7 (Chan local compactness [7]) Let $(X,=_X,x_0,d)$ be an inhabited, complete metric space. X is *Chan locally compact*, if there exists a binary approximation $(A_n)_{n\in\mathbb{N}}$. The sequence $(\|A_n\|)_{n\in\mathbb{N}}$, where $\|\cdot\|$ is the finite cardinality operator, is called a *modulus of Chan local compactness* for X.

Next we show in detail that a locally compact space with a modulus of local compactness is Bishop locally compact. The converse requires the axiom of countable choice[6] (CC), which is generally accepted in BISH (see Bishop and Richman [6, page 12]), but not in Richman's subsystem RICH [26, 27] of BISH, which is exactly BISH without CC. As explained in the subsequent proof some form of finite choice is also required in the proof of the equivalence between our notion of local compactness and Chan local compactness.

Proposition 3.8 *Let $(X,=_X,x_0,d)$ be an inhabited metric space. The following are equivalent over BISH.*

- *(i) X is locally compact with some modulus of local compactness $\big((K_n,\iota^X_{K_n})_{n\in\mathbb{N}},\kappa\big)$.*
- *(ii) X is Bishop locally compact.*
- *(iii) X is Chan locally compact.*



[6]The use of CC in order to get our notion of local compactness from Bishop–LC rests also on the treatment of the powerset as a set, as one needs to use the separation scheme over it to define the 'set' of compact subsets of a metric space. This is avoided in the proof that a Bishop–LC space is a Chan–LC space, as this proof uses CC in the $\mathbb{N}-\mathbb{N}$ form.

Proof (i) $\Rightarrow$ (ii): Let (A, x_1, ι_A^X) be an inhabited set with a modulus of boundedness $M \in \mathbb{N}$. If $n := \mathcal{K}_{d(x_0, \iota_A^X(x_1))+M}$, the canonical bound of $d\left(x_0, \iota_A^X(x_1)\right) + M$, then $(A, \iota_A^X) \subseteq [d_{x_0} \leq n]$, as for all $x \in A$ we have that $d\left(x_0, \iota_A^X(x)\right) \leq d\left(x_0, \iota_A^X(x_1)\right) + d\left(\iota_A^X(x_1), \iota_A^X(x)\right) \leq d\left(x_0, \iota_A^X(x_1)\right) + M \leq n$. By definition of a modulus of local compactness we get $(A, \iota_A^X) \subseteq [d_{x_0} \leq n] \subseteq (K_{\kappa(n)}, \iota_{\kappa(n)})$.

(ii) $\Rightarrow$ (iii): In Chan [7, Proposition 3.2.3] it is shown (with the use of countable choice) that every Bishop locally compact space has a binary approximation.

(iii) $\Rightarrow$ (i): Let $(A_n)_{n\in\mathbb{N}}$ be a binary approximation of X and $(\|A_n\|)_{n\in\mathbb{N}}$ a modulus of Chan local compactness. For every $n, m \in \mathbb{N}$ with $n \leq m$ we have that A_m is a 2^{-m}–approximation of $[d_{x_0} \leq 2^n]$, therefore $[d_{x_0} \leq 2^n]$ is compact. If we define $K_n := [d_{x_0} \leq 2^n]$, for every $n \in \mathbb{N}$, we can show[7] that $\left((K_n, \iota_n)_{n\in\mathbb{N}}, \mathrm{id}_{\mathbb{N}}\right)$ is a modulus of local compactness for X. □

Next we recall Mandelkern's definition of local compactness for metric spaces.

Definition 3.9 (Mandelkern local compactness [14]) A metric space X is *Mandelkern locally compact*, if there is an sequence $(H_k)_{k\in\mathbb{N}}$ of compact subsets of X, such that $X = \bigcup_{k\in\mathbb{N}} H_k$ and for every $k \in \mathbb{N}$ we have that H_{k+1} is a uniform neighbourhood of H_k, ie there is some $r_n > 0$, such that $\bigcup_{x\in H_n}[d_x < r_n] \subseteq H_{n+1}$.

While Definition 3.3 employs a similar notion of a chain of compact sets that tend towards the whole space, it does not enforce that the chain is in fact (strictly) ascending element-wise. It is therefore not clear how to force the modulus of local compactness to be strictly ascending, each being a uniform neighbourhood of the previous subspace. Conversely, it is not clear how fast the chain of uniform neighbourhoods is tending towards the whole space. By introducing a modulus of convergence, or similarly requiring H_{n+1} to include a $\left(\frac{1}{n}\right)$–neighbourhood of H_n, it is possible to pick an appropriate m, such that $[d_{x_0} \leq n] \subseteq H_m$.

4 Integration Theory of Locally Compact Metric Spaces

In this section we reconstruct predicatively the basic integration theory of locally compact metric spaces relying on the notion of an integration space in BCMT. We use

[7] Getting the 2^{-m} approximation requires (finite) choice in order to get a point of the approximation that is at most 2^{-m} away, since there might be multiple points of the approximation in that range.

our previous definitions as well as an amendment to the definition of an integration space as presented in Petrakis [19], in order to remove the impredicativity of BCMT that was discussed in the Introduction.

Continuous functions with compact support play an important role in the integration theory of locally compact metric spaces. First we present their set $C^{\text{supp}}(X)$ and the corresponding set of partial functions $\text{Supp}(X)$ over $C^{\text{supp}}(X)$, in the sense of Section 2.

Definition 4.1 Let $(X, =_X, d)$ be a metric space, (S, ι_S^X) a located subset of X and $f\colon X \to \mathbb{R}$. S is called a *support* of f, if for all $x \in X$ such that $d(x, S) > 0$ we have that $f(x) = 0$. Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $\big((K_n, \iota_{K_n}^X)_{n\in\mathbb{N}}, k\big)$, $f \in C(X)$, and $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$, such that K_n is a support of f, then f is called a *function with compact support* in X with *modulus of compact support* n. Let $C^{\text{supp}}(X)$ be the set of continuous functions with compact support in X. The elements of $C^{\text{supp}}(X)$ are also called *test functions*.

Next proposition is the adaptation of [3, Proposition 6.15, page 119] in our framework, and its proof is omitted (the details can be found in Grubmüller [12, page 36]).

Proposition 4.2 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $\big((K_n, \iota_{K_n}^X)_{n\in\mathbb{N}}, k\big)$. If (K, ι) is a compact subset of X and $\varepsilon \in \mathbb{R}^+$, there is $N \in \mathbb{N}$ and for every $j \in \mathbb{N}^{<N}$ there is a non-negative test function $f_j \in C^{\text{supp}}(X)$ and a compact subset (K_j, ι_j) with $\text{diam}(K_j) < \varepsilon$ that is a support for f_j, such that $\sum_{k=0}^{N} f_k \leq 1$ and $\sum_{k=0}^{N} f_k(\iota_k(x)) = 1$, for all $x \in K$.*

Definition 4.3 Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $\big((K_n, \iota_{K_n}^X)_{n\in\mathbb{N}}, k\big)$. Following Proposition 2.1, let $\text{Supp}(X) := (\lambda_0, \mathscr{E}^{C^{\text{supp}}(X)}, \lambda_1, \mathscr{P}^{\mathbb{R}})$ be the following $C^{\text{supp}}(X)$–set of partial functions:

- $\lambda_0\colon C^{\text{supp}}(X) \rightsquigarrow \mathbb{V}_0$ is the constant non-dependent assignment routine $\lambda_0(f) := X$, for all $f \in C^{\text{supp}}(X)$.
- $\mathscr{E}^{C^{\text{supp}}(X)}\colon \curlywedge_{i\in I} \mathcal{F}(X, X)$ is the constant dependent operation $\mathscr{E}^{C^{\text{supp}}(X)}(f) := \text{id}_X$, for all $f \in C^{\text{supp}}(X)$.
- $\lambda_1\colon \curlywedge_{(f,g)\in D(C^{\text{supp}}(X))} \mathcal{F}(X, X)$ is the constant dependent operation $\lambda_1(f, g) := \text{id}_X$, for all $f, g \in C^{\text{supp}}(X)$ with $f =_{C^{\text{supp}}(X)} g$.
- $\mathscr{P}^{\mathbb{R}}\colon \curlywedge_{f\in C^{\text{supp}}(X)}$ is the dependent operation $\mathscr{P}^{\mathbb{R}}(f) := f$, for every $f \in C^{\text{supp}}(X)$.

Next we define the notion of an integration space using the definition from [19] with minor changes regarding the use of moduli. Our main example of integration space will be a locally compact metric space with a modulus of local compactness, an indexed set of partial functions, and an integral.

Due to the nature of sets of partial functions, it is more convenient to define the integral on the index set rather than the actual set of partial functions. For this reason, in [19, 25] the resulting structure is called a 'pre-integration space' rather than an 'integration space'. Note however, that this definition can easily be pushed back onto the set of partial functions itself, albeit requiring rather cumbersome notation. To simplify notation further, we also define some of the most important arithmetic operations directly on the index set in the obvious way. After the definitions of what is called here a *metric integration space* and a positive measure on it, we prove some intermediate statements and our main result, Theorem 4.10.

Definition 4.4 Let $(X, =_X)$ be a set. We define some arithmetic operations on $\mathfrak{F}(X)$. If $\tilde{f} := (A_f, \iota_f, f)$ and $\tilde{g} := (A_g, \iota_g, f)$ are partial functions from X to $\mathbb{R}$, let

$$\tilde{f} \,\square\, \tilde{g} := (A_f \cap A_g, \iota^X_{A_f \cap A_g}, f \,\square\, g)$$

where $\square$ is one of $+, -, \cdot$ and $\wedge$. We also define the absolute value $\left|\tilde{f}\right| := (A_f, \iota_f, |f|)$, as well as the scalar multiplication with some $\alpha \in \mathbb{R}$ by $\alpha \cdot \tilde{f} := (A_f, \iota_f, \alpha \cdot f)$.

If $(I, =_I)$ is a set, let $(\lambda_0, \mathscr{E}, \lambda_1, \mathscr{P})$ be an I–set of partial functions from X to $\mathbb{R}$. We define the same arithmetic operations on the index set I. For $i, j, k \in I$ such that

$$(\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i) \,\square\, (\lambda_0(j), \mathscr{E}_j, \mathscr{P}_j) =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$$

let $i \,\square\, j := k$, where $\square$ is one of $+, -, \cdot$ and $\wedge$. Additionally, if k is such that $|(\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i)| =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$, then we define $|i| := k$. If $\alpha \in \mathbb{R}$ and k is such that $\alpha \cdot (\lambda_0(i), \mathscr{E}_i, \mathscr{P}_i) =_{\mathfrak{F}(X)} (\lambda_0(k), \mathscr{E}_k, \mathscr{P}_k)$, we define $\alpha \cdot i := k$.

Definition 4.5 (Metric integration space with a modulus of continuity and unity) Let $(X, =_X, x_0, d)$ be a locally compact metric space with a modulus of local compactness, $(I, =_I)$ a set, $L := (\lambda_0, \mathscr{E}, \lambda_1, \mathscr{P})$ an I–set of partial functions from X to $\mathbb{R}$, and $\int \colon I \to \mathbb{R}$ a function. Consider additionally a function $c \colon I \times \mathcal{F}(\mathbb{N}, I) \to X$ and some $p \in I$. We call the structure $(X, I, L, \int)$ a *metric integration space* with a *modulus of continuity and unity* (c, p), if the following properties are satisfied:

(a) For all $i, j \in I$ and $\alpha, \beta \in \mathbb{R}$, there exists $k \in I$ such that $\alpha \cdot i + \beta \cdot j =_I k$ and $\int k = \alpha \int i + \beta \int j$. Also, there exists $l \in I$, such that $|i| =_I l$, as well as $m \in I$, such that $f \wedge 1 =_I m$, where 1 denotes also the constant function 1.

(b) If $i \in I$ and $(i_n)_{n\in\mathbb{N}} \subseteq I$, then we have the following: if for all $n \in \mathbb{N}$ and all $x \in \lambda_0(i_n)$, where $\mathscr{P}_{i_n}(x)$ is non-negative and $\sum_{n\in\mathbb{N}} \int i_n$ converges, as well as $\sum_{n\in\mathbb{N}} \int i_n < \int i$, then $\sum_{n\in\mathbb{N}} \mathscr{P}_{i_n}\big(c(i,(i_n)_{n\in\mathbb{N}})\big)$ converges and

$$\sum_{n\in\mathbb{N}} \mathscr{P}_{i_n}\big(c(i,(i_n)_{n\in\mathbb{N}})\big) < \mathscr{P}_{i}\big(c(i,(i_n)_{n\in\mathbb{N}})\big).$$

(c) $\int p = 1$.

(d) For every $i \in I$ and $m \in \mathbb{N}$ there are $j, k \in I$, such that $i \wedge m =_I j$, as well as $|i| \wedge m^{-1} =_I k$, where m and m^{-1} denote the respective constant functions. Moreover, we have that $\lim_{n\to\infty} \int(i \wedge n) = \int i$, and $\lim_{n\to\infty} \int(|i| \wedge n^{-1}) = 0$.

Remark For the relation of Definition 4.5 to the notion of an integration space in [3, page 217], and its use in [3, Theorem 1.10, page 220], we notice the following. First, we explicitly use the partial functions that correspond to the total functions used in [3], in order to be compatible to the definition of an integration space. Second, we explicitly use a modulus, both for the point of X in the original continuity condition[8] (1.1.2) in [3, page 217], as well as for the function with integral 1 in the corresponding condition (1.1.3).

Definition 4.6 Let $(X, =_X, x_0, d)$ be a locally compact metric space with a modulus of local compactness, $\mu\colon C^{\mathrm{supp}}(X) \to \mathbb{R}$ a linear map and $u \in C^{\mathrm{supp}}(X)$. We call μ a *positive measure* with *modulus of unity* u, if $\mu(u) = 1$, and for all non-negative functions $f \in C^{\mathrm{supp}}(X)$ we have that $\mu(f) \geq 0$. We also write $\int f d\mu := \mu(f)$.

The above definition is equivalent to the definition of [3], where instead the existence of some $f \in C^{\mathrm{supp}}(X)$, such that $\mu(f) > 0$ is required. To arrive at Definition 4.6, we merely need to define $u := \frac{f}{\mu(f)}$.

Lemma 4.7 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $((K_n, \iota_{K_n})_{n\in\mathbb{N}}, \kappa)$, μ a positive measure on X, $f \in C^{\mathrm{supp}}(X)$ with modulus of compact support n and $(f_n)_{n\in\mathbb{N}} \subseteq C^{\mathrm{supp}}(X)$, where $f_n \geq 0$, for every $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu$ converges and*

$$\sum_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu < \int f \,\mathrm{d}\mu.$$

If $\varepsilon \in \mathbb{R}^+$, there is a non-negative $g \in C^{\mathrm{supp}}(X)$ and a compact $K \subseteq X$ with $\mathrm{diam}(K) < \varepsilon$ that is a support for g, $\sum_{n\in\mathbb{N}} \int f_n g \,\mathrm{d}\mu$ converges, and $\sum_{n\in\mathbb{N}} \int f_n g \,\mathrm{d}\mu < \int fg \,\mathrm{d}\mu$.

[8] This condition is the constructive version of Daniell's continuity condition in the definition of a Daniell space. The passage from the classical theory of Daniell spaces to the constructive theory of integration spaces is analysed in Petrakis [23].

Proof The proof is based on Proposition 4.2, and it is omitted, as it is similar to the proof of [3, Lemma 1.8, page 219]. □

Lemma 4.8 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $((K_n, \iota_{K_n})_{n\in\mathbb{N}}, \kappa)$, $f \in C^{supp}(X)$ a test function with modulus of compact support $n \in \mathbb{N}$. If $\int f \,\mathrm{d}\mu > 0$, then there is $x \in K$ such that $f(x) > 0$.*

Proof Let $M := \sup\,\{|f(x)|\,;\, x \in X\}$. By the definition of supremum it suffices to show that $M > 0$. For this, we first define the auxiliary function g for every $x \in X$ by $g(x) := \max(1 - d\,(x, K_n)\,, 0)$.

We show[9] that $h := M \cdot g - f \geq 0$. First, we observe that h is uniformly continuous. By [3, Proposition 6.12, page 117], a function that vanishes at infinity on X (see [3, Definition 6.10, page 116]) is uniformly continuous, and every function with compact support also vanishes at infinity. Moreover, g is uniformly continuous, as the maximum of two uniformly continuous functions, because the mapping $x \mapsto d(x, K_n)$ is uniformly continuous; actually, it suffices for K_n to be a located subset (see [3, page 96]). Let $D := K_n \cup -K_n$, where $-K_n := \{x \in X; d(x, K_n) > 0\}$ is the metric complement of K_n. It is immediate to show that D is a dense subset of X (see also [3, page 88]). Next we show that the restriction of h to D is non-negative.

If $x \in K_n$, then $d(x, K_n) = 0$ and $h(x) = M - f(x) \geq 0$. If $x \in -K_n$, then $d(x, K_n) > 0$ and $f(x) = 0$, hence $h(x) = Mg(x) \geq 0$. As the uniformly continuous h is non-negative on the dense subset D of X, it is also non-negative on the whole space X. To show this, we suppose that there is some $x_0 \in X$, such that $h(x_0) < 0$, and by the uniform continuity of h it is immediate to get a contradiction. By [3, Lemma 2.18, page 26], we get $h(x_0) \geq 0$, and since x_0 is arbitrary, $h \geq 0$.

Due to the monotonicity of the integral, $0 \leq \int h \,\mathrm{d}\mu = M \int g \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu$, which is equivalent to $\int f \,\mathrm{d}\mu \leq M \int g \,\mathrm{d}\mu$. Since $\int f \,\mathrm{d}\mu > 0$, by assumption, and $\int g \,\mathrm{d}\mu \geq 0$, it follows that $M > 0$. □

Lemma 4.9 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $((K_n, \iota_{K_n})_{n\in\mathbb{N}}, \kappa)$, $f \in C^{supp}(X)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C^{supp}(X)$, such that $f_n \geq 0$, as well as $\sum_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu$ exists, and*

$$\sum_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu < \int f \,\mathrm{d}\mu.$$

Then there is $x \in X$, such that for all $m \in \mathbb{N}$ we have that $\sum_{n=1}^{m} f_n(x) \leq f(x)$.

[9] The proof of this fact is omitted in the corresponding proof of [3, Lemma 1.9, page 219]. As this proof is not trivial, we include it here.

Proof By iterated application of Lemma 4.7 we define $(g_n)_{n\in\mathbb{N}} \subseteq C^{\text{supp}}(X)$ recursively:

(a) $g_0 := f$

(b) Let $m \in \mathbb{N}$ and assume $g_k \in C^{\text{supp}}(X)$ has already been defined for all $k \in \mathbb{N}, k < m$. Then by Lemma 4.7 we can construct $g_m \in C^{\text{supp}}(X)$, such that g_m has a compact support K_m, where $\text{diam}(K_m) < m^{-1}$ and:

$$\sum_{n\in\mathbb{N}} \int f_n g_1 \cdots g_m \, \mathrm{d}\mu < \int f g_1 \cdots g_m \, \mathrm{d}\mu$$

Cutting off the outer series, for every $m \in \mathbb{N}$ we have that

$$\sum_{n=1}^{m} \int f_n g_1 \cdots g_m < \int f g_1 \cdots g_m \, \mathrm{d}\mu$$

and applying Lemma 4.8 on $fg_1 \cdots g_m - \sum_{n=1}^{m} f_n g_1 \cdots g_m$ yields $x_m \in X$ with:

$$\Big[0 \leq \Big] \sum_{n=1}^{m} (f_n g_1 \cdots g_m)(x_m) < (fg_1 \cdots g_m)(x_m)$$

Because of $(fg_1 \cdots g_m)(x_m) > 0$ for every $m \in \mathbb{N}$, it holds that for all $k \in \mathbb{N}, k \leq m$ we have $g_k(x_m) > 0$, and therefore by induction it follows that $x_m \in K_k$. Since $\text{diam}(K_k) < k^{-1}$ for all $k \in \mathbb{N}$ this means that specifically $d(x_m, x_k) < k^{-1}$, ie $(x_m)_m$ is a Cauchy sequence in X. Due to the completeness of X, $(x_m)_m$ converges to some $x \in X$.

Due to the fact that for all $m \in \mathbb{N}$ we have that $g_m(x_m) > 0$, it follows that $\sum_{n=1}^{m} f_n(x_m) < f(x_m)$, ie $\sum_{n=1}^{m} f_n(x) \leq f(x)$. □

The next theorem is the main result in the integration theory of locally compact metric spaces that is reconstructed here within our predicative and proof-relevant framework.

Theorem 4.10 *Let $(X, =_X, x_0, d)$ be a locally compact metric space with modulus of local compactness $((K_n, \iota_{K_n})_{n\in\mathbb{N}}, \kappa)$, $f \in C^{supp}(X)$, and μ a positive measure on X with modulus of unity u. There is $c\colon I \times \mathcal{F}(\mathbb{N}, C^{supp}(X)) \to X$, such that $(X, C^{supp}(X), \text{Supp}(X), \mu)$ is a metric integration space with modulus of continuity and unity (c, u).*

Proof First, we note that all objects fulfill the respective required signatures. We therefore need to show points (a)–(d) of Definition 4.5.

(a) Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^{\text{supp}}(X)$ with moduli of compact support $n_f, n_g \in \mathbb{N}$ respectively. Then $\max(n_f, n_g)$ is a modulus of compact support for the function

$\alpha f + \beta g$ ie $\alpha f + \beta g \in C^{\text{supp}}(X)$. By definition of a positive measure, μ is linear ie $\int \alpha f + \beta g \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu + \beta \int g \,\mathrm{d}\mu$. Additionally, n_f is also the modulus of compact support for the functions $|f|$ as well as $f \wedge 1$ ie $|f|,\ f \wedge 1 \in C^{\text{supp}}(X)$.

(c) By definition we have that $\int u \,\mathrm{d}\mu = 1$.

(d) Let $f \in C^{\text{supp}}(X)$ with modulus of compact support n. For every $r \in \mathbb{R}^+$ we have that n is also a modulus of compact support for $f \wedge r$. Restricting f to the compact set K_n allows us to use [3, Corollary 4.3, page 94], according to which $\sup f \restriction K_n := \sup\{f(\iota_n(x));\, x \in K_n\}$ exists. Since, if $f(x) > 0$ for some $x \in X$, then $k \in K_n$ such that $x = \iota_{K_n}$, we have that $\sup\{f(x);\, x \in X\} = \sup f \restriction K_n$. Now consider $k := \mathcal{K}_{\sup\{f(x);\, x \in X\}}$ ie the canonical bound of $\sup\{f(x);\, x \in X\}$. Then we have for all $m \in \mathbb{N}$ with $m \geq k$ that $f \wedge m =_{\mathcal{F}(X)} f$ ie the sequence $(\int f \wedge m \,\mathrm{d}\mu)_{m \in \mathbb{N}}$ becomes constant and therefore $\lim_{m\to\infty} \int f \wedge m \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu$.

For the other part, let $m \in \mathbb{N}$ and define a function $g\colon X \to \mathbb{R}$ by setting $g(x) := \max(1 - d(x, K_n), 0)$ for $x \in X$. Assume that there is $x \in X$ such that $|f(x)| \wedge m^{-1} > m^{-1} g(x)$. By observing that in any case we have that $|f(x)| \wedge m^{-1} \leq m^{-1}$, we arrive at the following inequality chain: $m^{-1} \geq |f(x)| \wedge m^{-1} > m^{-1}(1 - d(x, K_n))$. By elementary transformations it is equivalent to $0 \leq m(|f(x)| \wedge m^{-1}) - 1 < d(x, K_n)$, and specifically $d(x, K_n) > 0$. Due to the definition of n, it follows that $f(x) = 0$ and therefore $|f(x)| \wedge m^{-1} = 0$, which contradicts our assumption. It follows that $|f| \wedge m^{-1} \leq m^{-1} g$. Due to μ being a positive measure, it follows that $0 \leq \int |f| \wedge m^{-1} \,\mathrm{d}\mu \leq m^{-1} \int g \,\mathrm{d}\mu$, and particularly $0 \leq \lim_{m\to\infty} \int |f| \wedge m^{-1} \,\mathrm{d}\mu \leq \lim_{m\to\infty} m^{-1} \int g \,\mathrm{d}\mu = 0$ ie the second part of (d).

(b) Let $f \in C^{\text{supp}}(X)$ and $(f_n)_{n \in \mathbb{N}}$ a sequence in $C^{\text{supp}}(X)$ such that $f_n \geq 0$ as well as $\sum_{n \in \mathbb{N}} \int f_n \,\mathrm{d}\mu$ exists and $\sum_{n \in \mathbb{N}} \int f_n \,\mathrm{d}\mu < \int f \,\mathrm{d}\mu$. Additionally, consider the previously defined function g given by $g(x) := \max(1 - d(x, K_n), 0)$ and define

$$\alpha := \frac{1}{2} \frac{\int f \,\mathrm{d}\mu - \sum_{n \in \mathbb{N}} \int f_n \,\mathrm{d}\mu}{2 + \int g \,\mathrm{d}\mu}$$

such that
$$\sum_{n \in \mathbb{N}} \int f_n \,\mathrm{d}\mu + \alpha \cdot \left(2 + \int g \,\mathrm{d}\mu\right) < \int f \,\mathrm{d}\mu.$$

Then let $(N(n))_{n \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers, such that $\sum_{k=N(n)}^{N(n+1)} \int f_n \,\mathrm{d}\mu < 2^{-2n}\alpha$, which we can find, as the series converges. Finally, let the sequence $(f_n')_{n \in \mathbb{N}}$, defined by:

$$f_n' := \begin{cases} \alpha g & n = 0 \\ f_{n'} & n = 2n' \text{ for some } n' \in \mathbb{N}^+ \\ 2^{n'} \sum_{k=N(n')}^{N(n'+1)} f_k & n = 2n' + 1 \text{ for some } n' \in \mathbb{N}^+ \end{cases}$$

ie $(f'_n)_{n\in\mathbb{N}} = (\alpha g, f_0, 2^0\sum_{k=N(0)}^{N(1)} f_k, f_1, 2^1\sum_{k=N(1)}^{N(2)} f_k, \dots)$.

Then we have that:

$$\begin{aligned}
\sum_{n\in\mathbb{N}}\int f'_n \,\mathrm{d}\mu &= \int \alpha g \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}}\int f_n \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}}\int 2^n \sum_{k=N(n)}^{N(n+1)} f_k \,\mathrm{d}\mu \\
&\leq \alpha \int g \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}}\int f_n \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}} 2^n \sum_{k=N(n)}^{\infty}\int f_k \,\mathrm{d}\mu \\
&< \alpha \int g \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}} f_n \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}} 2^n \cdot 2^{-2n}\alpha \\
&= \alpha \int g \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}}\int f_n \,\mathrm{d}\mu + \alpha\sum_{n\in\mathbb{N}} 2^{-n} \\
&= \alpha \int g \,\mathrm{d}\mu + \sum_{n\in\mathbb{N}}\int f_n \,\mathrm{d}\mu + 2\alpha \\
&= \sum_{n\in\mathbb{N}}\int f_n \,\mathrm{d}\mu + \alpha\cdot\left(2 + \int g \,\mathrm{d}\mu\right) \\
&< \int f \,\mathrm{d}\mu
\end{aligned}$$

Now we can apply Lemma 4.9 and construct a point $x \in K$, such that for every $m \in \mathbb{N}$

$$\alpha g(x) + \sum_{n=0}^{m} f_n(x) + \sum_{n=0}^{m} 2^n \sum_{k=N(n)}^{N(n+1)} f_k(x) \leq f(x)$$

ie in particular:

$$\alpha g(x) + \sum_{n=0}^{m} f_n(x) + 2^m \sum_{k=N(m)}^{N(m+1)} f_k(x) \leq f(x)$$

It follows that

$$\sum_{k=N(m)}^{N(m+1)} f_k(x) \leq 2^{-m} f(x)$$

ie $\sum_{n\in\mathbb{N}} f_n(x)$ converges and that $\alpha g(x) + \sum_{n\in\mathbb{N}} f_n(x) \leq f(x)$. Since $x \in K$, $\alpha g(x) = \alpha > 0$, hence $\sum_{n\in\mathbb{N}} f_n(x) < f(x)$, as required. Thus, we define $c(f, (f_n)_n) :\,= x$. □

5 Concluding comments

Here we presented a predicative and proof-relevant formulation of the constructive integration theory of locally compact metric spaces within BCMT. Our work is a chapter in predicative BCMT (PBCMT), which is initiated in Petrakis [19], Zeuner [30] as well as Petrakis and Zeuner [25]. First, we introduced locally compact metric spaces with a modulus of local compactness and discussed their relation to Bishop–LC spaces, Chan–LC spaces and Mandelkern–LC spaces. The addition of moduli that witness certain properties of spaces or functions to the constructive theory of metric spaces facilitates the formalisation of this theory eg in type theory, and the extraction of its computational content. Moreover, it provides a choice-free formulation of concepts and results that makes possible the development of the corresponding theory within the stronger subsystem RICH of BISH.

As predicativity should be, in our view, a property of a constructive mathematical theory, it is necessary to have a predicative reconstruction of BCMT, the most developed constructive theory of measure and integration. The avoidance of quantification over proper classes was achieved through the use of the theory of set-indexed families of sets, subsets and partial functions. These concepts, which were only mentioned by Bishop at a very basic, intuitive level, are defined explicitly and elaborated in Petrakis [19, 20, 21], providing a fruitful mathematical extension of Bishop's original theory of sets. Although most of the results in Section 4 have their analogue in Chapter 6 of Bishop and Bridges [3], their development here is motivated by the need of a predicative formulation of BCMT and a uniform, proof-relevant presentation of the constructive theory of metric spaces based on the use of moduli.

It is expected that the further study of the (integration) theory of locally compact metric spaces with a modulus of local compactness will provide examples of avoidance of countable choice, as in the cases of the theory of L^p–spaces in Petrakis and Zeuner [25]. We hope to present such examples in a subsequent work. As already mentioned, the question whether an integration theory of Mandelkern–LC spaces is possible is still open. As Mandelkern–LC is more general than Bishop–LC, and, because of Proposition 3.8, it is also more general than local compactness of Definition 3.3, the development of an integration theory of Mandelkern–LC spaces is an important future task. Consequently, the question whether BCMT, or a variation of it, could serve as a generalisation of the integration theory of Mandelkern–LC spaces is also open.

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