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Hyperreal differentiation with an idempotent ultrafilter

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Abstract: In the hyperreals constructed using a free ultrafilter on $\mathbb{R}$, where $[f]$ is the hyperreal represented by $f : \mathbb{R} \to \mathbb{R}$, it is tempting to define a derivative operator by $[f]' = [f']$, but unfortunately this is not generally well-defined. We show that if the ultrafilter in question is idempotent and contains $(0, \epsilon)$ for arbitrarily small real ϵ then the desired derivative operator is well-defined for all f such that $[f']$ exists. We also introduce a hyperreal variation of the derivative from finite calculus, and show that it has surprising relationships to the standard derivative. We give an alternate proof, and strengthened version, of Hindman's Theorem.

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1 Introduction

There is a long tradition (Shelly [13], Barbeau [2], Ufnarovski and Åhlander [16], Buium [4, 5, 6], Stay [14], Kovic [11], Pasten [12]) of attempting to differentiate numbers in various ways. Much attention was focused on derivatives of numbers when Jeffries' paper on the subject appeared in the Notices of the AMS late last year [10]; almost simultaneously (and apparently independently), Tossavainen et al's survey on the subject appeared in the College Math Journal [15].

Why should the reader care about differentiating numbers? In general, any time a new theory is introduced, it is natural to seek *numerical* structures satisfying that theory: thus, when the theory of groups is introduced, it is natural to introduce examples like $(\mathbb{Q}, +)$ and $(\mathbb{R}^+, \cdot)$. Theories about elementary calculus functions, in languages including the unary function symbol $\bullet'$, were originally modeled by structures whose universes consisted of elementary calculus functions, *not* numbers. We can at least try to find numerical models for these theories. The act of interpreting $\bullet'$ in a structure whose universe is a number system is the act of "differentiating numbers". We are hopeful

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that generalizing models of elementary calculus function theories could eventually be fruitful just like generalizing models of permutation sets led to the abstract theory of groups.

When it comes to numerically modeling theories, different theories might require different number systems: both $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Q}, +, \cdot)$ are rings, but only the latter is a field. By gaining knowledge about which number systems are needed for which theories, we gain insight into those theories. We hope that a greater knowledge about which number systems are needed to model various subtheories of elementary calculus functions, will eventually give us insight into those subtheories.

If the only axiom we care about is the Leibniz rule (and the nontriviality axiom $\exists x[x' \neq 0]$), we can interpret $\bullet'$ on $\mathbb{N}$ so as to satisfy that. That is the approach of [2] and [16]. But their $(\mathbb{N}, \bullet')$ does not even satisfy the linearity axiom. Our interpretation (in Section 3) of $\bullet'$ on a subset of the hyperreals will satisfy far more axioms of the theory of elementary calculus functions. And in Section 3.2, we will introduce a stricter subset of the hyperreals where not only $\bullet'$ can be elegantly interpreted, but $\circ$ as well (in other words, there is an elegant way to define the "composition" of two numbers there), in such a way as to numerically satisfy even the chain rule.

The key idea behind our numerical interpretation of $\bullet'$ is to commute the derivative operation with the operation of taking a function's equivalence class in the hyperreals, in other words, define $[f]' = [f']$ (we will spell out the details below). Unfortunately, this is not well-defined in general. However, the idea can be salvaged in several different ways, by making use of certain idempotent ultrafilters. In Section 4 we will use the same approach to well-define a hyperreal variation of the derivative from finite calculus, and as an application of that, we will give a new proof of Hindman's Theorem and also strengthen said theorem.

2 Preliminaries

Throughout the paper, we write $\beta\mathbb{R}$ for the set of ultrafilters on $\mathbb{R}$.

Definition 1 *(Hyperreals) For each free $p \in \beta\mathbb{R}$, let ${}^*\mathbb{R}_p$ be the hyperreals constructed using p. For every $f : \mathbb{R} \to \mathbb{R}$, let $[f]_p$ be the hyperreal represented by f. If p is clear from context, we will write ${}^*\mathbb{R}$ and $[f]$ for ${}^*\mathbb{R}_p$ and $[f]_p$, respectively.*

Convention 2 *If $p \in \beta\mathbb{R}$ is free and f is a function with codomain $\mathbb{R}$ and with domain* $\mathrm{dom}(f) \in p$, *we will write $[f]_p$ for $[\hat{f}]_p$ where $\hat{f} : \mathbb{R} \to \mathbb{R}$ is the extension of f defined* *by* $\hat{f}(x) = 0$ *for all* $x \in \mathbb{R}\backslash\mathrm{dom}(f)$*. If* $\mathrm{dom}(f) \notin p$*, we say that* $[f]_p$ does not exist. *If* p *is clear from context, we will write* $[f]$ *for* $[f]_p$*.*

Definition 3 *For each* $f : \mathbb{R} \to \mathbb{R}$*, let* ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ *be the nonstandard extension of* f*. Let* $\Omega = [x \mapsto x]$ *be the hyperreal represented by the identity function.*

For every $f : \mathbb{R} \to \mathbb{R}$, there are two ways of viewing f in nonstandard analysis. It can be viewed as the number $[f]$ or as the function ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$. The two are related via Ω. Namely: $[f] = {}^*f(\Omega)$.

Unfortunately, the following proposition shows that the idea of defining $[f]' = [f']$ does not work in general.

Proposition 4 *(Ill-definedness)*

(1) *There exists a free* $p \in \beta\mathbb{R}$ *and everywhere-differentiable* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $[f]_p = [g]_p$ *but* $[f']_p \neq [g']_p$*.*
(2) *For every free* $p \in \beta\mathbb{R}$*, for all* $f : \mathbb{R} \to \mathbb{R}$ *such that* $[f']_p$ *exists, there exists* $g : \mathbb{R} \to \mathbb{R}$ *such that* $[f]_p = [g]_p$ *but* $[g']_p$ *does not exist.*

Proof (1) Let $p \in \beta\mathbb{R}$ such that $\mathbb{N} \in p$. The claim is witnessed by $f(x) = 0$ and $g(x) = \sin \pi x$.

(2) Let $D \subseteq \mathbb{R}$ be dense and co-dense. Assume $D \in p$ (if not, then $D^c \in p$ and a similar argument applies). The claim is witnessed by $g(x) = f(x) + \chi_{D^c}(x)$. □

In light of Proposition 4, we cannot expect the definition $[f]'_p = [f']_p$ to work for every free $p \in \beta\mathbb{R}$ even if we restrict our attention to everywhere-differentiable f; and, if we do not so restrict our attention, then we can *expect* the definition $[f]'_p = [f']_p$ to fail for every p. We will show that if we restrict attention to those f such that $[f']$ exists, then the definition $[f]'_p = [f']_p$ *does* work provided p is idempotent and contains $(0, \epsilon)$ for every $\epsilon > 0$.

3 Differentiating hyperreals $[f]$ such that $[f']$ exists

Definition 5 *(Idempotent ultrafilters on* $\mathbb{R}$*)*

(1) *For each* $S \subseteq \mathbb{R}$ *and any* $y \in \mathbb{R}$*,* $S - y$ *is defined to be* $\{x - y : x \in S\}$*.*
(2) *An ultrafilter* $p \in \beta\mathbb{R}$ *is* idempotent *if* $p = \{S \subseteq \mathbb{R} : \{y \in \mathbb{R} : S - y \in p\} \in p\}$*.*

Definition 6 *By 0^+ we mean the set of ultrafilters $p \in \beta\mathbb{R}$ such that p satisfies the following requirement. For every real $\epsilon > 0$, the open interval $(0, \epsilon) \in p$.*

Lemma 7 *0^+ contains an idempotent ultrafilter.*

Proof By Lemma 13.29(a) and Theorem 13.31 of Hindman and Strauss [9]. □

Clearly an ultrafilter in 0^+ is free. The following lemma illustrates the power of ultrafilters in 0^+.

Lemma 8 *Let $p \in 0^+$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 then $\mathrm{st}([f]_p) = f(0)$.*

Proof Let $\epsilon > 0$ be real. By continuity of f at 0, $\exists\delta > 0$ such that $|f(0) - f(x)| < \epsilon$ whenever $x \in (0, \delta)$. Since $p \in 0^+$, $(0, \delta) \in p$. Thus, f is within ϵ of $f(0)$ ultrafilter often, so $[f]_p$ is within ϵ of $f(0)$. □

For the rest of this section, we fix an idempotent $p \in 0^+$. The following theorem shows that this suffices to make the definition $[f]' = [f']$ well-defined if we restrict it to functions such that $[f']$ exists. Note that since $p \in 0^+$, the existence of $[f']$ is equivalent to the statement that for all real $\epsilon > 0$, there exists real $\delta \in (0, \epsilon)$ such that $f'(\delta)$ exists.

Theorem 9 *For all $f, g : \mathbb{R} \to \mathbb{R}$ such that $[f']$ and $[g']$ exist, if $[f] = [g]$ then $[f'] = [g']$.*

Proof Since $[f] = [g]$, there is some $S_0 \in p$ such that $f = g$ on S_0. Let $S = S_0 \cap \mathrm{dom}(f') \cap \mathrm{dom}(g')$. Existence of $[f']$ and $[g']$ means $\mathrm{dom}(f') \in p$ and $\mathrm{dom}(g') \in p$, thus $S \in p$. Since p is idempotent, $\{x \in \mathbb{R} : S - x \in p\} \in p$. Thus $S \cap \{x \in \mathbb{R} : S - x \in p\} \in p$. To show $[f'] = [g']$, we will show that $f' = g'$ on $S \cap \{x \in \mathbb{R} : S - x \in p\}$. Let $x \in S \cap \{x \in \mathbb{R} : S - x \in p\}$. In particular, $S - x \in p$. We must show $f'(x) = g'(x)$.

Claim: For all $h \in S - x$, $(f(x+h) - f(x))/h = (g(x+h) - g(x))/h$. Indeed, let $h \in S - x$. This means $h = y - x$ for some $y \in S$. Compute

$$\begin{aligned}
(f(x+h) - f(x))/h &= (f(x+y-x) - f(x))/h && (h = y - x)\\
&= (f(y) - f(x))/h && \text{(algebra)}\\
&= (g(y) - g(x))/h && (f = g \text{ on } S)\\
&= (g(x+y-x) - g(x))/h && \text{(algebra)}\\
&= (g(x+h) - g(x))/h && (h = y - x)
\end{aligned}$$

proving the claim.

Since $p \in 0^+$ and $S - x \in p$, it follows that for all real $\epsilon > 0$, $(S - x) \cap (0, \epsilon) \in p$, thus is nonempty. So $S - x$ contains h arbitrarily near 0. Since $x \in \mathrm{dom}(f') \cap \mathrm{dom}(g')$, $\lim_{h \to 0}(f(x+h) - f(x))/h$ and $\lim_{h \to 0}(g(x+h) - g(x))/h$ exist. Since both limits exist and since there are h arbitrarily near 0 such that $(f(x+h) - f(x))/h = (g(x+h) - g(x))/h$, the limits must be equal, that is, $f'(x) = g'(x)$. □

Corollary 10 *Let $\mathcal{D} = \{[f] : f : \mathbb{R} \to \mathbb{R}$ and $[f']$ exists$\}$. The derivative operation $\bullet' : \mathcal{D} \to {}^*\mathbb{R}$ defined by $[f]' = [f']$ (for all f such that $[f']$ exists) is well-defined.*

We do not yet know whether $\mathcal{D}$ (from Corollary 10) is a proper subset of ${}^*\mathbb{R}$. In other words: can every hyperreal be written in the form $[f]$ where $[f']$ exists? Does this depend on p?

If $\mathcal{D}$ is as in Corollary 10 then it follows that the structure $(\mathcal{D}, 1, \Omega, +, \cdot, \bullet')$ satisfies every positive formula in the theory of elementary calculus functions in the language $(1, \mathrm{id}, +, \cdot, \bullet')$, where, by *positive formula*, we mean a formula that can be built up without using $\neg$ or $\neq$. In particular this includes the Leibniz rule axiom, linearity, and the power rule schema. The structure also satisfies the nontriviality axiom $\exists x[x' \neq 0]$. All this remains true if constant symbols for other individual functions (such as sin and cos), besides just the identity function, are added to the language, interpreted in $\mathcal{D}$ by the hyperreals represented thereby (such as $[\sin]$ and $[\cos]$), provided those hyperreals' derivatives exist.

In terms of nonstandard extensions, Theorem 9 says that there is a well-defined map which sends every ${}^*f(\Omega)$ to ${}^*f'(\Omega)$. For example, this map sends $e^\Omega + \Omega^3 + \cos 2\Omega$ to $e^\Omega + 3\Omega^2 - 2\sin 2\Omega$.

One might intuitively wonder whether $[f]' = 0$ implies f is constant (at least ultrafilter often). The following proposition provides a counterexample.

Proposition 11 *There exists $f : \mathbb{R} \to \mathbb{R}$ such that $[f]' = 0$ but for every $r \in \mathbb{R}$, $\{x \in \mathbb{R} : f(x) = r\} \notin p$.*

Proof By Theorem 1.14 of Bankston and McGovern [1] there exist disjoint Cantor sets on $\mathbb{R}$. An ultrafilter cannot contain two disjoint sets, so there is some Cantor set C on $\mathbb{R}$ with $C \notin p$. Let f be the devil's staircase based on C. Then $f'(x) = 0$ for all $x \notin C$ so $[f]' = 0$, but f is increasing, and is not flat on $(0, \epsilon)$ for any $\epsilon > 0$, implying (since $p \in 0^+$) that $\{x \in \mathbb{R} : f(x) = r\} \notin p$ for all r. □

We have proven Corollary 10 under the assumption that p is idempotent and in 0^+. Similar reasoning would hold if p were idempotent and in 0^- (ie, if p were required to contain $(-\epsilon, 0)$ for every positive real ϵ). We currently do not know whether Corollary 10 holds for any other type of ultrafilter.

3.1 Differential equations and the secant method

Since $\bullet'$ takes (a subset of) ${}^*\mathbb{R}$ to ${}^*\mathbb{R}$, one can attempt to solve (or approximately solve) differential equations by using the secant method from numerical analysis, which is traditionally only used to solve non-differential equations. This is interesting because as far as we know, the secant method has not previously been applicable to differential equations. We illustrate this with an example in which the method finds a correct solution in one step.

Example 12 *Solve the differential equation $y' - 2x = 0$ using the secant method, with initial guesses $y_0 = x^2 + x^3$ and $y_1 = x^2 - x^3$.*

Solution. Define $\alpha : \subseteq {}^*\mathbb{R} \to {}^*\mathbb{R}$ by $\alpha([f]) = [f]' - 2\Omega$ whenever $[f]'$ is defined. We desire a solution $[f]$ of the equation $\alpha([f]) = 0$. Since $\Omega = [x \mapsto x]$, it follows that $[f]' - 2\Omega = [x \mapsto f'(x) - 2x]$, so any such solution $[f]$ will yield a solution $y = f(x)$ to the differential equation $y' - 2x = 0$ (at least p–ae). Compute:

$$\begin{aligned}
[f_0] &= \Omega^2 + \Omega^3 && \text{(initial guess } y_0 = x^2 + x^3\text{)}\\
[f_1] &= \Omega^2 - \Omega^3 && \text{(initial guess } y_1 = x^2 - x^3\text{)}\\
[f_2] &= [f_1] - \alpha([f_1]) \frac{[f_1] - [f_0]}{\alpha([f_1]) - \alpha([f_0])} && \text{(secant method)}\\
\alpha([f_1]) &= (\Omega^2 - \Omega^3)' - 2\Omega = (2\Omega - 3\Omega^2) - 2\Omega\\
\alpha([f_0]) &= (\Omega^2 + \Omega^3)' - 2\Omega = (2\Omega + 3\Omega^2) - 2\Omega
\end{aligned}$$

It follows that $[f_2] = \Omega^2$. This yields a solution $y = x^2$ to the original differential equation. □

We have not yet found any examples where this approach is more practical than other approximate methods in differential equations. We hope that either such examples can be found later, or, if not, that the lack of such examples might provide insight into limitations of the secant method itself. The point of this subsection is not so much to focus on the secant method, but rather to illustrate the kind of things we hope might be possible by numerically interpreting the theory of elementary calculus functions.

3.2 The well-definability of composition on a subset of the hyperreals

The work we have presented above is relevant to numerically modeling subtheories of the theory of elementary calculus functions in a language containing a unary function symbol $\bullet'$ for differentiation. But one key axiom is missing from that theory, namely the chain rule, since the chain rule also involves a binary function symbol $\circ$ for composition. In this section, we introduce a subset of the hyperreals suitable for numerically modeling subtheories of the theory of elementary calculus functions in a language containing $\bullet'$ and $\circ$. The following definition is motivated by the theory of complex analysis.

Definition 13 *(Entire numbers)*

(1) *A function $f:\mathbb{R}\to\mathbb{R}$ is* entire *if f is infinitely differentiable at* 0 *and* $\forall x\in\mathbb{R}$, $f(x)=\sum_{k=0}^{\infty}f^{(k)}(0)x^k/k!$.

(2) *A hyperreal number is* entire *if it can be written as $[f]$ for some entire $f:\mathbb{R}\to\mathbb{R}$.*

Proposition 14 *For all entire $f,g:\mathbb{R}\to\mathbb{R}$, if $[f]=[g]$ then $f=g$.*

Proof By well-known results from analysis, f and g are infinitely differentiable everywhere, so $[f^{(k)}]$ and $[g^{(k)}]$ exist for all k. By repeated applications of Theorem 9, each $[f^{(k)}]=[g^{(k)}]$. By Lemma 8, it follows that each $f^{(k)}(0)=g^{(k)}(0)$. Since f and g are entire, this implies $f=g$. □

In the same way that we can think of the differential equation $2x+2yy'=0$ as describing the family of all circles centered at the origin, we can think of the equation $y=\sum_{k=0}^{\infty}y^{(k)}(0)x^k/k!$ as describing the family of all entire functions. The latter is much worse behaved than the former: no two circles centered at the origin ever intersect each other, but for all $(x_0,y_0)\in\mathbb{R}^2$ there exist distinct entire functions whose graphs intersect at (x_0,y_0). In this sense, we can say that the real plane contains no "critical points" of $2x+2yy'=0$, but that every point of the real plane is a "critical point" of $y=\sum_{k=0}^{\infty}y^{(k)}(0)x^k/k!$. We can interpret Proposition 14 as saying that in the hyperreal plane, every point on the vertical line $x=\Omega$ is a "non-critical point" of the family of all entire functions, for the proposition says that no two distinct entire function graphs intersect anywhere on this vertical line.

Corollary 15 *The operation $\circ$ defined on the entire numbers by $[f]\circ[g]=[f\circ g]$ (whenever f and g are entire) is well-defined.*

Clearly with $\circ$ defined as in Corollary 15 and with $\bullet'$ defined as in Corollary 10, the entire numbers satisfy the chain rule axiom, $\forall x\forall y\,(x\circ y)'=(x'\circ y)y'$.

3.3 The approximately space-filling nature of differentiation

Clearly $\bullet'$ is linear over $\mathbb{R}$, in the sense that for all $[f], [g] \in {}^*\mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$, $(\lambda[f]+\mu[g])' = \lambda[f]' + \mu[g]'$ if the derivatives in question exist. So how badly behaved could the graph $y = x'$ be? We will show that it is approximately space-filling, in the following sense.

Definition 16 *A subset $C \subseteq ({}^*\mathbb{R})^2$ is* approximately space-filling *if for all $(x_0, y_0) \in \mathbb{R}^2$, there exists some $(x, y) \in C$ such that $\mathrm{st}(x) = x_0$ and $\mathrm{st}(y) = y_0$.*

It is not hard to find functions ${}^*\mathbb{R} \to {}^*\mathbb{R}$ which are linear over $\mathbb{R}$ and whose graphs are approximately space-filling. For example, if we consider ${}^*\mathbb{R}$ as a vector space over $\mathbb{R}$ and let $\mathcal{B}$ be a basis for it with an infinitesimal basis element v, then the projection $\pi(\cdots + \lambda v + \cdots) = \lambda$ is one such function. Nevertheless, we find it interesting (even if the proof is quite simple) that our derivative operator also has these properties.

Proposition 17 *The hyperreal graph $y = x'$, ie, the set of all $([f], [g]) \in ({}^*\mathbb{R})^2$ such that $[g] = [f]'$, is approximately space-filling.*

Proof For any $(x_0, y_0) \in \mathbb{R}^2$, let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable everywhere with $f(0) = x_0$ and $f'(0) = y_0$. By Lemma 8, $\mathrm{st}([f]) = f(0) = x_0$ and $\mathrm{st}([f]') = \mathrm{st}([f']) = f'(0) = y_0$. □

Having established that the hyperreal graph $y = x'$ is approximately space-filling, we will proceed to state two additional results about approximately space-filling sets in general, which therefore apply in particular to said graph.

Proposition 18 *If $C \subseteq ({}^*\mathbb{R})^2$ is approximately space-filling, then for any $X \subseteq \mathbb{R}^2$, there exists some $S \subseteq {}^*\mathbb{R}$ such that*

$$X = \{(\mathrm{st}(x), \mathrm{st}(y)) \,:\, (x, y) \in C \text{ and } x \in S\}.$$

In particular, for any $X \subseteq \mathbb{R}^2$, there exists some $S \subseteq {}^*\mathbb{R}$ such that

$$X = \{(\mathrm{st}(x), \mathrm{st}(x')) \,:\, x \in S\}.$$

Proof Straightforward. □

Proposition 19 *Suppose $C \subseteq ({}^*\mathbb{R})^2$ is approximately space-filling where C is the graph (over ${}^*\mathbb{R}$) of $y = F(x)$ for some $F : \subseteq {}^*\mathbb{R} \to {}^*\mathbb{R}$. Let $X \subseteq \mathbb{R}^2$ be the graph of the equation $y = f(x)$ for some everywhere-differentiable $f : \mathbb{R} \to \mathbb{R}$. If S is as in Proposition 18, then $F|_S$ has the same slope as $y = f(x)$ in the following sense: for every $x \in S$, for every real $\epsilon > 0$, there exists some real $\delta > 0$ such that for all $h \in {}^*\mathbb{R}$, if $0 < |h| < \delta$ and $x + h \in S$ then $|f'(\mathrm{st}(x)) - (F(x+h) - F(x))/h| < \epsilon$. In particular, this is true when $F = \bullet'$.*

Proof Straightforward. □

4 A variant of finite calculus using an idempotent ultrafilter on $\mathbb{N}$

In so-called finite calculus, one considers the "derivative" $f(x+1) - f(x)$ of f, see Graham, Knuth and Patashnik [8, Section 2.6]. In this section, we will investigate a variant of this derivative, namely $\Delta f(x) = {}^*f(x + \Omega) - f(x)$ where $\Omega = [n \mapsto n]$ is the canonical hyperreal in the hyperreals constructed using an idempotent ultrafilter on $\mathbb{N} = \{1, 2, \ldots\}$. (Note that we omit 0 from $\mathbb{N}$.) We will show that this finite derivative Δ has unexpected connections to the standard derivative, and that the equivalence class (in an iterated ultrapower construction) of $(\Delta f)|_{\mathbb{N}}$ is well-defined as a function of $[f|_{\mathbb{N}}]$. (We elaborate what we mean by "in an iterated ultrapower construction" in Remark 28 below.)

The following Definitions 20 and 22, and Lemma 21 are $\mathbb{N}$–focused analogies of the $\mathbb{R}$–focused Definitions 5 and 1, and Lemma 7 above, respectively. We prefer this slight redundancy (instead of defining everything in higher generality) for the sake of concreteness.

Definition 20 *(Idempotent ultrafilters on $\mathbb{N}$)*

(1) *If $S \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $S - n = \{x - n : x \in S \text{ and } x - n \in \mathbb{N}\}$.*
(2) *An ultrafilter q on $\mathbb{N}$ is* idempotent *if $q = \{S \subseteq \mathbb{N} : \{n \in \mathbb{N} : S - n \in q\} \in q\}$.*

Lemma 21 *Idempotent ultrafilters on $\mathbb{N}$ exist and are free.*

Proof See Hindman and Strauss [9]. □

Definition 22 *If q is a free ultrafilter on $\mathbb{N}$, we write ${}^*\mathbb{R}_q$ for the hyperreal numbers constructed using q in the usual way, and for each $f : \mathbb{N} \to \mathbb{R}$ we write $[f]_q$ for the hyperreal represented by f. If q is clear from context, we will write ${}^*\mathbb{R}$ for ${}^*\mathbb{R}_q$, $[f]$ for $[f]_q$, and ${}^*f$ for the nonstandard extension ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ of $f : \mathbb{R} \to \mathbb{R}$.*

For the remainder of the paper, we fix an idempotent ultrafilter q on $\mathbb{N}$. Lemma 24 below will replace 0^+.

Definition 23 *For each $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$, we define $\mathrm{rm}_\gamma : \mathbb{N} \to (-\gamma/2, \gamma/2)$ as follows. (We pronounce rm as "remainder".) For each $n \in \mathbb{N}$, we define $\mathrm{rm}_\gamma(n) = n - k\gamma$ where $k\gamma$ is the closest integer multiple of γ to n (there is a unique such closest integer multiple of γ because γ is irrational and $n \in \mathbb{N}$).*

Note that the following lemma depends on q being idempotent.

Lemma 24 *Let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$. For every real $\epsilon > 0$, $\{n \in \mathbb{N} : |\mathrm{rm}_\gamma(n)| < \epsilon\} \in q$.*

Proof Follows from Theorem 7.2 of Bergelson [3]. □

For the rest of the section, let $\Omega = [n \in \mathbb{N} \mapsto n]$ (the canonical element of ${}^*\mathbb{R} \backslash \mathbb{R}$).

Definition 25 *(Finite derivative) For each $f : \mathbb{R} \to \mathbb{R}$, we define the finite derivative $\Delta f : \mathbb{R} \to {}^*\mathbb{R}$ by $\Delta f(x) = {}^*f(x + \Omega) - f(x)$.*

The following theorem shows that, when restricted to γ–periodic functions for a fixed irrational real number $\gamma > 0$, Δ is a constant multiple of $\bullet'$ (up to an infinitesimal error), at least where f' exists. And even when f' does not exist, Δ (times the same constant multiple) sometimes still provides information about the slope in question.

Theorem 26 *Let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$, and let $f : \mathbb{R} \to \mathbb{R}$ be γ–periodic. Let $x \in \mathbb{R}$.*

(1) *If $f'(x)$ exists, then $\mathrm{st}(\Delta f(x)/[\mathrm{rm}_\gamma]) = f'(x)$.*
(2) *If $f'(x)$ fails to exist because $\lim_{h \to 0}(f(x+h) - f(x))/h$ diverges to ∞ (respectively $-\infty$) then $\Delta f(x)/[\mathrm{rm}_\gamma]$ is infinite (respectively negative infinite).*
(3) *Let $g : \mathbb{R} \to \mathbb{R}$ be γ–periodic. If both $f'(x)$ and $g'(x)$ fail to exist because $\lim_{h \to 0}(f(x+h) - f(x))/h$ and $\lim_{h \to 0}(g(x+h) - g(x))/h$ both diverge to ∞ but $\lim_{h \to 0}(f(x+h) - f(x))/h$ diverges to ∞ faster (ie, there exists $\delta > 0$ such that $(f(x+h) - f(x))/h > (g(x+h) - g(x))/h$ whenever $0 < |h| < \delta$) then $\Delta f(x)/[\mathrm{rm}_\gamma] > \Delta g(x)/[\mathrm{rm}_\gamma]$. Similarly for $-\infty$.*

Proof (1) Fix $x \in \mathbb{R}$ such that $f'(x)$ exists. Define $g : \mathbb{N} \to \mathbb{R}$ by $g(n) = (f(x+n) - f(x))/\mathrm{rm}_\gamma(n)$, then $[g] = \Delta f(x)/[\mathrm{rm}_\gamma]$. To show $\mathrm{st}([g]) = f'(x)$, it suffices to show that for every real $\epsilon > 0$, $\{n \in \mathbb{N} : |f'(x) - g(n)| < \epsilon\} \in q$. Fix ϵ. By definition of f', there is some $\delta > 0$ such that $|f'(x) - (f(x+h) - f(x))/h| < \epsilon$ whenever $0 < |h| < \delta$. Let $S = \{n \in \mathbb{N} : |\mathrm{rm}_\gamma(n)| < \delta\}$. By Lemma 24, $S \in q$. We claim that $|f'(x) - g(n)| < \epsilon$ for all $n \in S$. Let $n \in S$. Then:

$$\begin{aligned}
|f'(x) - g(n)| &= |f'(x) - (f(x+n) - f(x))/\mathrm{rm}_\gamma(n)| && \text{(definition of } g\text{)} \\
&= |f'(x) - (f(x+\mathrm{rm}_\gamma(n)) - f(x))/\mathrm{rm}_\gamma(n)| && (f \text{ is } \gamma\text{–periodic}) \\
&< \epsilon && (0 < |\mathrm{rm}_\gamma(n)| < \delta)
\end{aligned}$$

The proofs of (2) and (3) are similar to (and easier than) the proof of (1). □

In particular, $\Delta(f \circ g)(x)/[\mathrm{rm}_\gamma] \approx (\Delta f(g(x))/[\mathrm{rm}_\gamma])(\Delta g(x)/[\mathrm{rm}_\gamma])$ (with infinitesimal error) provided $f'(g(x))$ and $g'(x)$ exist and g is γ–periodic; thus, $\Delta/[\mathrm{rm}_\gamma]$ satisfies a chain rule. Contrast this with Graham, Knuth and Patashnik's claim that "there's no corresponding chain rule of finite calculus" [8].

Without the idempotency requirement, it would be possible to find free $q \in \beta\mathbb{N}$ so as to falsify Theorem 26. For example, one could choose q such that $\{n \in \mathbb{N} : \mathrm{rm}_\gamma(n) \in (\frac{\gamma}{3}, \frac{2\gamma}{3})\} \in q$. Then for any γ–periodic $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = f'(0) = 0$ and $f(x) = 1$ for all $x \in (\frac{\gamma}{3}, \frac{2\gamma}{3})$, it would follow that $\Delta f(0) = 1$, implying $\frac{3}{2\gamma} < \Delta f(0)/[\mathrm{rm}_\gamma] < \frac{3}{\gamma}$, making the conclusion of Theorem 26 (part 1) impossible at $x = 0$.

As in Section 3, we would like to transform the derivative operation $\Delta : \mathbb{R}^\mathbb{R} \to ({}^*\mathbb{R})^\mathbb{R}$ into a well-defined derivative on hyperreals. But since $\Delta f(x)$ is itself hyperreal, we will need to be careful.

Definition 27 *For every $f : \mathbb{N} \to {}^*\mathbb{R}$, we write $[\![f]\!]$ for the equivalence class of f modulo the equivalence relation defined by declaring that $h, g : \mathbb{N} \to {}^*\mathbb{R}$ are equivalent if and only if $\{n \in \mathbb{N} : h(n) = g(n)\} \in q$. We identify each $r \in \mathbb{R}$ with $[\![n \mapsto r]\!]$.*

Remark 28 *Suppose $f : \mathbb{N} \to {}^*\mathbb{R}$. Using Proposition 6.5.2 of Chang and Keisler [7], one can in fact think of $[\![f]\!]$ as being a hyperreal number, but in a different copy of the hyperreals. To be more precise, one can think of $[\![f]\!]$ as a hyperreal in ${}^*\mathbb{R}_{q\times q}$, where $\times$ is the filter product defined shortly before Proposition 6.2.1 of [7].*

The following theorem will allow us to well-define $\Delta[f] = [\![\Delta f]\!]$.

Theorem 29 *Let $f, g : \mathbb{R} \to \mathbb{R}$. If $[f|_{\mathbb{N}}] = [g|_{\mathbb{N}}]$ then $[\![\Delta f|_{\mathbb{N}}]\!] = [\![\Delta g|_{\mathbb{N}}]\!]$.*

Proof Assume $[f|_{\mathbb{N}}] = [g|_{\mathbb{N}}]$. We must show $\{n \in \mathbb{N} : \Delta f(n) = \Delta g(n)\} \in q$. Since $[f|_{\mathbb{N}}] = [g|_{\mathbb{N}}]$, there is some $S \in q$ such that $f(n) = g(n)$ for all $n \in S$. Since q is idempotent, $\{n \in \mathbb{N} : S - n \in q\} \in q$. Thus $S \cap \{n \in \mathbb{N} : S - n \in q\} \in q$. We claim $\Delta f(n) = \Delta g(n)$ for all $n \in S \cap \{n \in \mathbb{N} : S - n \in q\}$. Fix any such n. By construction, $S - n \in q$. By a similar argument as in the proof of Theorem 9, for all $h \in S - n$, $f(n+h) - f(n) = g(n+h) - g(n)$. Thus $[h \mapsto f(n+h) - f(n)] = [h \mapsto g(n+h) - g(n)]$, in other words ${}^*f(n+\Omega) - f(n) = {}^*g(n+\Omega) - g(n)$, in other words $\Delta f(n) = \Delta g(n)$. □

Corollary 30 *The finite derivative $\Delta : {}^*\mathbb{R} \to {}^*\mathbb{R}_{q\times q}$ defined by $\Delta[f] = [\![\Delta f]\!]$ is well-defined.*

One might intuitively expect that $\Delta[f] = 0$ should imply that f is constant (at least ultrafilter often); we present a counterexample disproving this intuition and replacing it with a characterization related to periodicity.

Definition 31 *Let $f : \mathbb{N} \to \mathbb{R}$. We say f is q–*ae constant *if $\exists r \in \mathbb{R}$ such that $\{n \in \mathbb{N} : f(n) = r\} \in q$. We say f is q–*ae Ω–periodic *if $\{n \in \mathbb{N} : {}^*f(n+\Omega) = f(n)\} \in q$.*

Theorem 32
(1) *For all $[f] \in {}^*\mathbb{R}$, $\Delta[f] = 0$ if and only if f is q–ae Ω–periodic.*
(2) *For all $[f] \in {}^*\mathbb{R}$, if f is q–ae constant then $\Delta[f] = 0$.*
(3) *There exists $f : \mathbb{N} \to \mathbb{R}$ such that $\Delta[f] = 0$ but f is not q–ae constant.*

Proof (1) and (2) are straightforward. For (3), let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$ and define $g : (-\gamma/2, \gamma/2) \to \mathbb{R}$ as follows. For $x \neq 0$, let $g(x) = \lfloor 1/|x| \rfloor$ where $\lfloor \bullet \rfloor$ is the greatest integer function. Let $g(0) = 0$. We claim $f = g \circ \mathrm{rm}_\gamma$ witnesses (3). By Lemma 24, $\{n \in \mathbb{N} : |\mathrm{rm}_\gamma(n)| < \epsilon\} \in q$ for all real $\epsilon > 0$. Since $\lim_{x\to 0} g(x) = \infty$, it follows that $[f]$ is infinite, thus f is not q–ae constant.

By Lemma 24, $S = \{n \in \mathbb{N} : |\mathrm{rm}_\gamma(n)| < \gamma/4\} \in q$. We claim $\Delta f(n) = 0$ for all $n \in S$, whence $\Delta[f] = [\![\Delta f]\!] = 0$. Let $n \in S$. Since γ is irrational, it follows that $\mathrm{rm}_\gamma(n)$ is not one of the jump discontinuity points of g, thus $\exists \delta > 0$ such that $g(x) = g(\mathrm{rm}_\gamma(n))$ whenever $|x - \mathrm{rm}_\gamma(n)| < \delta$. By Lemma 24, $T = \{m \in \mathbb{N} : |\mathrm{rm}_\gamma(m)| < \min(\delta, \gamma/4)\} \in q$. We claim $f(n+m) - f(n) = 0$ for all $m \in T$, which establishes $\Delta f(n) = 0$. Let $m \in T$. Since $|\mathrm{rm}_\gamma(n)| < \gamma/4$ and $|\mathrm{rm}_\gamma(m)| < \gamma/4$, it follows that $\mathrm{rm}_\gamma(n+m) = \mathrm{rm}_\gamma(n) + \mathrm{rm}_\gamma(m)$. Thus $|\mathrm{rm}_\gamma(n+m) - \mathrm{rm}_\gamma(n)| = |\mathrm{rm}_\gamma(m)| < \delta$, so $f(n+m) = g(\mathrm{rm}_\gamma(n+m)) = g(\mathrm{rm}_\gamma(n)) = f(n)$ by choice of δ. □

4.1 An alternate proof and strengthening of Hindman's Theorem

The well-definedness in Corollary 30 can be used to prove Hindman's Theorem (Theorem 34 below). Formally, our proof of Hindman's Theorem is basically identical to the usual proof, but informally it appears different because all references to idempotent ultrafilters are hidden underneath the innocent-looking fact that the derivative of a constant function is zero. But the idempotency of the underlying ultrafilter was used to prove that the derivative in question is well-defined.

Lemma 33 *Let $c \in \mathbb{R}$. If $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{R}$ are such that each $[f_i] = c$, then there exist arbitrarily large $n \in \mathbb{N}$ such that the following requirement holds. For each $i \in \{1, \ldots, k\}$, ${}^*f_i(n + \Omega) = f_i(n) = c$.*

Proof By Corollary 30 each $\Delta[f_i] = \Delta c = 0$, thus $[\![n \mapsto {}^*f_i(n+\Omega) - f_i(n)]\!] = 0$, ie $S_i = \{n \in \mathbb{N} : {}^*f_i(n+\Omega) = f_i(n)\} \in q$. Since each $[f_i] = c$, each $T_i = \{n \in \mathbb{N} : f_i(n) = c\} \in q$. Thus $\bigcap_i (S_i \cap T_i) \in q$. The elements thereof witness the lemma. □

Theorem 34 *(Hindman's Theorem) If $f : \mathbb{N} \to \mathbb{R}$ has finite range, then there exists some $c \in \mathbb{R}$ and some infinite $S \subseteq \mathbb{N}$ such that for all finite nonempty $X \subseteq S$, $f(\sum X) = c$.*

Proof By the maximality of ultrafilters, $[f] = c$ for some $c \in \mathbb{R}$. For every finite $X \subseteq \mathbb{N}$, define $f_X : \mathbb{N} \to \mathbb{R}$ by $f_X(n) = f((\sum X) + n)$ (note that $f_\emptyset = f$).

Inductively, suppose we have defined $n_1 < \cdots < n_k$ (an empty list if $k = 0$) such that:

(1) For each nonempty $X \subseteq \{n_1, \ldots, n_k\}$, $f(\sum X) = c$.
(2) For each $X \subseteq \{n_1, \ldots, n_k\}$, $[f_X] = c$.

By Lemma 33, pick $n_{k+1} > \max\{n_1, \ldots, n_k\}$ such that

$$(*) \text{ for each } X \subseteq \{n_1, \ldots, n_k\},\ {}^*f_X(n_{k+1} + \Omega) = f_X(n_{k+1}) = c.$$

For each $X \subseteq \{n_1, \ldots, n_{k+1}\}$ with $n_{k+1} \in X$, we have (by $*$) $f(\sum X) = f_{X \backslash \{n_{k+1}\}}(n_{k+1}) = c$ because $X \backslash \{n_{k+1}\} \subseteq \{n_1, \ldots, n_k\}$. And for each $X \subseteq \{n_1, \ldots, n_k\}$, since (by $*$) ${}^*f_X(n_{k+1} + \Omega) = c$, it follows that ${}^*f_{X \cup \{n_{k+1}\}}(\Omega) = c$, ie, $[f_{X \cup \{n_{k+1}\}}] = c$. So $n_1 < \cdots < n_{k+1}$ also satisfy 1–2. By induction, we obtain $n_1 < n_2 < \cdots$ with the above properties, which clearly proves the theorem. □

In the above proof, we proved more than was required. This leads to the following strengthening of Hindman's Theorem.

Theorem 35 *(Ω as universal Hindman number) If $f : \mathbb{N} \to \mathbb{R}$ has finite range, then there exists some $c \in \mathbb{R}$ and some infinite $S \subseteq \mathbb{N}$ such that for all finite nonempty $X \subseteq S \cup \{\Omega\}$, ${}^*f(\sum X) = c$.*

Proof In the proof of Theorem 34, we proved the existence of $c \in \mathbb{R}$ and $n_1, n_2, \ldots \in \mathbb{N}$ such that (1) for each finite nonempty $X \subseteq \{n_1, n_2, \ldots\}$, $f(\sum X) = c$, and (2) for each finite $X \subseteq \{n_1, n_2, \ldots\}$, $[f_X] = c$, where $f_X : \mathbb{N} \to \mathbb{R}$ is defined by $f_X(n) = f((\sum X)+n)$. For $X \subseteq \mathbb{N}$, the statement $[f_X] = c$ is equivalent to ${}^*f_X(\Omega) = c$, which is equivalent to ${}^*f(\sum(X \cup \{\Omega\})) = c$. □

In a heuristical sense, our proof of Theorem 34 seems to suggest that the idempotency requirement might be indispensable for Corollary 30: if the corollary could be proven using weaker assumptions about q, then we would have a non-idempotent ultrafilter proof of Hindman's Theorem, which seems like it would be surprising. But of course, this is not a rigorous proof, and we do not actually know whether there exists any non-idempotent ultrafilter for which Corollary 30 holds.

4.2 Differentiating by $[\mathrm{rm}_\gamma]$

In Theorem 26 we established a deep connection between our finite derivative and the usual derivative from elementary calculus. But the theorem was limited to γ–periodic functions for some irrational γ. At first glance, this seems very limiting. But since $[\![\Delta f]\!]$ only depends on $f|_{\mathbb{N}}$, the following lemma shows that the γ–periodic hypothesis in Theorem 26 does not limit the $\bullet'$–like nature of the derivative from Corollary 30 at all.

Lemma 36 *Let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$. For every $f : \mathbb{R} \to \mathbb{R}$, there exists a γ–periodic function $\hat{f} : \mathbb{R} \to \mathbb{R}$ such that $f|_{\mathbb{N}} = \hat{f}|_{\mathbb{N}}$.*

Proof Define $\hat{f} : \mathbb{R} \to \mathbb{R}$ by

$$\hat{f}(x) = \begin{cases} f(n) & \text{if } x = m\gamma + n \text{ for some } m \in \mathbb{Z}, n \in \mathbb{N}, \\ 0 & \text{in any other case.} \end{cases}$$

Since γ is irrational, there do not exist distinct ways to write $x = m\gamma + n$, thus $\hat{f}(x)$ is well-defined. Clearly $\hat{f}$ has the desired properties. □

Thus if $f : \mathbb{R} \to \mathbb{R}$ then $\Delta[f|_\mathbb{N}] = \Delta[\hat{f}|_\mathbb{N}] = [\![n \mapsto \Delta\hat{f}(n)]\!]$ encodes (by Theorem 26) information not about the derivative of f but rather about the derivative of $\hat{f}$. Since $[f|_\mathbb{N}] = [\hat{f}|_\mathbb{N}]$ this means that $\Delta[f|_\mathbb{N}]$ does indeed encode information about the hyperreal $[f|_\mathbb{N}]$, just not necessarily about f. For example, if $f(x) = x$ for all $x \in \mathbb{R}$, then $\hat{f}$ is exotic and $\Delta[f|_\mathbb{N}]/[\mathrm{rm}_\gamma]$ is far from the derivative $x' = 1$ we might expect. Nonetheless, we can use Theorem 26 to obtain some other derivatives for which the familiar rules of elementary calculus apply.

Definition 37 *For every $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$, for every $f : \mathbb{N} \to \mathbb{R}$, define $D_\gamma f : \subseteq\mathbb{N} \to \mathbb{R}$ by $D_\gamma f(n) = \mathrm{st}(\Delta f(n)/[\mathrm{rm}_\gamma])$ for all n such that $\mathrm{st}(\Delta f(n)/[\mathrm{rm}_\gamma])$ is defined.*

Definition 38 *For every $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$, define $D_\gamma : \subseteq{}^*\mathbb{R} \to {}^*\mathbb{R}$ as follows. For every $[f] \in {}^*\mathbb{R}$ (so $f : \mathbb{N} \to \mathbb{R}$), define $D_\gamma[f] = [n \in \mathbb{N} \mapsto D_\gamma f(n)]$ provided $[n \in \mathbb{N} \mapsto D_\gamma f(n)]$ is defined.*

Proposition 39 *Let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$. The operator $D_\gamma : \subseteq{}^*\mathbb{R} \to {}^*\mathbb{R}$ of Definition 38 is well-defined.*

Proof Suppose $f, g : \mathbb{N} \to \mathbb{R}$ are such that $[f] = [g]$. By Theorem 29, $[\![\Delta f]\!] = [\![\Delta g]\!]$. Thus, $\exists S \in q$ such that $\Delta f(n) = \Delta g(n)$ for all $n \in S$. Thus for all $n \in S$, $\Delta f(n)/[\mathrm{rm}_\gamma] = \Delta g(n)/[\mathrm{rm}_\gamma]$, and $\mathrm{st}(\Delta f(n)/[\mathrm{rm}_\gamma])$ is defined if and only if $\mathrm{st}(\Delta g(n)/[\mathrm{rm}_\gamma])$ is defined. Thus $D_\gamma[f] = D_\gamma[g]$ (and either both sides are defined, or both sides are undefined). □

Theorem 40 *Let $\gamma \in \mathbb{R}^+ \backslash \mathbb{Q}$. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $(-\gamma/2, \gamma/2)$, then $D_\gamma[f \circ \mathrm{rm}_\gamma] = [f' \circ \mathrm{rm}_\gamma]$.*

Proof Let $S = \mathbb{R} \backslash \{(k + \frac{1}{2})\gamma : k \in \mathbb{Z}\}$. Define $\overline{\mathrm{rm}}_\gamma$ the same way rm_γ was defined (Definition 23) except define it for all $x \in S$ instead of only $x \in \mathbb{N}$. Clearly $\overline{\mathrm{rm}}_\gamma|_\mathbb{N} = \mathrm{rm}_\gamma|_\mathbb{N}$ and $\overline{\mathrm{rm}}_\gamma$ is differentiable with $\overline{\mathrm{rm}}'_\gamma(x) = 1$ on S. Since f is differentiable on $(-\gamma/2, \gamma/2) = \overline{\mathrm{rm}}_\gamma(S)$, we can apply the chain rule and see $f(\overline{\mathrm{rm}}_\gamma(x))' = f'(\overline{\mathrm{rm}}_\gamma(x))\overline{\mathrm{rm}}'_\gamma(x) = f'(\overline{\mathrm{rm}}_\gamma(x))$ on S $(*)$. For every $n \in \mathbb{N}$,

$$\begin{aligned}
D_\gamma(f \circ \mathrm{rm}_\gamma)(n) &= \mathrm{st}(\Delta(f \circ \mathrm{rm}_\gamma)(n)/[\mathrm{rm}_\gamma]) && \text{(Definition 37)}\\
&= \mathrm{st}(\Delta(f \circ \overline{\mathrm{rm}}_\gamma)(n)/[\mathrm{rm}_\gamma]) && (\mathrm{rm}_\gamma|_\mathbb{N} = \overline{\mathrm{rm}}_\gamma|_\mathbb{N})\\
&= (f \circ \overline{\mathrm{rm}}_\gamma)'(n) && \text{(Theorem 26)}\\
&= f'(\overline{\mathrm{rm}}_\gamma(n)) && (\text{by } *)\\
&= f'(\mathrm{rm}_\gamma(n)), && (\mathrm{rm}_\gamma|_\mathbb{N} = \overline{\mathrm{rm}}_\gamma|_\mathbb{N})
\end{aligned}$$

so $[n \mapsto D_\gamma(f \circ \mathrm{rm}_\gamma)(n)] = [n \mapsto f'(\mathrm{rm}_\gamma(n))]$, ie, $D_\gamma[f \circ \mathrm{rm}_\gamma] = [f' \circ \mathrm{rm}_\gamma]$. □

For example:

$$D_\gamma(e^{[\mathrm{rm}_\gamma]} + [\mathrm{rm}_\gamma]^3 + \cos 2[\mathrm{rm}_\gamma]) = e^{[\mathrm{rm}_\gamma]} + 3[\mathrm{rm}_\gamma]^2 - 2\sin 2[\mathrm{rm}_\gamma]$$

Thus, D_γ follows the familiar derivative rules from elementary calculus as long as we consider functions not of the continuous variable $x \in \mathbb{R}$, but rather of the discrete variable $\mathrm{rm}_\gamma(n) \in \mathrm{rm}_\gamma(\mathbb{N})$. In a sense, one can "differentiate by $[\mathrm{rm}_\gamma]$"; it might even be tempting to write D_γ as $d/d[\mathrm{rm}_\gamma]$. This is spiritually similar to how the prime numbers play the role of dimensions, and how one differentiates by prime numbers, in Jeffries' paper [10] in the Notices.

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