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On the complexity of spectra of bounded analytic functions

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Abstract: The spectrum of a bounded analytic function on the unit disk $\mathbb{D}$ is the set of the accumulation points of its zeros. We investigate the computability-theoretic complexity of spectra of computable bounded analytic functions. While the spectrum of a bounded analytic function on $\mathbb{D}$ is Σ^0_3–closed, we show the converse fails. At the same time, we construct a bounded analytic function on $\mathbb{D}$ whose spectrum is Σ^0_3–complete. We also show that there exists a Σ^0_2–closed set of unimodular points which is not the spectrum of any bounded analytic function on $\mathbb{D}$, while every Π^0_2–closed set of unimodular points is. We then turn to uniform Frostman functions. We prove an effective version of a theorem of Matheson. Namely, every computably closed and nowhere dense set of unimodular points is the spectrum of a computable uniform Frostman function.

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1 Introduction

Let $\mathbb{D}$ denote the unit disk; that is $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Fix a nonconstant analytic function $f\colon \mathbb{D} \to \mathbb{C}$. Let $\mathcal{Z}(f)$ denote the multiset consisting of the zeros of f (each zero repeated according to its multiplicity), and let $\Sigma_f = \sum_{z\in\mathcal{Z}(f)} 1 - |z|$. By the Identity Theorem, $\mathcal{Z}(f)$ does not have an accumulation point in $\mathbb{D}$. That is, all accumulation points of $\mathcal{Z}(f)$ belong to $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set of all accumulation points of $\mathcal{Z}(f)$ is known as the *spectrum* of f and we denote it spec(f).

$H^\infty(\mathbb{D})$ denotes the set of bounded analytic functions on $\mathbb{D}$. By a well-known theorem (see eg Rudin [13, 15.22]), when $f \in H^\infty(\mathbb{D})$, $\Sigma_f < \infty$. Furthermore, if $A \subseteq \mathbb{D}$ is a multiset so that $\sum_{a\in A} 1 - |a| < \infty$, then there exists $f \in H^\infty(\mathbb{D})$ so that $\mathcal{Z}(f) = A$.

Matheson and McNicholl [9] and McNicholl [10] proved computability-theoretic analogs of these results. Namely, if $f \in H^\infty(\mathbb{D})$ is computable, then $\mathcal{Z}(f)$ and Σ_f are computable. Conversely, $f \in H^\infty(\mathbb{D})$ is computable if $\mathcal{Z}(f)$ and Σ_f are computable.

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Matheson and McNicholl did not consider the complexity of the spectra of bounded analytic functions. It is well-known that each closed subset of $\mathbb{T}$ is the spectrum of a bounded analytic function. Here, we consider the effective content of this simple result; namely, we consider the complexity of $\mathrm{spec}(f)$. Our first result (Proposition 4.1) is that when $f \in H^\infty(\mathbb{D})$ is computable, $\mathrm{spec}(f)$ is a Σ^0_3–closed set. That is, there is a Σ^0_3 formula that defines the set of rational open arcs that intersect $\mathrm{spec}(f)$. On the one hand, we prove this result is optimal by constructing a computable $f \in H^\infty(\mathbb{D})$ for which this set of rational open arcs is Σ^0_3–complete. However, we also show there is a Σ^0_2–closed subset of $\mathbb{T}$ that is not the spectrum of any computable function in $H^\infty(\mathbb{D})$. We then show that every Π^0_2–closed subset of $\mathbb{T}$ is the spectrum of such a function.

For readers who find our construction of these complex analytically well-behaved objects which are nonetheless incomputable as intriguing as we do, we also recommend the work of Braverman and Yampolsky [3] wherein they construct locally connected Julia sets which are not computable. Similarly, for those interested in constructions of complex analytical objects with arbitrarily high time complexity, Dudko and Yampolsky [6] provide an example in their construction of Cremer Julia sets, as do Rojas and Yampolsky [12] in their construction of real quadratic Julia sets. For an accessible exposition of these results and others, Rojas and Yampolsky [11] provide a comprehensive survey of computable geometric complex analysis and dynamics.

Following these constructions, we turn our attention to the investigation of the uniform Frostman condition and its effect on the spectrum of a computable function in $H^\infty(\mathbb{D})$. This condition is defined as follows. When $f \in H^\infty(\mathbb{D})$, let

$$\sigma_f = \sup_{|\zeta|=1} \sum_{z \in \mathcal{Z}(f)} \frac{1-|z|}{|\zeta - z|}.$$

The inequality $\sigma_f < \infty$ is called the *uniform Frostman condition*. Matheson [8] showed that if $f \in H^\infty(\mathbb{D})$ satisfies the uniform Frostman condition (ie is *uniform Frostman*), then $\mathrm{spec}(f)$ is nowhere dense. Furthermore, he showed that every closed nowhere dense subset of $\mathbb{T}$ is the spectrum of a uniform Frostman function.

We prove two theorems on the spectra of computable uniform Frostman functions. We say that a unimodular point ζ is *accessible* if there is a computable monotone sequence $(\theta_n)_{n\in\mathbb{N}}$ of rational numbers so that $\zeta = \lim_n e^{i\theta_n}$. Our first result on computable uniform Frostman functions is the following.

Theorem 1.1 *If ζ is an accessible unimodular point, then for every $k \in \mathbb{N}$ there is a computable $f \in H^\infty(\mathbb{D})$ so that $\mathrm{spec}(f) = \{\zeta\}$ and so that $\sigma_f < 1 + 2^{-k}$.*

By taking $\zeta = e^{i\Omega}$, where Ω is Chaitin's Ω, we see that there is a computable uniform Frostman $f \in H^\infty(\mathbb{D})$ whose spectrum consists of a single Martin–Löf random point.

Our second theorem on computable uniform Frostman functions is an effective version of Matheson's second result.

Theorem 1.2 *If $S \subseteq \mathbb{T}$ is computably closed and nowhere dense, then for every $k \in \mathbb{N}$ there is a computable $f \in H^\infty(\mathbb{D})$ so that $\operatorname{spec}(f) = S$ and so that $\sigma_f < 1 + 2^{-k}$.*

The paper is organized as follows. Relevant background from complex and computable analysis is summarized in Section 2. In Section 3, we prove a number of results on identifying connected components of c.e. open subsets of $\mathbb{T}$ which will be used in the proof of Theorem 1.2. Our theorems on the complexity of spectra of computable functions in $H^\infty(\mathbb{D})$ are proven in Section 4. Theorem 1.1 is proven in Section 5. Finally, we prove Theorem 1.2 in Section 6.

2 Background

2.1 Background from complex analysis

Let $D_r(z)$ denote the open disk with center z and radius r.

Let Λ denote Lebesgue measure on $\mathbb{T}$. When $p, q \in \mathbb{T}$ are distinct, let

$$d_\Lambda(p, q) = \frac{1}{\pi} \min\{\Lambda(C_1), \Lambda(C_2)\}$$

where C_1 and C_2 are the connected components of $\mathbb{T} - \{p, q\}$. Let $d_\Lambda(p, p) = 0$. It follows that d_Λ is a metric on $\mathbb{T}$. The balls of this metric space are arcs. Accordingly, we let $A(z; r) = \{w \in \mathbb{T} \ : \ d_\Lambda(z, w) < r\}$.

When $\Theta = (\theta_n)_{n\in\mathbb{N}}$ is a sequence of unimodular points, we define the *limit set* of Θ to be

$$\operatorname{Lim} \Theta := \{z \in \mathbb{T} : \forall r > 0\ \exists^\infty n\ \theta_n \in A(z; r)\}.$$

Equivalently, $\zeta \in \operatorname{Lim} \Theta$ if and only if there is a subsequence of Θ that converges to ζ.

A *chain* is a sequence $(C_0, \ldots, C_m)$ of sets so that $C_j \cap C_{j+1} \neq \emptyset$ when $j < m$. If $p \in C_0$ and $q \in C_m$, a chain $(C_0, \ldots, C_m)$ is said to be a *chain from p to q*.

The following is a consequence of Hocking and Young [7, Theorem 3–4].

Lemma 2.1 *Suppose $U \subseteq \mathbb{T}$ is open and connected, and let $\mathcal{J}$ be a set of open arcs so that $U = \bigcup \mathcal{J}$. Then, whenever $p, q \in U$ are distinct, there is a chain $(C_0, \ldots, C_m)$ from p to q so that $C_j \in \mathcal{J}$ for all $j \leq m$.*

The following may be found in Cima, Matheson, and Ross [4, Lemma 1.12.1].

Lemma 2.2 *If $|\zeta| = 1$, and if $1/2 < |z| < 1$, then $|\zeta - z| \geq \frac{\pi}{3} d_\Lambda(\zeta, \frac{z}{|z|})$.*

In the introduction, we discussed the well-known theorem which states that if $A \subseteq \mathbb{D}$ is a multiset so that $\sum_{a \in A} 1 - |a| < \infty$, then there is a function $f \in H^\infty(\mathbb{D})$ so that $\mathcal{Z}(f) = A$. In fact, for every such set A, there exists a canonical function with this property which we define now.

For $a \in \mathbb{D}$, let

$$b_a(z) = \begin{cases} \frac{|a|}{a} \frac{a-z}{1-\bar{a}z} & \text{if } a \neq 0, \\ z & \text{if } a = 0. \end{cases}$$

The function b_a is known as a *Blaschke factor*. When $A \subseteq \mathbb{D}$ is a multiset, let:

$$\Sigma_A = \sum_{a \in A} 1 - |a|$$

$$B_A = \prod_{a \in A} b_a$$

B_A is called a *Blaschke product*, and Σ_A is called the *Blaschke sum* of A. The following theorem states the fundamental facts about Blaschke products. A proof can be found in Rudin [13].

Theorem 2.3 *Let $A \subseteq \mathbb{D}$ be a multiset.*

(1) $B_A \in H^\infty(\mathbb{D})$.
(2) *If $\Sigma_A = \infty$, then B_A is identically zero.*
(3) *If $\Sigma_A < \infty$, then $A = \mathcal{Z}(B_A)$.*

2.2 Background from computability theory and computable analysis

2.2.1 Definitions from computability theory, computable analysis, and computable metric spaces

We assume the reader is modestly familiar with the essentials of computability theory and computable analysis as expounded in Cooper [5], Brattka and Hertling [2], and Weihrauch [14], however we provide a brief introduction to the relevant concepts here.

Definition 2.4 A function f whose domain is a subset of $\mathbb{N}$ and whose codomain is $\mathbb{N}$ (denoted $f\colon \subseteq \mathbb{N} \to \mathbb{N}$) is a *computable partial function* if there exists an algorithm which, on input $n \in \mathrm{dom}(f)$, halts and outputs $f(n)$ and which, on input $n \notin \mathrm{dom}(f)$, does not halt.

If $\mathrm{dom}(f) = \mathbb{N}$, then f is a *computable function*.

We note that this definition is immediately extendable to functions whose domains and codomains are subsets of $\mathbb{N}^n$. We may further extend this definition to any countable domain and codomain (eg $\mathbb{Q}$) by fixing an effective enumeration thereof.

Definition 2.5 A set $A \subseteq \mathbb{N}$ is *computably enumerable (c.e.)* if there exists a computable function $f\colon \subseteq \mathbb{N} \to \mathbb{N}$ so that $A = \mathrm{ran}(f)$.

Equivalently, A is c.e. if it is the domain of a computable partial function. In both formulations of the definition, the most useful property of a c.e. set is that there exists an algorithm which, given $n \in \mathbb{N}$, halts if and only if $n \in A$. In parallel, we may ask such an algorithm "is n an element of A?" for $n = 0, 1, 2, \ldots$ and return the n for which the answer is "yes," hence the term "computably enumerable."

We fix an effective enumeration $(W_e)_{e\in\mathbb{N}}$ of the c.e. subsets of $\mathbb{N}$. $W_{e,s}$ denotes the set of natural numbers enumerated into W_e in at most s steps.

Definition 2.6 A is *computable* if both A and its complement are c.e.

In this case, we may ask both the algorithm which enumerates A and the one which enumerates $\mathbb{N} - A$ whether an input $n \in \mathbb{N}$ is an element of A or $\mathbb{N} - A$. One of these computations will halt, which allows us to determine whether $n \in A$. For this reason, an alternative term for computable sets is the more computer scientific *decidable*, in that the *decision problem* of determining whether a given n is an element of A is solvable algorithmically.

The classes of c.e. sets, sets whose complements are c.e., and computable sets form the first three levels of the arithmetical hierarchy, which we define somewhat informally as follows.

For $n \in \mathbb{N}$, we say that a set $A \subseteq \mathbb{N}$ is Σ^0_n (formally, $A \in \Sigma^0_n$) if there exists a computable set $B \subseteq \mathbb{N}^{n+1}$ so that

$$m \in A \iff (\exists k_1)(\forall k_2)\cdots(\Box k_n)\,(m, k_1, \ldots, k_n) \in B$$

where the quantifiers alternate between existential and universal, and the $\Box$ denotes the quantifier which differs from the previous one. Similarly, A is Π^0_n if there exists a computable set $B \subseteq \mathbb{N}^{n+1}$ so that

$$m \in A \iff (\forall k_1)(\exists k_2)\cdots(\Box k_n)\,(m, k_1, \ldots, k_n) \in B.$$

Equivalently, $\Pi^0_n = \{\mathbb{N} - A : A \in \Sigma^0_n\}$. We finally define $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$. For a more formal definition, see eg Cooper [5].

We observe crucially that, for all $e \in \mathbb{N}$, the set $\{(n, s) : n \in W_{e,s}\}$ is computable, and so

$$n \in W_e \iff (\exists s)\,(n, s) \in \{(n, s) : n \in W_{e,s}\}.$$

Hence, the class of c.e. sets is Σ^0_1, the class of sets whose complements are c.e. is Π^0_1, and the class of computable sets is Δ^0_1. By allowing for the addition of quantifiers, the arithmetical hierarchy provides the promised generalization of these classes.

We now turn our attention to the computability of objects in classical analysis. Consider, for example, the real number π. We may famously approximate π to any desired precision using a variety of algorithms, and so we would like to say that π is, in some sense, computable. This leads to the natural definition of computability for real numbers.

Definition 2.7 A real number x is *computable* if there exists a computable function $f\colon \mathbb{N} \to \mathbb{Q}$ so that, for all $n \in \mathbb{N}$, $|f(n) - x| < 2^{-n}$.

This definition extends naturally to complex numbers through the equivalent definitions of a complex number as a pair of real numbers both of which are computable, or by taking f in the above definition to have codomain $\mathbb{Q}[i]$.

The extension of this definition to sequences of real (or complex) numbers is nearly as straightforward, bar a few technicalities. We may wish to define a computable sequence of reals as one whose terms are computable. However, this permissiveness leads to some undesirable consequences. For example, if $A \subseteq \mathbb{N}$ is incomputable, then we had ought not to regard $(\chi_A(n))_{n\in\mathbb{N}}$ as a computable sequence even though each term is practically as computable as can be. To avoid this, we define a computable sequence as follows.

Definition 2.8 A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is *computable* if there exists a computable function $f\colon \mathbb{N}^2 \to \mathbb{Q}$ so that, for all $n, r \in \mathbb{N}$, $|f(n, r) - x_n| < 2^{-r}$.

As with computable reals, this definition extends naturally to sequences of complex numbers.

More tricky is the definition of computability for functions from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{C}$ to $\mathbb{C}$). As with real numbers, we wish to say, in a colloquial sense, that a function $f\colon \mathbb{R} \to \mathbb{R}$ is computable if it is arbitrarily approximable by an algorithm. The seemingly most immediate way to do this is to say that f is computable if there exists an algorithm which, given an algorithm approximating a real number x and a precision parameter $r \in \mathbb{N}$, outputs $f(x)$ to within 2^{-r}. While this property will hold for computable real-valued functions as we will soon define them, it is not restrictive enough for a definition. The reason why lies in the fact that each computable real number has infinitely many algorithms to approximate it (consider again the example of π, for which there is famously great interest in finding new algorithms to approximate it efficiently). Consider, for example, the task of evaluating $f(0)$ where

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Both $g_0(n) = -2^{-n}$ and $g_1(n) = 2^{-n}$ are both algorithmically specifiable functions from $\mathbb{N}$ to $\mathbb{Q}$ which approximate 0, but any algorithm which alleges to compute f would return 0 given the g_0 approximating algorithm for 0 and 1 given the g_1 approximating algorithm for 0. Thus, our definition of computability for a real-valued function must, in some way, incorporate an effective notion of continuity. For this reason (amongst others), we define computable real-valued functions as follows.

Definition 2.9 (cf [1, page 107]) Let $\mathcal{B}$ denote the set of all rational open balls in $\mathbb{R}$. Suppose $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$. We say that f is *computable* if there exists an algorithm P with the following properties.

(1) On input $B \in \mathcal{B}$, if P halts, it returns $B' \in \mathcal{B}$ so that $f[B] \subseteq B'$.

(2) For all neighborhoods V of $f(x)$, there exists $B \in \mathcal{B}$ such that $x \in B$ and, on input B, P halts with output $B' \in \mathcal{B}$ such that $B' \subseteq V$.

The first condition is referred to as the *approximation property*, and the second as the *convergence property*.

As with the previous definitions, this one is readily extensible to $\mathbb{C}$ by replacing open rational intervals with balls with rational centers and radii.

The careful reader with an eye toward generalization will note that all of our comments about ready extensions to $\mathbb{C}$ could apply to any separable metric space having an "algorithmically-graspable" dense subset.

Definition 2.10 (see eg [11, Definition 5.2.1]) A separable metric space (X, d) is *computable* if there exists a dense sequence of points $\{s_i : i \in \mathbb{N}\}$ for which there exists a computable function $f\colon \mathbb{N}^3 \to \mathbb{Q}$ for which

$$|d(s_i, s_j) - f(i, j, r)| < 2^{-r}.$$

With this definition in hand, we may extend our notions of computability for objects in real and complex analysis to any separable metric space. In the next section, we will see how this extension allows us to define computability for closed subsets of $\mathbb{T}$.

2.2.2 Applications to bounded analytic functions on $\mathbb{D}$

We say $\zeta \in \mathbb{T}$ is *rational* if there is a rational number q so that $\zeta = e^{i\pi q}$. Let $\mathbb{T}_{\mathbb{Q}}$ denote the set of rational points of $\mathbb{T}$. Fix an effective enumeration $(\rho_n)_{n\in\mathbb{N}}$ of the rational points of $\mathbb{T}$. Then, $(\mathbb{T}, (\rho_n)_{n\in\mathbb{N}})$ is a computable presentation of the metric space $(\mathbb{T}, d_\Lambda)$. That is, $(m, n) \mapsto d_\Lambda(\rho_m, \rho_n)$ is computable. We identify this presentation with $\mathbb{T}$. We refer to the rational balls of $\mathbb{T}$ as *rational arcs*. Fix an effective enumeration $(A_n)_{n\in\mathbb{N}}$ of the open rational arcs.

When $S \subseteq \mathbb{T}$ is closed, let $\mathcal{I}(S) = \{n \in \mathbb{N} : A_n \cap S \neq \emptyset\}$. S is completely characterized by $\mathcal{I}(S)$. As in Andreev and McNicholl [1], we define the complexity of S to be the complexity of $\mathcal{I}(S)$. For example, S is Σ^0_n–closed if $\mathcal{I}(S)$ is Σ^0_n. A Σ^0_1–closed set is also known as a *c.e. closed set*.

When $U \subseteq \mathbb{T}$ is open, let $\mathcal{I}(U) = \{n : \overline{A_n} \subseteq U\}$. U is said to be *c.e. open* if $\mathcal{I}(U)$ is c.e. This is equivalent to the existence of a c.e. set of open arcs whose union is U. A closed set $S \subseteq \mathbb{T}$ is said to be *computably closed* if it is c.e. closed and if $\mathbb{T} - S$ is c.e. open.

For the reader familiar with the theory and terminology of computable metric spaces (eg as expounded by Rojas and Yampolsky in [11, Subsection 5.2.1]), our definition of c.e. closed sets coincides precisely with that of lower-computable closed sets. We claim that our definition of c.e. open sets is also equivalent to that of lower-computable open sets, reproduced below.

Definition 2.11 ([11, Definition 5.2.4]) An open set U is *lower-computable* if there is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ so that

$$U = \bigcup_{n\in\mathbb{N}} A_{f(n)}.$$

The fact that, if $U \subseteq \mathbb{T}$ is c.e. open, then it is lower-computable, is immediate by taking the f in Definition 2.11 to be a computable enumeration of $\mathcal{I}(U)$. Conversely, if U is a lower-computable open set with function f as in the definition, then the fact that the $\overline{A_m}$ are closed implies that, for any $m \in \mathcal{I}(U)$, there exists a witness $N \in \mathbb{N}$ so that $\overline{A_m} \subseteq \bigcup_{n=0}^{N} A_{f(n)}$; this in turn implies that $\mathcal{I}(U)$ is c.e. and so U is c.e. open. Just as we define a computably closed set as one which is c.e. closed and whose complement is c.e. open, [11, Definitions 5.2.8 and 5.2.5] defines it as one which is lower-computable closed and whose complement is lower-computable open. Our choice of terminology and definitions is motivated by our desire to allow for ready extension to other classes within the arithmetical hierarchy.

These definitions naturally extend to the product topology on $\mathbb{T} \times \mathbb{T}$. For example, we say that an open set $U \subseteq \mathbb{T} \times \mathbb{T}$ is c.e. open if $\{(m, n) : \overline{A_m} \times \overline{A_n} \subseteq U\}$ is c.e. Again, this is equivalent to the existence of a c.e. set $F \subseteq \mathbb{N}^2$ so that $U = \bigcup_{(m,n) \in F} A_m \times A_n$.

Given the one-to-one correspondence between a Blaschke B product and the multiset of its zeros $\mathcal{Z}(B)$, it is natural to ask which computability conditions (if any) on $\mathcal{Z}(B)$ guarantee the computability of B, and vice versa. To begin to answer this question, we must first define a notion of computability which may be applied to such zero sets. If $A \subseteq \mathbb{C}$ is a multiset, and if $(a_n)_{n \in \mathbb{N}}$ is a sequence of complex numbers, then $(a_n)_{n \in \mathbb{N}}$ *represents* A if the elements of A are precisely the terms of $(a_n)_{n \in \mathbb{N}}$ and if the multiplicity of each $z \in A$ is the number of n so that $a_n = z$. We then say that A is *computable* if it is finite or if it has a computable representative. We can now state a theorem that characterizes the computable Blaschke products.

Theorem 2.12 (Matheson and McNicholl [9] and McNicholl [10]) *A non-constant Blaschke product B is computable if and only if $\mathcal{Z}(B)$ and Σ_B are computable.*

We note that Theorem 2.12 is uniform in both directions. That is, from an index of a computable Blaschke product B, it is possible to compute an index of a representative of $\mathcal{Z}(B)$ and an index of Σ_B. Furthermore, from an index of Σ_B and a representative of $\mathcal{Z}(B)$, it is possible to compute an index of B.

The following is a corollary of Matheson and McNicholl [9, Lemma 3.3 and Theorem 3.4].

Theorem 2.13 *If f is a computable analytic function which is not identically zero, then $\mathcal{Z}(f)$ is computable.*

3 Preliminaries

Matheson's [8] proof that any closed, nowhere dense subset S of $\mathbb{T}$ is the spectrum of some uniform Frostman Blaschke product relies on the construction of a uniform Frostman Blaschke product for each connected component of $\mathbb{T} - S$. Effectivizing this construction thus requires a notion of computable enumerability of connected components of subsets of $\mathbb{T}$. In this section, we define such a notion and show where it sits in the computability hierarchy discussed in Section 2. We start by considering the problem of deciding whether two points of $\mathbb{T}$ belong to the same connected component of a given subset.

When $C \subseteq \mathbb{T}$ and $\zeta, \zeta' \in C$, we write $\zeta \equiv_C \zeta'$ if ζ and ζ' belong to the same connected component of C; write $\zeta|_C\zeta'$ if $\zeta \not\equiv_C \zeta'$.

Definition 3.1 Suppose $C \subseteq \mathbb{T}$. C is *computably decomposable* if $\equiv_C$ and $|_C$ are c.e. open subsets of $\mathbb{T} \times \mathbb{T}$.

Thus, if $C \subseteq \mathbb{T}$ is computably decomposable, then C is c.e. open.

Our first result is that computable decomposability of c.e. open subsets of $\mathbb{T}$ can be reduced to consideration of the rational points.

Proposition 3.2 *Suppose $C \subseteq \mathbb{T}$ is c.e. open. Then, C is computably decomposable if and only if $\{(\zeta_0, \zeta_1) \in \mathbb{T}_\mathbb{Q} \times \mathbb{T}_\mathbb{Q} \ : \ \zeta_0, \zeta_1 \in C \ \wedge \ \zeta_0|_C\zeta_1\}$ is c.e.*

Proof sketch Let $m, n \in \mathbb{N}$, and suppose $A_m \cup A_n \subseteq C$. Since A_m and A_n are connected, it follows that $A_m \times A_n \subseteq |_C$ if and only if there exist $\zeta_0 \in A_m$ and $\zeta_1 \in A_n$ so that $\zeta_0 \not\equiv_C \zeta_1$. It then suffices to consider the density of the rational points. □

We construct a c.e. open $C \subseteq \mathbb{T}$ that is not computably decomposable as follows. Fix a c.e. incomputable set $A \subseteq \mathbb{N}$. Let $r_n = 2^{-(n+3)}$. Define c_n and w_n by simultaneous recursion by setting:

$$\begin{aligned} c_0 &= e^{i(3\pi/8)} \\ w_n &= c_n e^{-i\pi r_n} \\ c_{n+1} &= w_n e^{-i\pi r_{n+1}} \end{aligned}$$

Let X_n be the open arc with center c_n and radius r_n (with respect to the metric d_Λ). Thus, $w_n = \partial X_n \cap \partial X_{n+1}$. Let

$$C = \bigcup_{n\in\mathbb{N}} X_n \ \cup \ \{w_n \ : \ n \in A\}.$$

It follows that C is c.e. open. However, $c_n \not\equiv_C c_{n+1}$ if and only if $n \notin A$.

We now consider the following definition, which will provide a means for enumerating the connected components of a given subset of $\mathbb{T}$.

Definition 3.3 Suppose $C \subseteq \mathbb{T}$ and $\phi\colon\ \subseteq \mathbb{N}^2 \to \mathbb{N}$ is computable. ϕ is a *computable decomposition* of C if it satisfies the following.

(1) If $(m, n+1) \in \mathrm{dom}(\phi)$, then $(m,n) \in \mathrm{dom}(\phi)$ and $A_{\phi(m,n)} \subseteq A_{\phi(m,n+1)}$.
(2) $C = \bigcup\{A_{\phi(m,n)}\ :\ (m,n) \in \mathrm{dom}(\phi)\}$.
(3) If $(m_0,n_0),(m_1,n_1) \in \mathrm{dom}(\phi)$, and if $m_0 \neq m_1$ then $A_{\phi(m_0,n_0)} \cap A_{\phi(m_1,n_1)} = \emptyset$.

If ϕ is a computable decomposition of C, then let

$$C_{\phi,m} = \bigcup\{A_{\phi(m,n)}\ :\ (m,n) \in \mathrm{dom}(\phi)\}.$$

The following makes sense of the claim that a computable decomposition of C enumerates the computable components of C.

Proposition 3.4 *Suppose ϕ is a computable decomposition of $C \subseteq \mathbb{T}$. Then,*

$$\{C_{\phi,m}\ :\ (m,0) \in \mathrm{dom}(\phi)\}$$

consists precisely of the connected components of C.

Proof Suppose $(m,0) \in \mathrm{dom}(\phi)$. By definition, $C_{\phi,m} \subseteq C$ is connected. Suppose U is a connected component of C so that $U \supseteq C_{\phi,m}$. We claim $U - C_{\phi,m}$ is open. For, let $p \in U - C_{\phi,m}$. Then, there exists m', n so that $p \in A_{\phi(m',n)}$. Since U is a connected component, $A_{\phi(m',n)} \subseteq U$. Since $p \notin C_{\phi,m}$, $m' \neq m$. Hence, $A_{\phi(m',n)} \cap C_{\phi,m} = \emptyset$. Thus, $U - C_{\phi,m}$ is open. Since U is connected, it follows that $U = C_{\phi,m}$.

Conversely, suppose U is a connected component of C. Let $p \in U$. Then, there exists m so that $p \in C_{\phi,m}$. Hence, $(m,0) \in \mathrm{dom}(\phi)$. Since $C_{\phi,m}$ is a connected component of C, $C_{\phi,m} = U$. □

Given that the definitions of computable decomposition and computably decomposable both seem to decide a similar problem (namely, the problem of separating the connected components of a given set), it is natural to ask if they have the same computational strength. The following theorem answers this in the affirmative.

Theorem 3.5 *Suppose $\emptyset \neq C \subseteq \mathbb{T}$ is open. Then, C is computably decomposable if and only if C has a computable decomposition.*

Proof Suppose C is computably decomposable. Thus, C is c.e. open, and so there is a computable $f\colon \mathbb{N} \to \mathbb{N}$ so that $C = \{A_{f(n)} \;:\; n \in \mathbb{N}\}$. Let

$$S = \{(\zeta_0, \zeta_1) \in \mathbb{T}_{\mathbb{Q}} \times \mathbb{T}_{\mathbb{Q}} \;:\; \zeta_0, \zeta_1 \in C \;\wedge\; \zeta_0 |_C \zeta_1\}.$$

By Proposition 3.2, A is c.e.; let $(S_t)_{t\in\mathbb{N}}$ be a computable enumeration of S.

Suppose $A_m \cup A_n \subseteq C$. We say A_m, A_n are *witnessed to be included in different components of C at stage t* if $((A_m \times A_n) \cup (A_n \times A_m)) \cap S_t \neq \emptyset$. It follows that A_m and A_n are included in different connected components of C if and only if there exists t at which they are witnessed to be included in different components of C.

Now, we say that A_m and A_n are *witnessed at stage t to be included in the same component of C* if there exist $n_0, \ldots, n_k \leq t$ so that $(A_{f(n_0)}, \ldots, A_{f(n_k)})$ is a chain and $\bigcup_{s\leq k} A_{f(n_s)} \cap A_j \neq \emptyset$ for each $j \in \{m, n\}$. Again, if A_m and A_n are witnessed to be included in the same component of C at stage t, then A_m and A_n are in fact included in the same component of C. Conversely, suppose X is a connected component of C and $A_m \cup A_n \subseteq X$. Let $p \in A_m$, and let $q \in A_n$. Then, by Lemma 2.1, there exist $n_0, \ldots, n_k$ so that $(A_{f(n_0)}, \ldots, A_{f(n_k)})$ is a chain, $p \in A_{f(n_0)}$, and $q \in A_{f(n_k)}$. Thus, A_m and A_n are witnessed at t to be included in the same connected component of C when $t \geq \max\{n_0, \ldots, n_k\}$.

We define a function ψ by recursion as follows. Set $\psi(0) = 0$. If $\psi(n)$ is defined, then let $\psi(n+1)$ be the least integer k so that for some $t \in \mathbb{N}$ and each $j \leq n$, $A_{f(k)}$ and $A_{f(\psi(j))}$ are witnessed not to be included in the same component of C at t. If there is no such k, then $\psi(n+1)\uparrow$.

We are now ready to define our computable decomposition of C. For each $m \in \mathrm{dom}(\psi)$, set $\phi(m, 0) = \psi(f(m))$. Define $\phi(m, n+1)$ so that

$$A_{\phi(m,n+1)} = A_{\phi(m,n)} \cup \{A_{f(j)} \;:\; j \leq n \;\wedge\; A_{f(j)} \cap A_{\phi(m,n)} \neq \emptyset\}.$$

By construction ϕ satisfies conditions (1) and (3) of Definition 3.3. Let $\zeta \in C$, and let X denote the connected component of ζ in C. We claim there is an $m \in \mathrm{dom}(\psi)$ so that $A_{f(\psi(m))} \subseteq X$. By way of contradiction, suppose otherwise. Let k be the least number so that $A_{f(k)} \subseteq X$. Thus, $k > 0$. By definition, ψ is increasing. Let m be the largest number in $\mathrm{dom}(\psi)$ so that $\psi(m) < k$. It follows there exists $j \leq m$ so that $A_{f(k)}$ is included in the same component of C as $A_{f(\psi(j))}$, a contradiction.

We now claim $\zeta \in \bigcup_n A_{\phi(m,n)}$. Let $q \in A_{\phi(m,0)}$. Thus, by what has just been shown, $q \in X$. Then, there exists $n_0, \ldots, n_k$ so that $(A_{f(n_0)}, \ldots, A_{f(n_k)})$ is a chain from q to ζ. By construction, for each $j \leq k$, $A_{f(n_j)} \subseteq A_{\phi(m,\max\{n_0,\ldots,n_j\})}$. Hence, $\zeta \in \bigcup_n A_{\phi(m,n)}$, and so ϕ is a computable decomposition of C.

Now, for the sake of the converse, suppose $\phi\colon\ \subseteq \mathbb{N}^2 \to \mathbb{N}$ is a computable decomposition of C. Let

$$S = \{(\phi(m_1, n_1), \phi(m_2, n_2)) \ : \ (m_1, n_1), (m_2, n_2) \in \mathrm{dom}(\phi) \ \wedge \ m_1 \neq m_2\}.$$

It follows $\bigcup_{(k_1,k_2)\in S}(A_{k_1}, A_{k_2}) \subseteq \ |_C$. Suppose $p_1|_C\, p_2$. Let X_j denote the connected component of p_j in C. Then, there exists m_j so that $X_j = C_{\phi,m_j}$. Since $p_1|_C\, p_2$, $m_1 \neq m_2$. Thus, $(p_1, p_2) \in \bigcup\{(A_m, A_n) \ : \ (m, n) \in S\}$. □

Given the effective nature of the decomposition in the previous theorem, it is desirable to determine the complexity of such computably decomposable sets. Since Matheson's construction of a uniform Frostman Blaschke product with a desired spectrum relied heavily on the decomposition of the complement of the spectrum into connected components, we are particularly interested in classifying closed sets $S \subseteq \mathbb{T}$ whose complements are computably decomposable.

Theorem 3.6 *If $S \subseteq \mathbb{T}$ is computably closed, then $\mathbb{T} - S$ is computably decomposable.*

Proof Let $C = \mathbb{T} - S$, and let $G = C \cap \mathbb{T}_{\mathbb{Q}}$. Since C is c.e. open, G is c.e. Set

$$V = \{(\zeta_0, \zeta_1) \in G \times G \ : \ \exists n_0, n_1 \in \mathbb{N}\, A_{n_0} \cap S \neq \emptyset \ \wedge \ A_{n_1} \cap S \neq \emptyset \ \wedge$$
$$A_{n_0}, A_{n_1} \text{ are included in different components of } \mathbb{T} - \{\zeta_0, \zeta_1\}\}.$$

Thus, since S is c.e. closed, V is computably enumerable. However,

$$V = \{(\zeta_0, \zeta_1) \in G \times G \ : \ \zeta_0 \not\equiv_C \zeta_1\}.$$

Thus, by Proposition 3.2, C is computably decomposable. □

One of Matheson's main results is that, if $S \subseteq \mathbb{T}$ is closed, then it is the spectrum of a uniform Frostman Blaschke product if and only if it is nowhere dense. Thus, it is natural to analyze the complexity of such S whose complement is computably decomposable.

Theorem 3.7 *If $S \subseteq \mathbb{T}$ is closed and nowhere dense, and if $\mathbb{T} - S$ is computably decomposable, then S is computably closed.*

Proof Suppose $S \subseteq \mathbb{T}$ is closed and nowhere dense, and suppose $\mathbb{T} - S$ is computably decomposable. Let $C = \mathbb{T} - S$. It follows that C is c.e. open. Thus, we may assume S is infinite since otherwise every point of S is computable.

We make use of the following principle: if $p, q \in C$ are distinct, then $p|_C\, q$ if and only if each connected component of $\mathbb{T} - \{p, q\}$ contains a point of S.

Since $|_C$ is c.e. open, it follows that the set of all pairs of rational points of $\mathbb{T}$ that belong to distinct components of C is computably enumerable. Let $k \in V$ if and only if there exist rational points in A_k that belong to distinct components of C. Thus, V is computably enumerable. If $k \in V$, then $A_k \cap S \neq \emptyset$. Conversely, suppose $p \in A_k \cap S$. There exists $\eta \in (0, 1/2)$ so that $B := \{\zeta \in \mathbb{T} \ : \ d_\Lambda(p, \zeta) < \eta\} \subseteq A_k$. Since $\eta < 1/2$, $B - \{p\}$ has 2 connected components; label these components B_1 and B_2. Since S is nowhere dense, $B_j \cap C \neq \emptyset$. Since $B_j \cap C$ is open, and since the rational points of $\mathbb{T}$ are dense in $\mathbb{T}$, it follows that there is a rational point $\zeta_j \in B_j \cap C$. Because S is infinite, we may choose η small enough so that $(\mathbb{T} - \overline{B}) \cap S \neq \emptyset$ (otherwise $S = \{p\}$). Thus, both components of $\mathbb{T} - \{\zeta_1, \zeta_2\}$ contain a point of S. Therefore, $\zeta_1 |_C \zeta_2$. Hence, $k \in V$. □

Finally, in order to effectivize Matheson's construction, we need a means to compute the endpoints of the connected components of $\mathbb{T} - S$, since these (along with their limit points) will comprise the spectrum S.

Proposition 3.8 *Suppose $S \subseteq \mathbb{T}$ is closed and nowhere dense, and suppose ϕ is a computable decomposition of $\mathbb{T} - S$. Then, from $m \in \mathbb{N}$ so that $(m, 0) \in \mathrm{dom}(\phi)$, it is possible to compute the length and the endpoints of $C_{\phi,m}$.*

Proof Suppose $(m, 0) \in \mathrm{dom}(\phi)$. Clearly, if the length of $C_{\phi,m}$ can be computed, then so can its endpoints.

Let ϕ_s be the stage s approximation of ϕ. For each $s \in \mathbb{N}$, set:

$$\begin{aligned} U_s &= \bigcup \{A_{\phi(m',n)} \ : \ (m', n) \in \mathrm{dom}(\phi_s)\} \\ C_{m,s} &= \bigcup \{A_{\phi(m,n)} \ : \ (m, n) \in \mathrm{dom}(\phi_s)\} \end{aligned}$$

Given $k \in \mathbb{N}$, wait for s so that for each endpoint p of $C_{m,s}$, the normalized length of the connected component of p in $\partial\mathbb{D} - U_s$ is smaller than 2^{-k+1}. The existence of s follows from the meagerness of S. It follows that $\Lambda(C_{\phi,m}) \in [\Lambda(C_{m,s}), \Lambda(C_{m,s}) + 2^{-k}]$. □

4 Complexity of spectra

We begin with an upper bound on the complexity of spectra of computable functions in $H^\infty(\mathbb{D})$.

Proposition 4.1 *If $f \in H^\infty(\mathbb{D})$ is computable, then* spec(f) *is* Σ_3^0*– closed.*

Proof We note that $A_n \cap \text{spec}(f) \neq \emptyset$ if and only if A_n has a closed rational subarc whose subtended wedge contains infinitely many zeros of f (and thus a point of spec(f)). It follows from Theorem 2.13 that $\mathcal{Z}(f) - \{0\}$ is computable. Without loss of generality, we assume $\mathcal{Z}(f)$ is infinite. Let $(a_n)_{n\in\mathbb{N}}$ be a computable representative of $\mathcal{Z}(f) - \{0\}$. Thus,

$$n \in \mathcal{I}(\text{spec}(f)) \iff (\exists m \in \mathbb{N})(\exists^\infty j \in \mathbb{N})\, \overline{A_m} \subset A_n \wedge \frac{a_j}{|a_j|} \in A_m.$$

The latter is clearly a Σ_3^0 statement. □

We note that so long as $\Sigma_f < \infty$, spec(f) depends only on the arguments of the points in $\mathcal{Z}(f)$. That is, since $\Sigma_f < \infty$ implies that the zeros accumulate to the boundary, the spectrum is determined only by the arguments of the zeros. Thus, when considering the spectra of computable bounded analytic functions, we may restrict our attention to the sequence of arguments of the zeros.

Lemma 4.2 *Let $S \subseteq \mathbb{T}$. The following are equivalent.*

(1) *S is the spectrum of a computable $f \in H^\infty(\mathbb{D})$.*
(2) *$S = \emptyset$ or there is a computable sequence Θ of unimodular points so that $S = \text{Lim}\,\Theta$.*

Moreover, given a computable sequence Θ of unimodular points and any computable $\epsilon > 0$, there exists a computable $f \in H^\infty(\mathbb{D})$ with $\text{Lim}\,\Theta = \text{spec}(f)$ *and* $\Sigma_f = \epsilon$.

Proof On the one hand, suppose S is the spectrum of a computable $f \in H^\infty(\mathbb{D})$. Without loss of generality, suppose $S \neq \emptyset$. Thus, $\mathcal{Z}(f)$ is infinite. By Theorem 2.13, $\mathcal{Z}(f)$ has a computable representative. Since $\mathcal{Z}(f)$ is infinite, it then follows that $\mathcal{Z}(f) - \{0\}$ has a computable representative $(z_n)_{n\in\mathbb{N}}$. Set $\theta_n = z_n/|z_n|$, and let $\Theta = (\theta_n)_{n\in\mathbb{N}}$. It follows that $\text{Lim}\,\Theta = S$.

Conversely, suppose (2) holds. If $S = \emptyset$, we may take f to be the constant function 1. Suppose $S \neq \emptyset$, and let $\theta = (\theta_n)_{n\in\mathbb{N}}$ be a computable sequence of unimodular points so that $S = \text{Lim}\,\theta$.

Suppose $\epsilon > 0$ is computable. Set $a_n = (1 - 2^{-(n+1)}\epsilon)\theta_n$. Hence, $\sum_n (1 - a_n) = \epsilon$. Set $f = \prod_n b_{a_n}$. Hence, by Theorem 2.3, $f \in H^\infty(\mathbb{D})$, and by Theorem 2.12, f is computable. By construction, spec(f) $= S$. □

In light of Proposition 4.1, it is natural to ask whether this upper bound on the complexity of spec(f) is optimal. On the one hand, the following theorem indicates that it is.

Theorem 4.3 *For every computable $\epsilon > 0$, there is a computable $f \in H^\infty(\mathbb{D})$ so that $\mathcal{I}(\mathrm{spec}(f))$ is Σ^0_3–complete and so that $\Sigma_f = \epsilon$.*

Proof By Lemma 4.2, it suffices to show there is a computable sequence $\Gamma = (\gamma_j)_{j\in\mathbb{N}}$ of unimodular points so that $\mathrm{Cof} \leq_m \mathcal{I}(\mathrm{Lim}\,\Gamma)$. Fix a computable sequence $(F_m)_{m\in\mathbb{N}}$ of rational open arcs that omit 1 and so that $(\overline{F_m})_{m\in\mathbb{N}}$ is pairwise disjoint. For each $m \in \mathbb{N}$, let

$$P_m : F_m \cap \mathrm{Lim}\,\Gamma \neq \emptyset \iff m \in \mathrm{Cof}\,.$$

It suffices to construct Γ so that P_m holds for each $m \in \mathbb{N}$.

For each $m \in \mathbb{N}$, select a sequence $(\theta^{(m)}_n)_{n\in\mathbb{N}}$ of distinct rational points in F_m so that $\lim_n \theta^{(m)}_n \in \partial F_m$. We choose these points so that $\theta^{(m)}_n$ is computable from m and n. Let $S = \{\theta^{(m)}_n \ : \ m, n \in \mathbb{N}\}$. We also choose these points so as to ensure S is discrete.

For each $m, n \in \mathbb{N}$, we ensure the following requirements:

$$P_{m,n} : \theta^{(m)}_n \in \mathrm{Lim}\,\Gamma \iff \mathbb{N} - W_m \subseteq \{0, \dots, n\}$$

$$N_m : \mathrm{Lim}\,\Gamma \cap F_m \subseteq \{\theta^{(m)}_n \ : \ n \in \mathbb{N}\}$$

For each $m, n, s \in \mathbb{N}$, let

$$\ell(m, n, s) = \max\{x \leq s \ : \ \forall n \leq y < x\; y \in W_{m,s}\}.$$

Thus, $\ell(m, n, s) \leq \ell(m, n, s + 1)$. Furthermore, $\lim_s \ell(m, n, s) = \infty$ if and only if $\mathbb{N} - W_m \subseteq \{0, \dots, n\}$. We say $P_{m,n}$ *needs attention at stage* s if $s \in \mathbb{N}^{[\langle m,n\rangle]}$ and if

$$\ell(m, n, s) > \max(\{-\infty\} \cup \{\ell(m, n, r) \ : \ r < s \wedge r \in \mathbb{N}^{\langle m,n\rangle}\}).$$

Thus, at each stage s, at most one requirement needs attention.

If no requirement needs attention at stage s, then set $\gamma_s = 1$. If $P_{m,n}$ needs attention at s, then set $\gamma_s = \theta^{(m)}_n$.

We now show $P_{m,n}$ is satisfied. On the one hand, suppose $\theta^{(m)}_n \in \mathrm{Lim}\,\Gamma$. Then, there is an increasing sequence $(s_k)_{k\in\mathbb{N}}$ of natural numbers so that $\theta^{(m)}_n = \lim_k \gamma_{s_k}$. Since $\theta^{(m)}_n \neq 1$, $\gamma_{s_k} \neq 1$ for all sufficiently large k. Hence, by construction, $\gamma_{s_k} \in S$ for all sufficiently large k. Thus, since S is discrete, $\gamma_{s_k} = \theta^{(m)}_n$ for all sufficiently large k. By construction, $P_{m,n}$ needs attention at s_k for all sufficiently large k, and so $\mathbb{N} - W_m \subseteq \{0, \dots, n\}$.

Conversely, suppose $\mathbb{N} - W_m \subseteq \{0, \ldots, n\}$. Then, $\lim_s \ell(m,n,s) = \infty$. Since $(\ell(m,n,s))_{s\in\mathbb{N}}$ is non-decreasing, it follows that $P_{m,n}$ requires attention at infinitely many s. By construction, $\theta_n^{(m)} = \gamma_s$ for infinitely many s. Hence, $\theta_n^{(m)} \in \operatorname{Lim}\Gamma$.

We now show N_m is satisfied for each m. Suppose $\zeta \in F_m \cap \operatorname{Lim}\Gamma$. There is an increasing sequence $(s_k)_{k\in\mathbb{N}}$ of natural numbers so that $\lim_k \gamma_{s_k} = \zeta$. Since $(\overline{F_r})_{r\in\mathbb{N}}$ is pairwise disjoint, $\gamma_{s_k} \in F_m$ for all sufficiently large k. Hence, by construction, $\gamma_{s_k} \in F_m \cap S$ for all sufficiently large k. Since $\lim_n \theta_n^{(m)} \in \partial F_m$, and since S is discrete, there exists n so that $\gamma_{s_k} = \theta_n^{(m)}$ for all sufficiently large k. Therefore, $\zeta = \theta_n^{(m)}$.

Finally, we show R_m is satisfied. Suppose $m \in \mathrm{Cof}$. Then, there exists n so that $\mathbb{N} - W_m \subseteq \{0, \ldots, n\}$. Hence, by $P_{m,n}$, $\theta_n^{(m)} \in F_m \cap \operatorname{Lim}\Gamma$. Conversely, suppose $F_m \cap \operatorname{Lim}\Gamma \neq \emptyset$. By N_m, $\theta_n^{(m)} \in F_m \cap \operatorname{Lim}\Gamma$ for some n. Therefore, by $P_{m,n}$, $\mathbb{N} - W_m \subseteq \{0, \ldots, n\}$. □

In contrast with Theorem 4.3, the following theorem suggests that Proposition 4.1 is far from optimal.

Theorem 4.4 *There exists a Σ_2^0–closed set $S \subseteq \mathbb{T}$ which is not the spectrum of any computable function in $H^\infty(\mathbb{D})$.*

To prove Theorem 4.4, we need the following lemma.

Lemma 4.5 *Let A be an open rational arc. Suppose ρ_n is a Cauchy name of θ_n for each $n \in \mathbb{N}$. If $A \cap \operatorname{Lim}(\theta_n)_{n\in\mathbb{N}} \neq \emptyset$, then $A(\rho_{n,n}; 2^{-n}) \subseteq A$ for infinitely many n.*

Proof Set $\Theta = (\theta_n)_{n\in\mathbb{N}}$. Suppose $z \in A \cap \operatorname{Lim}\Theta$. Choose $k \in \mathbb{N}$ so that $A(z; 2^{-k}) \subseteq A$. Let $N \in \mathbb{N}$, and set $N' = \max\{N, k+1\}$. There exists $n \in \mathbb{N}$ so that $\theta_n \in A(z; 2^{-(k+1)})$ and so that $n > N'$. Suppose $\zeta \in A(\rho_{n,n}; 2^{-n})$. Then, $d_\Lambda(\rho_{n,n}, \theta_n) \leq 2^{-n} < 2^{-(k+1)}$. Hence, $d_\Lambda(\zeta, z) < 2^{-k}$. □

Proof of Theorem 4.4 By Lemma 4.2, it suffices to construct a Σ_2^0–closed set $S \subseteq \mathbb{T}$ that is not the limit set of any computable sequence of unimodular points.

Let $(\theta^{(e)})_{e\in\mathbb{N}}$ be an effective enumeration of the computable partial sequences of points in $\mathbb{T}$. That is:

(1) $\theta^{(e)}$ is a computable partial function from $\mathbb{N}^2$ into $\mathbb{T}_\mathbb{Q}$.
(2) If $(m, n+1) \in \operatorname{dom}(\theta^{(e)})$, then $(m,n) \in \operatorname{dom}(\theta^{(e)})$ and $d_\Lambda(\theta^{(e)}_{m,n}, \theta^{(e)}_{m,n+1}) < 2^{-(n+1)}$.

If $\theta^{(e)}_{m,n}\downarrow$ for all $n \in \mathbb{N}$, then we let $\overline{\theta}^{(e)}_m = \lim_n \theta^{(e)}_{m,n}$. Thus, if $\Theta = (\theta_n)_{n\in\mathbb{N}}$ is a sequence of unimodular points, then Θ is computable if and only if $\Theta = \overline{\theta}^{(e)}$ for some $e \in \mathbb{N}$.

Let $(T^{(e)})_{e\in\mathbb{N}}$ be an effective enumeration of rational open arcs in $\mathbb{T}$ such that, if $e_1 \neq e_2$, then $d_\Lambda(\overline{T^{(e_1)}}, \overline{T^{(e_2)}}) \geq 2^{-e_1-3}$. For each $e \in \mathbb{N}$, let $(F^{(e)}_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint rational open sub-arcs of $T^{(e)}$ which accumulate to an endpoint of $T^{(e)}$ and are uniformly computable from e and k. Let $c_{e,k}$ denote the center of $F^{(e)}_k$.

For each $k, e \in \mathbb{N}$, let

$$J^{(e)}_{k,t} = \{n > t : \theta^{(e)}_n(n)\downarrow \wedge A(\theta^{(e)}_{n,n}; 2^{-n}) \subseteq F^{(e)}_k\}.$$

Further, let $S_0 = \{c_{e,k} : \exists t\, J^{(e)}_{k,t} = \emptyset\}$, and let $S = \overline{S_0}$.

Suppose $e \in \mathbb{N}$ is such that $\theta^{(e)}$ is total. As a first case, further suppose there exist $k, t \in \mathbb{N}$ such that $J^{(e)}_{k,t} = \emptyset$. Then $c_{e,k} \in S_0$, so $F^{(e)}_k \cap S \neq \emptyset$. Furthermore, by Lemma 4.5, $A \cap \operatorname{Lim}\theta^{(e)} \neq \emptyset$.

On the other hand, suppose $J^{(e)}_{k,t} \neq \emptyset$ for all k, t. Define $(n_k)_{k\in\mathbb{N}}$ by setting:

$$n_0 = \min J^{(e)}_{0,0}$$
$$n_{k+1} = \min J^{(e)}_{k+1,n_k}$$

We note that, for all k, $n_k < n_{k+1}$ and $\overline{\theta}^{(e)}_{n_k} \subseteq F^{(e)}_k$. Thus, $(\theta^{(e)}_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $\theta^{(e)}$ which converges to some $\zeta \in \partial T^{(e)}$. Furthermore, $\zeta \in \partial T^{(e)} \cap \operatorname{Lim}\theta^{(e)}$. Let $s \in S_0$. Then, $s = c_{e',k}$ for some e', k. We note that for all $k \in \mathbb{N}$, $c_{e,k} \notin S_0$, and so $e' \neq e$. Hence, $d_\Lambda(\partial T^{(e)}, s) \geq 2^{-(e+3)}$. Hence, $\partial T^{(e)} \cap S = \emptyset$.

Fix $n \in \mathbb{N}$. Since A_n is open, we have that

$$\begin{aligned} A_n \cap S \neq \emptyset &\iff A_n \cap S_0 \neq \emptyset \\ &\iff \exists e, k\; c_{e,k} \in A_n \cap S_0 \\ &\iff \exists e, k, t\; c_{e,k} \in A_n \wedge J^{(e)}_{k,t} = \emptyset \\ &\iff \exists e, k, t\, \forall n > t\; c_{e,k} \in A_n \wedge \neg(\theta^{(e)}_{n,n}\downarrow \wedge A(\theta^{(e)}_{n,n}; 2^{-n}) \subseteq F^{(e)}_k). \end{aligned}$$

The statement $\theta^{(e)}_{n,n}\downarrow \wedge A(\theta^{(e)}_{n,n}; 2^{-n}$ is Σ^0_1. Thus, S is Σ^0_2–closed. □

Unlike with Σ^0_2 sets, the situation for Π^0_2 sets is more positive. Namely, we show the following.

Theorem 4.6 *If $S \subseteq \mathbb{T}$ is Π^0_2–closed, and if $\epsilon > 0$ is computable, then there is a computable $f \in H^\infty(\mathbb{D})$ so that* $\operatorname{spec}(f) = S$ *and so that $\Sigma_f = \epsilon$.*

Proof In light of Lemma 4.2, it suffices to show there is a computable sequence θ of unimodular points whose limit set is S. We construct such a sequence as follows.

Let $(C_{m,n})_{m\in\mathbb{N},n<K_m}$ be a computable array of open rational arcs with the following properties:

(1) $K_m < \omega$, and $m \mapsto K_m$ is computable.
(2) $\mathbb{T} \subseteq \bigcup_{n<K_m} C_{m,n}$.
(3) If $n < K_m$, then $\operatorname{diam} C_{m,n} < 2^{-m}$.
(4) The covering $\{C_{m+1,n} \ : \ n < K_{m+1}\}$ refines $\{C_{m,n} \ : \ n < K_m\}$.

Since S is Π^0_2–closed, there exists a uniformly computable sequence $(D^s)_{s\in\mathbb{N}}$ of finite subsets of $\mathbb{N}^2$ such that

$$C_{m,n} \cap S \neq \emptyset \iff (m,n) \in \bigcap_{t=0}^{\infty} \bigcup_{s=t}^{\infty} D^s$$

(see eg Cooper [5, Exercise 16.2.15]).

For all $m,n \in \mathbb{N}$ with $n < K_m$, let

$$f(m,n) = \{(j,k) : j \leq m \wedge k < K_j \wedge C_{m,n} \subseteq C_{j,k}\}.$$

Then, let

$$T = \left\{ (m,n) : (\exists s)\, f(m,n) \subseteq \bigcap_{t=0}^{m} \bigcup_{r=t}^{s} D^r \right\}.$$

It follows that $(m,n) \in T$ whenever $C_{m,n} \cap S \neq \emptyset$. Thus, T is infinite, and by definition T is c.e. Accordingly, let $(J_n)_{n\in\mathbb{N}}$ be an effective one-to-one enumeration of T. Denote the center of each $C_{m,n}$ by $c_{m,n}$, and let $\theta = (c_{J_n})_{n\in\mathbb{N}}$.

We claim that $\operatorname{Lim}\theta = S$. On the one hand, let $z \in S$, and let $\delta > 0$. Let $m \in \mathbb{N}$ so that $2^{-m} < \delta$. Then there exists $n < K_m$ so that $z \in C_{m,n}$. Furthermore, for all $(j,k) \in f(m,n)$, we have that $z \in C_{m,n} \subseteq C_{j,k}$. In other words, $(m,n) \in T$ and so $c_{m,n}$ is a term of θ. We note that, since $z, c_{m,n} \in C_{m,n}$, we have that $d_\Lambda(z, c_{m,n}) < 2^{-m}$. Hence, there are infinitely many k so that $d_\Lambda(z, \theta_k)) < \delta$. That is, $z \in \operatorname{Lim}\theta$.

Conversely, suppose $z \in \operatorname{Lim}\theta$. Let $\delta > 0$, and choose $a,b \in \mathbb{N}$ so that $2^{-a} < \delta$ and $z \in C_{a,b}$. Since $z \in \operatorname{Lim}\theta$, there is an increasing $(k_j)_{j\in\mathbb{N}}$ so that $z = \lim_j \theta_{k_j}$. Since $C_{a,b}$ is open, for almost all $j \in \mathbb{N}$, $C_{J_{k_j}} \subseteq C_{a,b}$. Let $J_{k_j} = (m_j, n_j)$, and for each $j \in \mathbb{N}$ choose $s_j \in \mathbb{N}$ so that

$$f(m_j, n_j) \subseteq \bigcap_{t=0}^{m_j} \bigcup_{r=t}^{s_j} D^r.$$

When j is sufficiently large, $a \leq m_j$, and so $(a,b) \in f(m_j, n_j)$. It follows that $C_{a,b} \cap S \neq \emptyset$. Since S is closed and δ is arbitrary, we conclude $z \in S$. □

5 Proof of Theorem 1.1

Theorem 1.1 is a corollary of the following.

Theorem 5.1 *Suppose p is an endpoint of a c.e. open arc $I \subseteq \mathbb{T}$ so that $\Lambda(I) < 1/2$. Let $q \in I$ be computable. Then, for every $t \in \mathbb{N}$, there is a computable Blaschke product B so that $\mathrm{spec}(B) = \{p\}$ and so that for all $\zeta \in \mathbb{T}$,*

$$\sigma_B(\zeta) < \begin{cases} 1 + 2^{-t} & \zeta \in I \cap (\mathbb{T} - \{p, q\}) \\ 2^{-t} & \text{otherwise.} \end{cases}$$

The remainder of this section consists of the proof of Theorem 5.1. We begin with the construction of B. Let $J = I \cap (\partial D - \{p, q\})$, and let $p' \in \partial I - \{p\}$. Choose a rational point $\zeta_0 \in I \cap (\mathbb{T} - \{p, q\})$. Let $\alpha = \frac{2}{3}\pi d_\Lambda(q, \zeta_0)$.

Fix a computable branch arg of the argument so that $\bar{I} \subseteq \mathrm{dom}(\arg)$. Since I is c.e. open, $\arg(p)$ is either left-c.e. or right-c.e.; without loss of generality, suppose $\arg(p)$ is right-c.e. Then, there is a computable sequence $(p_n)_{n\in\mathbb{N}}$ of rational points in I so that $p_0 = \zeta_0$, $\arg(p_{n+1}) < \arg(p_n)$, and $\lim_n p_n = p$. Set:

$$\begin{aligned}
k &= \frac{2^t}{48} \\
r_n &= 1 - \frac{k2^{-t}\alpha}{4^n} \\
\theta_n &= \alpha 2^{-n} + \arg(p_n) \\
a_n &= r_n e^{i\theta_n} \\
\xi_n &= e^{i\theta_n} \\
A &= (a_n)_{n\in\mathbb{N}}
\end{aligned}$$

Thus, Σ_A is computable, and so B_A is computable. Let $B = B_A$. Thus, $\mathrm{spec}(B) = \{p\}$.

We divide the remainder of the proof into a sequence of lemmas.

Lemma 5.2 *If $\zeta \in J$, then $\sigma_B(\zeta) < 1 + 2^{-t}$.*

Proof Suppose $\zeta \in J$. Let $n_0 \in \mathbb{N}$ so that

$$\pi d_\Lambda(\zeta, \xi_{n_0}) = \min\{\pi d_\Lambda(\zeta, \xi_j) \ : \ j \in \mathbb{N}\}.$$

Let τ_n be the midpoint of $(\mathbb{T} - \{\xi_n, \xi_{n+1}\}) \cap I$.

Let $n \in \mathbb{N} - \{n_0\}$. We claim $\pi d_\Lambda(\zeta, \xi_n) \geq \alpha 2^{-(n+2)}$. We first consider the case $n < n_0$. Thus,

$$(\mathbb{T} - \{\zeta, \xi_n\}) \cap J \supseteq (\mathbb{T} - \{\tau_{n_0-1}, \xi_n\}) \supseteq (\mathbb{T} - \{\tau_n, \xi_{n+1}\}) \cap J.$$

Hence,

$$\begin{aligned} \pi d_\Lambda(\zeta, \xi_n) &\geq \frac{1}{2}(\pi d_\Lambda(\xi_n, \xi_{n+1})) \\ &\geq \alpha 2^{-(n+2)}. \end{aligned}$$

Now, suppose $n_0 < n$. Then,

$$(\mathbb{T} - \{\zeta, \xi_n\}) \supseteq (\mathbb{T} - \{\tau_{n-1}, \xi_n\}).$$

Hence,

$$\begin{aligned} \pi d_\Lambda(\zeta, \xi_n) &\geq \frac{1}{2}\pi d_\Lambda(\xi_{n-1}, \xi_n) \\ &\geq \frac{1}{2}\alpha 2^{-(n-1)} \\ &> \alpha 2^{-(n+2)}. \end{aligned}$$

Thus, by Lemma 2.2,

$$\begin{aligned} \sigma_B(\zeta) &= \frac{1 - r_{n_0}}{|\zeta - a_{n_0}|} + \sum_{n \neq n_0} \frac{1 - r_n}{|\zeta - a_n|} \\ &\leq 1 + \sum_{n=0}^{\infty} \frac{k 2^{-t}\alpha}{4^n} \frac{1}{3^{-1}\alpha 2^{-(n+2)}} \\ &= 1 + 3k2^{-t} \sum_{n=0}^{\infty} \frac{2^{n+2}}{4^n} \\ &= 1 + 24k2^{-t} \\ &< 1 + 2^{-t}. \end{aligned}$$

□

Lemma 5.3 *If $\zeta \notin J$, then $\sigma_B(\zeta) < 2^{-t}$.*

Proof We first claim $\pi d_\Lambda(\zeta, \xi_n) \geq \alpha 2^{-(n+2)}$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Let $T \subseteq \mathbb{T}$ be the shortest arc from ζ to ξ_n.

We first consider the case where ζ and q belong to the same component of $\mathbb{T} - \{\xi_n, -\xi_n\}$. It follows that $T \supseteq (\mathbb{T} - \{q, \xi_0\}) \cap J$. Hence,

$$\pi d_\Lambda(\zeta, \xi_n) \geq \pi d_\Lambda(q, \xi_0) = \frac{1}{2}\alpha \geq \alpha 2^{-(n+2)}.$$

Now, suppose ζ does not belong to the connected component of q in $\mathbb{T} - \{\xi_n, -\xi_n\}$. Then, $T \supseteq (\mathbb{T} - \{\xi_n, \xi_{n+1}\}) \cap J$. Hence,

$$\begin{aligned} \pi d_\Lambda(\zeta, \xi_n) &\geq \pi d_\Lambda(\xi_n, \xi_{n+1}) \\ &\geq \alpha 2^{-(n+1)} > \alpha 2^{-(n+2)}. \end{aligned}$$

As in the proof of Lemma 5.2, it follows that

$$\sigma_B(\zeta) = \sum_{n=0}^{\infty} \frac{1 - r_n}{|\zeta - a_n|} < 2^{-t}. \qquad \square$$

6 Proof of Theorem 1.2

Suppose $S \subseteq \mathbb{T}$ is computably closed and nowhere dense. Let $C = \mathbb{T} - S$. By Theorem 3.5, C is computably decomposable. Let ϕ be a computable decomposition of C, and let

$$U = \{m \in \mathbb{N} \; : \; (m, 0) \in \text{dom}(\phi)\}.$$

For each $m \in U$, let $J_m = \bigcup_{(m,n) \in \text{dom}(\phi)} A_{\phi(m,n)}$. By Proposition 3.8, the endpoints of J_m are computable uniformly in m. Let p_m, q_m denote the endpoints of J_m in counterclockwise order. We note that the q_m's are dense in S. For each $m \in U$, we compute a rational arc $I_m \subseteq J_m$ so that $\Lambda(I_m) < 1/2$ and so that $q_m \in \partial I_m$. We then compute, for each $m \in \mathbb{N}$, a rational $r_m \in I_m$.

Fix $t_0 \in \mathbb{N}$ so that $\sum_{j \in \mathbb{N}} 2^{-t_0 j} < 2^{-k}$. Let $(U_s)_{s \in \mathbb{N}}$ be a computable enumeration of U so that $\#(U_{s+1} - U_s) \leq 1$ for all $s \in \mathbb{N}$. For all $m \in U$, let $s(m)$ be the least number s so that $m \in U_s$.

By Theorem 5.1, uniformly in $m \in U$, we can compute a Blaschke product $\hat{B}_m$ so that $\text{spec}(\hat{B}_m) = \{q_m\}$ and so that

$$\sigma_{\hat{B}_m}(\zeta) < \begin{cases} 1 + 2^{-t_0 s(m)} & \zeta \in J_m \cap (\mathbb{T} - \{q_m, r_m\}) \\ 2^{-t_0 s(m)} & \text{otherwise.} \end{cases}$$

For each $m \in U$, we compute a representative $(\hat{a}_{m,n})_{n \in \mathbb{N}}$ of $\mathcal{Z}(\hat{B}_m)$ uniformly in m. We then compute, for each $m \in U$, an $N_m \in \mathbb{N}$ so that $\sum_{n \geq N_m} (1 - \hat{a}_{m,n}) < 2^{-s(m)}$. Let $a_{m,n} = \hat{a}_{m,n+N_m}$, and set $B_m = \prod_n b_{a_{m,n}}$. Thus, $\text{spec}(B_m) = \{q_m\}$ and $\sigma_{B_m} \leq \sigma_{\hat{B}_m}$. Set $B = \prod_{m \in U} B_m$.

It now follows from Theorem 2.12 that B is computable. By construction, $\text{spec}(B) = S$. Let $\zeta \in \mathbb{T}$. If $\zeta \in I_m$ for some m, then $\sigma_B(\zeta) < 1 + 2^{-k}$. Otherwise, $\sigma_B(\zeta) < 2^{-k}$.

7 Conclusion

The Blaschke product provides a convenient, constructible means to analyze the spectrum of any bounded analytic function on the unit disc solely by considering its zero sequence. Results of Matheson and McNicholl [9] and McNicholl [10] regarding computable Blaschke products were used to consider the set of limit points of the zero sequence of bounded analytic functions.

The problem of characterizing these spectra proved more interesting than originally anticipated and led to interesting results: namely, that all such spectra are Σ^0_3–closed, that there exists a Σ^0_3–complete spectrum, and that not all Σ^0_2–closed sets are spectra but all Π^0_2–closed sets are.

Of particular note is that many of the results in this paper extend well beyond zero sequences of bounded analytic functions. That is, the dependence of the spectrum on only a sequence of unimodular points may be generalized to sets of limit points of computable sequences of points in one-dimensional spaces other than just $\mathbb{T}$.

With regard to uniform Frostman Blaschke products, the definition of a computable decomposition of a subset of $\mathbb{T}$ and its equivalence to computable closure of the complement of a nowhere dense set allowed for a straightforward effectivization of Matheson's [8] construction of a uniform Frostman Blaschke product with a desired nowhere dense spectrum.

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References

[1] **V V Andreev**, **T H McNicholl**, *Computable complex analysis*, from: "Handbook of computability and complexity in analysis", Theory Appl. Comput., Springer, Cham (2021) 101–140; https://doi.org/10.1007/978-3-030-59234-9_4

[2] **V Brattka**, **P Hertling** (editors), *Handbook of computability and complexity in analysis*, Theory and Applications of Computability, Springer, Cham (2021); https://doi.org/10.1007/978-3-030-59234-9

[3] **M Braverman**, **M Yampolsky**, *Constructing locally connected non-computable Julia sets*, Comm. Math. Phys. 291 (2009) 513–532; https://doi.org/10.1007/s00220-009-0858-5

[4] **J A Cima**, **A L Matheson**, **W T Ross**, *The Cauchy transform*, volume 125 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI (2006); https://doi.org/10.1090/surv/125

[5] **S B Cooper**, *Computability theory*, Chapman & Hall/CRC, Boca Raton, FL (2004); https://doi.org/10.1201/9781315275789

[6] **A Dudko**, **M Yampolsky**, *On computational complexity of Cremer Julia sets*, Fund. Math. 252 (2021) 343–353; https://doi.org/10.4064/fm829-12-2019

[7] **J G Hocking**, **G S Young**, *Topology*, second edition, Dover Publications Inc., New York (1988)

[8] **A Matheson**, *Boundary spectra of uniform Frostman Blaschke products*, Proc. Amer. Math. Soc. 135 (2007) 1335–1341; https://doi.org/10.1090/S0002-9939-06-08470-X

[9] **A Matheson**, **T H McNicholl**, *Computable analysis and Blaschke products*, Proc. Amer. Math. Soc. 136 (2008) 321–332; https://doi.org/10.1090/S0002-9939-07-09102-2

[10] **T H McNicholl**, *Uniformly computable aspects of inner functions: estimation and factorization*, MLQ Math. Log. Q. 54 (2008) 508–518; https://doi.org/10.1002/malq.200710061

[11] **C Rojas**, **M Yampolsky**, *Computable geometric complex analysis and complex dynamics*, from: "Handbook of computability and complexity in analysis", Theory Appl. Comput., Springer, Cham (2021) 143–172; https://doi.org/10.1007/978-3-030-59234-9_5

[12] **C Rojas**, **M Yampolsky**, *Real quadratic Julia sets can have arbitrarily high complexity*, Found. Comput. Math. 21 (2021) 59–69; https://doi.org/10.1007/s10208-020-09457-w

[13] **W Rudin**, *Real and complex analysis*, third edition, McGraw-Hill Book Co., New York (1987)

[14] **K Weihrauch**, *Computable analysis*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin (2000)An introduction; https://doi.org/10.1007/978-3-642-56999-9

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