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Projective sets, intuitionistically

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Abstract: We try to develop intuitionistic descriptive set theory and study 'definable' subsets of Baire space $\mathcal{N} = \omega^\omega$. The logic of our arguments is intuitionistic and we also use L.E.J. Brouwer's Thesis on bars in ω^ω and his continuity axioms. We avoid the operation of taking the complement of a subset of ω^ω as much as possible, as the resulting sets, like negative statements, are not very useful in constructive mathematics.

A subset of ω^ω is *(positively) projective* if it results from a closed or an open subset of $\omega^\omega \times \omega^\omega (= \omega^\omega)$ by a finite number of applications of the two operations of *projection* and *universal projection* or *co-projection*. A subset of ω^ω is $\mathbf{\Sigma}^1_1$ or *analytic* if it is the projection of a closed subset of ω^ω. We give some examples of $\mathbf{\Sigma}^1_1$ subsets of ω^ω like the set of (the codes of) all closed subsets of ω^ω that are *positively uncountable* and also the set of (the codes of) all closed subsets of ω^ω containing an element coding a (positively) infinite subset of ω.

A subset of ω^ω is called *strictly analytic* if it is the projection of a *spread*, ie a closed and *located* subset of ω^ω. Some analytic subsets of ω^ω fail to be strictly analytic. We will see that Brouwer's Thesis on bars in ω^ω proves separation and boundedness theorems for strictly analytic subsets of ω^ω.

A subset of ω^ω is called $\mathbf{\Pi}^1_1$ or *co-analytic* if it is the co-projection of an open subset of $\omega^\omega \times \omega^\omega (= \omega^\omega)$. Most co-analytic sets are *not* the complement of an analytic set. There is no symmetry between analytic and co-analytic sets as there is in classical descriptive set theory. As an example of a $\mathbf{\Pi}^1_1$ set we consider the set of the codes of all closed subsets of ω^ω all of whose members code an *almost-finite* subset of ω.

We also study the set of the codes of closed and located subsets of ω^ω that are *almost-countable*, or, equivalently, *reducible in Cantor's sense*. This set is probably not $\mathbf{\Pi}^1_1$.

Finally, we explain the important fact that the (positive) projective hierarchy *collapses*: every (positively) projective set is $\mathbf{\Sigma}^1_2$ ie the projection of a co-analytic subset of ω^ω.

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1 Introduction

This paper on descriptive set theory is one in a series. We explore the field of study laid bare by *pre-intuitionists*[1] like R. Baire, É. Borel, H. Lebesgue, N. Lusin and M. Souslin, and consider it from L.E.J. Brouwer's *intuitionistic* point of view. In [35], we proved an intuitionistic Borel hierarchy theorem. In [36], we discovered the fine structure of the intuitionistic Borel hierarchy, and, in particular, the fine structure of the class $\mathbf{\Sigma}^0_2$, consisting of the countable unions of closed subsets of ω^ω. In both [35] and [36], the argument is far from classical and essential use is made of Brouwer's Continuity Principle.

We now are going to treat projective sets. Our earlier paper [33] already contains some surprising results on apparently simple analytic and co-analytic subsets of ω^ω.

This introductory section is divided into three parts. In the first part, we briefly present the basic assumptions of intuitionistic analysis and we agree on a number of notations. In the second part, we introduce *intuitionistic descriptive set theory*. The reader may decide to skip these first two parts and use them only if further reading makes it necessary to consult them. In the third part, we describe the further contents of the paper.

1.1 The language and axioms of intuitionistic analysis

The logical constants are used in their intuitionistic sense. A statement $P \vee Q$ is considered proven only if one either has a proof of P or a proof of Q. A statement $\exists x \in V[P(x)]$ is considered proven only if one is able to produce an element x of V with a proof of the fact that x has the property P.

Brouwer not only refined the language of mathematics but also introduced a number of assumptions one should call *axiomatic*. He was of course the first to use them, see [2, 3, 5, 6, 10]. The question how to state and defend them has been further discussed by others, see Heyting [12], Howard–Kreisel [13], Kleene–Vesley [16], Myhill [23], Troelstra [28], Troelstra–van Dalen [29], and Veldman [30, 32, 35, 34, 40]. One finds them below in Sections 1.1.3 (Axioms of Countable Choice), 1.1.6 (Brouwer's Continuity Principle and Axioms of Continuous Choice), 1.1.7 (the Fan Theorem), 1.1.8 (Stumps), 1.1.9 (Bar Induction) and 1.1.10 (the Creating Subject).

[1] Brouwer uses this term in [4, page 140] and [5, page 1].

1.1.1 Finite sequences of natural numbers

ω is the set of the natural numbers. We use $m, n, \ldots, s, t \ldots$ as variables over ω.

$S\colon \omega \to \omega$ is the successor function: $\forall n[S(n) = n + 1]$.

$p\colon \omega \to \omega$ is the function enumerating the primes: $p(0) = 2, p(1) = 3, p(2) = 5, \ldots$.

We code finite sequences of natural numbers by natural numbers: $\langle\ \rangle := 0$ is the (code number of) the empty sequence, and, for all $k > 0$, for all $m_0, m_1, \ldots m_{k-1}$, $\langle m_0, m_1, \ldots, m_{k-1}\rangle := \prod_{i<k} p(i)^{m_i} \cdot p(k-1) - 1$.

$\mathrm{length}(0) := 0$ and, for each $s > 0$, $\mathrm{length}(s) := 1 +$ the largest k such that $p(k)$ divides $s + 1$.

For each s, for each i, if $i < \mathrm{length}(s) - 1$, then $s(i) :=$ the largest m such that $p(i)^m$ divides $s + 1$; if $i = \mathrm{length}(s) - 1$, then $s(i) :=$ the largest m such that $p(i)^{m+1}$ divides $s + 1$; and, if $i \geq \mathrm{length}(s)$, then $s(i) := 0$. Observe that for each s, k, if $\mathrm{length}(s) = k$, then $s = \langle s(0), s(1), \ldots, s(k-1)\rangle$.

For each n,

$$\omega^n := \{s \mid \mathit{length}(s) = n\} \text{ and } [\omega]^n := \{s \in \omega^n \mid \forall i[i + 1 < n \to s(i) < s(i + 1)]\}.$$

$[\omega]^{<\omega} := \bigcup_n [\omega]^n$.

For all s and t, $s * t$ is the number u satisfying: $\mathrm{length}(u) = \mathrm{length}(s) + \mathrm{length}(t)$, $\forall i < \mathrm{length}(s)[u(i) = s(i)]$ and $\forall j < \mathrm{length}(t)[u(\mathrm{length}(s) + j) = t(j)]$.

For all s, n such that $n \leq \mathrm{length}(s)$, $\overline{s}(n) := \overline{s}n := \langle s(0), s(1), \ldots, s(n-1)\rangle$.

For all s, t:

$$s \sqsubseteq t \leftrightarrow \exists u[t = s * u] \qquad s \sqsubset t \leftrightarrow (s \sqsubseteq t \ \wedge\ s \neq t) \qquad s \sqsupset t \leftrightarrow t \sqsubset s$$

$$s <_{lex} t \leftrightarrow \exists n[n < \mathrm{length}(s) \ \wedge\ \overline{s}n \sqsubset t \ \wedge\ s(n) < t(n)]$$

$$s \perp t \leftrightarrow s \# t \leftrightarrow (s <_{lex} t \ \vee\ t <_{lex} s) \qquad s <_{KB} t \leftrightarrow (t \sqsubset s \ \vee\ s <_{lex} t)$$

$<_{KB}$ is a linear ordering of ω, the *Kleene–Brouwer-ordering*, also called the *Lusin–Sierpinski-ordering*; see Kechris [14, Section 2.G, page 11].

For all s, i, s^i is the number u satisfying: $\mathrm{length}(u) =$the least k such that $\langle i\rangle * k \geq \mathrm{length}(s)$ and $\forall j < \mathrm{length}(u)[u(j) = s(\langle i\rangle * j)]$. Note that, for each i, $\langle\ \rangle^i = \langle\ \rangle$. Note also that, for each p and i, $\langle p\rangle^i = \langle\ \rangle$.

For all n and m, $J(n, m) := (\langle n\rangle * m) - 1$. For each n, $K(n)$ and $L(n)$ are the numbers satisfying $n = J\big(K(n), L(n)\big)$.

For all s,t such that $\text{length}(s) = \text{length}(t)$, $\ulcorner s,t \urcorner$ is the number u satisfying $\text{length}(u) = \text{length}(s)$ and $\forall i < \text{length}(s) \left[u(i) = J\big(s(i),t(i)\big)\right]$.

For each u, u_I and u_{II} are the elements s,t of ω such that $u = \ulcorner s,t \urcorner$, ie $\text{length}(u_I) = \text{length}(u_{II}) = \text{length}(u)$ and $\forall i < \text{length}(u)[u_I(i) = K\big(u(i)\big) \ \wedge \ u_{II}(i) = L\big(u(i)\big)]$.

For each u, $u_{I,I} := (u_I)_I$, $u_{I,II} := (u_I)_{II}$, $u_{II,I} := (u_{II})_I$ and $u_{II,II} := (u_{II})_{II}$.

Bin $:= 2^{<\omega} := \{s \mid \forall i < \text{length}(s)[s(i) = 0 \ \vee \ s(i) = 1]\}$ is the set of the codes of finite binary sequences.

For each m, $\text{Bin}_m := \{s \in \text{Bin} \mid \text{length}(s) = m\}$.

For all $R \subseteq \omega$, $\forall m \forall n[mRn \leftrightarrow J(m,n) \in R]$.

For all $A,B \subseteq \omega$, $A \times B := \{J(m,n) \mid m \in A, n \in B\}$.

For all $A \subseteq \omega$, $n = \mu p[A(p)]$ if and only if $A(n)$ and $\forall p < n[\neg A(p)]$.

1.1.2 Infinite sequences of natural numbers

Baire space ω^ω is the set of all infinite sequences of natural numbers. We use $\alpha, \beta, \ldots, \varphi, \psi, \ldots \sigma, \tau, \ldots$ as variables over ω^ω.

An element of ω^ω is a function from ω to ω, and, given α, n we denote the result of applying α to n by $\alpha(n)$.

$[\omega]^\omega := \{\zeta \mid \forall n[\zeta(n) < \zeta(n+1)]\}$.

For every $X \subseteq \omega$, $X^\omega := \{\alpha \mid \forall n[\alpha(n) \in X]\}$.

For all α and β, $\alpha \circ \beta$ is the element γ of ω^ω satisfying $\forall n \left[\gamma(n) = \alpha\big(\beta(n)\big)\right]$.

For all α and t, $\alpha \circ t$ is the number u satisfying: $\text{length}(u) = \text{length}(t)$ and $\forall n < \text{length}(t) \left[u(n) = \alpha\big(t(n)\big)\right]$. In particular, for each t, $S \circ t$ is the number u satisfying: $\text{length}(u) = \text{length}(t)$ and $\forall n < \text{length}(t)\, [u(n) = t(n) + 1]$.

For all α and β, $\alpha \# \beta \leftrightarrow \alpha \perp \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$ and $\alpha = \beta \leftrightarrow \forall n\, [\alpha(n) = \beta(n)]$. It is a well-known fact that the relation #, called *apartness*, is *co-transitive*, ie for all α, β, γ, if $\alpha \# \beta$, then either $\alpha \# \gamma$ or $\gamma \# \beta$.

For each s, for each α, $s * \alpha$ is the element γ of ω^ω such that $\forall i < \text{length}(s)[\gamma(i) = s(i)]$ and $\forall i[\gamma\big(\mathit{length}(s) + i\big) = \alpha(i)]$.

For each s, for each $\mathcal{X} \subseteq \omega^\omega$, $s * \mathcal{X} := \{s * \alpha \mid \alpha \in \mathcal{X}\}$.

For each α, for each n, $\overline{\alpha}(n) := \overline{\alpha}n := \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$. $\overline{\alpha}(0) := \overline{\alpha}0 := \langle\ \rangle = 0$.

For all s and α, $s \sqsubset \alpha \leftrightarrow \exists n[s = \overline{\alpha}n]$ and $s \perp \alpha \leftrightarrow \alpha \perp s \leftrightarrow \neg(s \sqsubset \alpha)$. Note that for all a and b, for all γ, if $a \perp b$ then either $a \perp \gamma$ or $\gamma \perp b$.

For all s, $\omega^\omega \cap s := \{\alpha \mid s \sqsubset \alpha\}$.

For each m, $\underline{m}$ is the element γ of ω^ω such that $\forall n[\gamma(n) = m]$.

For all α and i, α^i is the element γ of ω^ω such that $\forall n[\gamma(n) = \alpha(\langle i\rangle * n)]$.

For all α, m and n, $\alpha^{m,n} := (\alpha^m)^n$. Note that for all m, n and p, $\alpha^{m,n}(p) = \alpha\big(\langle m, n\rangle * p\big)$.

For all α, for all s, ${}^s\alpha$ is the element γ of ω^ω such that $\forall n[\gamma(n) = \alpha(s * n)]$. Note that ${}^{\langle m\rangle}\alpha = \alpha^m$.

For every $\mathcal{X} \subseteq \omega^\omega$, $\mathcal{X}^\omega := \{\alpha \mid \forall n[\alpha^n \in \mathcal{X}]\}$.

For all α, β, $\ulcorner \alpha, \beta \urcorner$ is the element γ of ω^ω such that $\forall n[\gamma(n) = J\big(\alpha(n), \beta(n)\big)]$.

For each γ, γ_I and γ_{II} are the elements α, β of ω^ω such that $\gamma = \ulcorner \alpha, \beta \urcorner$, ie $\forall n[\gamma_I(n) = K\big(\gamma(n)\big) \ \wedge\ \gamma_{II}(n) = L\big(\gamma(n)\big)]$.

For each α, $\alpha_{I,I} := (\alpha_I)_I$, $\alpha_{I,II} := (\alpha_I)_{II}$, $\alpha_{II,I} := (\alpha_{II})_I$ and $\alpha_{II,II} := (\alpha_{II})_{II}$.

For all $\mathcal{R} \subseteq \omega^\omega$, $\forall \alpha \forall \beta[\alpha \mathcal{R} \beta \leftrightarrow \ulcorner \alpha, \beta \urcorner \in \mathcal{R}]$.

For all $\mathcal{R} \subseteq \omega^\omega$, $\forall \alpha \forall n[\alpha \mathcal{R} n \leftrightarrow n \mathcal{R} \alpha \leftrightarrow \langle n\rangle * \alpha \in \mathcal{R}]$.

For all $\mathcal{A} \subseteq \omega^\omega, B \subseteq \omega$, $\mathcal{A} \times B := B \times \mathcal{A} := \{\langle n\rangle * \alpha \mid \alpha \in \mathcal{A}, n \in B\}$.

For all $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$, $\mathcal{A} \times \mathcal{B}; = \{\ulcorner \alpha, \beta \urcorner \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$.

For all $\mathcal{A} \subseteq \omega^\omega$, for all n, $\mathcal{A} \upharpoonright n := \{\alpha \mid \langle n\rangle * \alpha \in \mathcal{A}\}$.

For all $\mathcal{X} \subseteq \omega^\omega$, for all n, $\mathcal{X}_n := \{\alpha \mid \langle n\rangle * \alpha \in \mathcal{X}\}$.

An infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \ldots$ of subsets of ω^ω is the *same* as the set $\mathcal{X} = \{\langle n\rangle * \alpha \mid n \in \omega, \alpha \in \mathcal{X}_n\}$.

For all $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$:

$$\mathcal{A} \subseteq \mathcal{B} \leftrightarrow \forall \alpha[\alpha \in \mathcal{A} \to \alpha \in \mathcal{B}]$$
$$\mathcal{A} \subsetneq \mathcal{B} \leftrightarrow \big(\mathcal{A} \subseteq \mathcal{B} \ \wedge\ \neg(\mathcal{B} \subseteq \mathcal{A})\big)$$
$$\mathcal{A} = \mathcal{B} \leftrightarrow (\mathcal{A} \subseteq \mathcal{B} \ \wedge\ \mathcal{B} \subseteq \mathcal{A})$$
$$\mathcal{A} \neq \mathcal{B} \leftrightarrow \neg(\mathcal{A} = \mathcal{B})$$

For all $\mathcal{X}_0, \mathcal{X}_1 \subseteq \omega^\omega$, $\mathcal{X}_0 \# \mathcal{X}_1 \leftrightarrow \forall\alpha[\forall i < 2[\alpha^i \in \mathcal{X}_i] \rightarrow \alpha^0 \# \alpha^1]$.

If $\mathcal{X}_0 \# \mathcal{X}_1$, then $\mathcal{X}_0 \cap \mathcal{X}_1 = \emptyset$, but the converse may fail to be true.

For every infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \dots$ of subsets of ω^ω, we define: $\#_n(\mathcal{X}_n) \leftrightarrow \forall\alpha\left[\forall n[\alpha^n \in \mathcal{X}_n] \rightarrow \exists i \exists j[\alpha^i \# \alpha^j]\right]$. If $\#_n(\mathcal{X}_n)$ then $\bigcap_n \mathcal{X}_n = \emptyset$, but the converse may fail to be true.

Cantor space $\mathcal{C} := 2^\omega := \{\alpha \mid \forall n[\alpha(n) < 2]\}$. (We use both notations.)

For each α,

$D_\alpha := \{n \mid \alpha(n) \neq 0\}$ is the subset of ω decided by α, and
$E_\alpha := \{m \mid \exists n[\alpha(n) = m + 1]\}$ is the subset of ω enumerated by α.

For each s,

$D_s := \{n < \text{length}(s) \mid s(n) \neq 0\}$ and
$E_s := \{m \mid \exists n < \text{length}(s)[s(n) = m + 1]\}$.

Note that for each α, $D_\alpha = \bigcup_n D_{\overline{\alpha}n}$ and $E_\alpha = \bigcup_n E_{\overline{\alpha}n}$.

For each $X \subseteq \omega$,

X is *inhabited* if and only if $\exists n[n \in X]$,
X is *decidable* if and only if $\exists\alpha[X = D_\alpha]$, and
X is *enumerable* if and only if $\exists\alpha[X = E_\alpha]$.

For each α, $T_\alpha := \{s \mid \forall t \sqsubset s[\alpha(t) = 0]\}$. T_α is called the tree determined by α. Note that $\forall\alpha[0 = \langle\,\rangle \in T_\alpha]$.

For all α and β, for all γ, we define:

$$\gamma : \alpha \leq^* \beta \leftrightarrow \big(\forall s[s \in T_\alpha \rightarrow \gamma(s) \in T_\beta] \ \wedge\ \forall s \forall t[s \sqsubset t \rightarrow \gamma(s) \sqsubset \gamma(t)]\big)$$

and

$$\gamma : \alpha <^* \beta \leftrightarrow \big(\forall s[s \in T_\alpha \rightarrow \gamma(s) \in T_\beta] \ \wedge\ \forall s \forall t[s \sqsubset t \rightarrow \gamma(s) \sqsubset \gamma(t)] \ \wedge\ \gamma(\langle\,\rangle) \neq \langle\,\rangle\big)$$

For all α, β, we define $\alpha <^* \beta \leftrightarrow \exists\gamma[\gamma : \alpha <^* \beta]$ and $\alpha \leq^* \beta \leftrightarrow \exists\gamma[\gamma : \alpha \leq^* \beta]$.

For each δ, $En_\delta := \{\delta^n \mid n \in \omega\}$ is the subset of ω^ω *enumerated by* δ.

1.1.3 Axioms of Countable Choice

First Axiom of Countable Choice:

AC$_{0,0}$: For all $R \subseteq \omega \times \omega$, if $\forall m \exists n\,[mRn]$, then $\exists\alpha\forall m[mR\alpha(m)]$.

Second Axiom of Countable Choice:

AC$_{0,1}$: For all $\mathcal{R} \subseteq \omega^\omega \times \omega$, if $\forall m \exists\alpha\,[m\mathcal{R}\alpha]$, then $\exists\alpha\forall m\,[mR\alpha^m]$.

1.1.4 Open and closed subsets of ω^ω, and spreads

For each β, $\mathcal{G}_\beta := \{\alpha \mid \exists n[\beta(\overline{\alpha}n) \neq 0]\}$ and $\mathcal{F}_\beta := \{\alpha \mid \forall n[\beta(\overline{\alpha}n) = 0]\}$.

The pair of sets $(\mathcal{G}_\beta, \mathcal{F}_\beta)$ is called a *complementary pair of rank 1*.

For each $\mathcal{X} \subseteq \omega^\omega$:

$\mathcal{X}$ is *open* or $\mathbf{\Sigma}^0_1$ if and only if $\exists\beta[\mathcal{X} = \mathcal{G}_\beta]$.
$\mathcal{X}$ is *closed* or $\mathbf{\Pi}^0_1$ if and only if $\exists\beta[\mathcal{X} = \mathcal{F}_\beta]$.
$\mathcal{X}$ is *inhabited* if and only if $\exists\gamma[\gamma \in \mathcal{X}]$.
$\mathcal{X}$ is *located* if and only if $\exists\gamma[D_\gamma = \{s \mid \exists\alpha \in \mathcal{X}[s \sqsubset \alpha]\}]$.
$\mathcal{X}$ is *semi-located* if and only if $\exists\gamma[E_\gamma = \{s \mid \exists\alpha \in \mathcal{X}[s \sqsubset \alpha]\}]$.

For every $\mathcal{X} \subseteq \omega^\omega$, $\overline{\mathcal{X}} := \{\alpha \mid \forall n \exists\gamma \in \mathcal{X}[\overline{\alpha}n \sqsubset \gamma]\}$. $\overline{\mathcal{X}}$ is called *the closure of* $\mathcal{X}$. $\overline{\mathcal{X}}$ is not necessarily $\mathbf{\Pi}^0_1$.[2]

One easily proves that for every $\mathcal{X} \subseteq \omega^\omega$, $\overline{\overline{\mathcal{X}}} = \overline{\mathcal{X}}$, and $\mathcal{X}$ is (semi-)located if and only if $\overline{\mathcal{X}}$ is (semi-)located.

$\mathcal{F} \subseteq \omega^\omega$ is a *spread* if and only if $\overline{\mathcal{F}} = \mathcal{F}$ and $\mathcal{F}$ is located.

For each β, we define: β *is a spread-law*, Spr(β), if and only if $\beta \in 2^\omega$ and $\forall s\left[\beta(s) = 0 \leftrightarrow \exists n[\beta(s * \langle n \rangle) = 0]\right]$. One easily proves that $\mathcal{F} \subseteq \omega^\omega$ is a spread if and only if $\exists\beta[\mathrm{Spr}(\beta) \wedge \mathcal{F} = \mathcal{F}_\beta]$.

Note that for all β, if Spr(β), then $\mathcal{F}_\beta = \emptyset$ if and only if $\beta(0) = 1$ if and only if $\beta = \underline{1}$, and $\exists\gamma[\gamma \in \mathcal{F}_\beta]$ ($\mathcal{F}_\beta$ is *inhabited*) if and only if $\beta(0) = 0$. The empty set $\emptyset$ thus is a spread, and one may decide, for every spread $\mathcal{F}$, either $\mathcal{F} = \emptyset$ or $\exists\gamma[\gamma \in \mathcal{F}]$.

Assume Spr(β) and $\beta(c) = 0$. We define: $\mathcal{F}_\beta \cap c := \{\gamma \in \mathcal{F}_\beta \mid c \sqsubset \gamma\}$. Note that $\mathcal{F}_\beta \cap c$ itself is a spread.

For each β, we define β *is a perfect-spread-law*, *Pfspr*(β), if and only if:

$$\mathrm{Spr}(\beta) \wedge \beta(0) = 0 \wedge \forall s\big[\beta(s) = 0 \rightarrow \exists t \exists u[s \sqsubset t \wedge s \sqsubset u \wedge t \perp u \wedge \beta(t) = \beta(u) = 0]\big]$$

$\mathcal{F} \subseteq \omega^\omega$ is a *perfect spread* if and only if $\exists\beta[\mathit{Pfspr}(\beta) \wedge \mathcal{F} = \mathcal{F}_\beta]$.

[2] One may see this as follows. For every α, define $\mathcal{Y}_\alpha := \{\gamma \mid \gamma = \underline{0} \wedge \alpha \# \underline{0}\}$ and note that $\mathcal{Y}_\alpha := \overline{\mathcal{Y}_\alpha}$. Assume that every $\mathcal{Y}_\alpha$ is $\mathbf{\Pi}^0_1$. Then $\forall\alpha\exists\beta\forall\gamma[\gamma \in \mathcal{Y}_\alpha \rightarrow \gamma \in \mathcal{F}_\beta]$, and therefore $\forall\alpha\exists\beta[\alpha \# \underline{0} \leftrightarrow \forall n[\beta(\overline{\underline{0}}n) = 0]]$. Using Axiom $\mathbf{AC}_{1,1}$ (see Section 1.1.6) one may derive a contradiction.

1.1.5 Continuous functions

For all φ, α, m, we define: φ *maps* α *onto* m, $\varphi\colon\ \alpha \mapsto m$, if and only if

$$\exists n\big[\varphi(\overline{\alpha}n) = m+1 \ \wedge\ \forall i < n[\varphi(\overline{\alpha}i) = 0]\big]$$

If $\exists m[\varphi\colon\ \alpha \mapsto m]$, we let $\varphi(\alpha)$ denote the unique m such that $\varphi\colon\ \alpha \mapsto m$.

For every $\mathcal{X} \subseteq \omega^\omega$, for all φ, we define: φ *codes a function from* $\mathcal{X}$ *to* ω, $\varphi\colon\ \mathcal{X} \to \omega$, if and only if $\forall \alpha \in \mathcal{X} \exists m[\varphi\colon\ \alpha \mapsto m]$.

$\varphi(\mathcal{X}) := \{m \mid \exists \alpha \in \mathcal{X}[\varphi\colon\ \alpha \mapsto m]\} = \{\varphi(\alpha) \mid \alpha \in \mathcal{X}\}$.

For every $\mathcal{X} \subseteq \omega^\omega$, $\omega^{\mathcal{X}} := \{\varphi \mid \varphi\colon\ \mathcal{X} \to \omega\}$.

For all φ, α, β, we define: φ *maps* α *onto* β, $\varphi\colon\ \alpha \mapsto \beta$, if and only if $\varphi(0) = \varphi(\langle\ \rangle) = 0$ and $\forall n[\varphi^n\colon\ \alpha \mapsto \beta(n)]$.

If $\exists \beta[\varphi\colon\ \alpha \mapsto \beta]$, we let $\varphi|\alpha$ denote the unique β such that $\varphi\colon\ \alpha \mapsto \beta$.

For every $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, for all φ, we define: φ *maps* $\mathcal{X}$ *into* $\mathcal{Y}$, $\varphi\colon\ \mathcal{X} \to \mathcal{Y}$, if and only if $\forall \alpha \in \mathcal{X} \exists \beta \in \mathcal{Y}[\varphi\colon\ \alpha \mapsto \beta]$.

$\varphi|\mathcal{X} := \{\beta \mid \exists \alpha \in \mathcal{X}[\varphi : \alpha \mapsto \beta]\} = \{\varphi|\alpha \mid \alpha \in \mathcal{X}\}$.

For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, for all φ, we define: φ *embeds* $\mathcal{X}$ *into* $\mathcal{Y}$, $\varphi\colon\ \mathcal{X} \rightarrowtail \mathcal{Y}$, if and only if $\varphi\colon\ \mathcal{X} \to \mathcal{Y}$ and $\forall \alpha \in \mathcal{X} \forall \beta \in \mathcal{X}[\alpha \mathrel{\#} \beta \to \varphi|\alpha \mathrel{\#} \varphi|\beta]$. $\mathrm{Emb}(\mathcal{X}, \mathcal{Y}) := \{\varphi \mid \varphi : : \mathcal{X} \rightarrowtail \mathcal{Y}\}$. For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, $\mathcal{X}$ *embeds into* $\mathcal{Y}$ if and only if $\exists \varphi[\varphi :: \ \mathcal{X} \rightarrowtail \mathcal{Y}]$.

For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, for all φ, we define: φ *is a surjective mapping from* $\mathcal{X}$ *onto* $\mathcal{Y}$, $\varphi\colon\ \mathcal{X} \twoheadrightarrow \mathcal{Y}$, if and only if $\varphi\colon\ \mathcal{X} \to \mathcal{Y}$ and $\forall \beta \in \mathcal{Y} \exists \alpha \in \mathcal{X}[\varphi|\alpha = \beta]$. $\mathcal{X}$ *maps onto* $\mathcal{Y}$ if and only if there exists a surjective mapping from $\mathcal{X}$ onto $\mathcal{Y}$.

For all $\mathcal{X} \subseteq \omega^\omega$, $(\omega^\omega)^{\mathcal{X}} := \{\varphi \mid \varphi\colon\ \mathcal{X} \to \omega^\omega\}$.

Note that $(\omega^\omega)^{(\omega^\omega)} = \{\varphi \mid \varphi\colon\ \omega^\omega \to \omega^\omega\} = \{\varphi \in \omega^{(\omega^\omega)} \mid \varphi(0) = 0\}$.

For all φ, s we let $\varphi|s$ be the largest number t such that $\mathrm{length}(t) \le \mathrm{length}(s)$ and $\forall j < \mathrm{length}(t) \exists p \le \mathrm{length}(s)[\varphi^j(\overline{s}p) = t(j) + 1 \ \wedge\ \forall i < p[\varphi^j(\overline{s}i) = 0]]$. Note that $\forall \varphi \forall s[\mathrm{length}(\varphi|s) \le \mathrm{length}(s)]$. Note that $\forall \varphi \forall \alpha \forall \beta[\varphi\colon\ \alpha \mapsto \beta \leftrightarrow \forall n \exists m[\overline{\beta}n \sqsubseteq \varphi|\overline{\alpha}m]]$.

For all φ, ψ in $(\omega^\omega)^{(\omega^\omega)}$, we define $\varphi \star \psi$ in $(\omega^\omega)^{(\omega^\omega)}$ such that, for all n, for all s, for all p, $\varphi^n(s) = p+1$ if and only if $n < \mathrm{length}\big(\varphi|(\psi|s)\big)$ and $\big(\varphi|(\psi|s)\big)(n) = p+1$. Note that $\forall \alpha[(\varphi \star \psi)|\alpha = \varphi|(\psi|\alpha)]$.

Let $\mathcal{F} \subseteq \omega^\omega$ be an inhabited spread. Find β such that $\mathrm{Spr}(\beta)$ and $\mathcal{F} = \mathcal{F}_\beta$. Now define $\rho\colon\ \omega^\omega \to \omega^\omega$ such that, for all α and m,

if $\beta(\overline{\rho|\alpha}m * \langle\alpha(m)\rangle) = 0$, then $(\rho|\alpha)(m) = \alpha(m)$, and
if $\beta(\overline{\rho|\alpha}m * \langle\alpha(m)\rangle) \neq 0$, then $(\rho|\alpha)(m) = \mu k[\beta(\overline{\rho|\alpha}m * \langle k\rangle) = 0]$.

ρ is called the *canonical retraction* of ω^ω onto $\mathcal{F}$. Note that $\forall\alpha[\rho|\alpha \in \mathcal{F}]$, $\forall\alpha[\rho|\alpha \mathrel{\#} \alpha \leftrightarrow \exists m[\beta(\overline{\alpha}m) \neq 0]]$, and $\forall\alpha \in \mathcal{F}[\rho|\alpha = \alpha]$.

Assume that $\mathrm{Spr}(\beta)$ and $B \subseteq \omega$ is a *bar* in $\mathcal{F}_\beta$, ie $\forall\gamma \in \mathcal{F}_\beta\exists n[\overline{\gamma}n \in B]$. Define $B' := B \cup \{s \mid \beta(s) \neq 0\}$. Then B' is a bar in ω^ω, ie $\forall\gamma\exists n[\overline{\gamma}n \in B']$. In order to see this, we use the canonical retraction ρ of ω^ω onto $\mathcal{F}_\beta$. Let γ be given. Find n such that $\overline{\rho|\gamma}n \in B$. Either $\overline{\rho|\gamma}n = \overline{\gamma}n$ and $\overline{\gamma}n \in B$; or, $\overline{\rho|\gamma}n \neq \overline{\gamma}n$ and $\exists m \leq n[\beta(\overline{\gamma}m) \neq 0]$. In both cases, $\overline{\gamma}n \in B'$.

1.1.6 Brouwer's Continuity Principle and the Axioms of Continuous Choice

Brouwer's Continuity Principle:

> **BCP**: For every spread $\mathcal{F}$, for every $\mathcal{R} \subseteq \mathcal{F} \times \omega$, if $\forall\alpha \in \mathcal{F}\exists n[\alpha\mathcal{R}n]$, then $\forall\alpha \in \mathcal{F}\exists m\exists n\forall\beta \in \mathcal{F}[\overline{\alpha}m \sqsubset \beta \rightarrow \beta\mathcal{R}n]$.

First Axiom of Continuous Choice:

> $\mathbf{AC}_{1,0}$: For every spread $\mathcal{F}$, for all $\mathcal{R} \subseteq \mathcal{F} \times \omega$, if $\forall\alpha \in \mathcal{F}\exists n[\alpha\mathcal{R}n]$, then $\exists\varphi[\varphi\colon \mathcal{F} \rightarrow \omega \ \wedge \ \forall\alpha \in \mathcal{F}[\alpha\mathcal{R}\varphi(\alpha)]]$.

Second Axiom of Continuous Choice:

> $\mathbf{AC}_{1,1}$: For every spread $\mathcal{F}$, for all $\mathcal{R} \subseteq \mathcal{F} \times \omega^\omega$, if $\forall\alpha \in \mathcal{F}\exists\beta[\alpha\mathcal{R}\beta]$, then $\exists\varphi[\varphi\colon \mathcal{F} \rightarrow \omega^\omega \ \wedge \ \forall\alpha \in \mathcal{F}[\alpha\mathcal{R}\varphi|\alpha]]$.

1.1.7 The Fan Theorem

For all $\mathcal{X} \subseteq \omega^\omega$, for all $B \subseteq \omega$, we define: $\mathrm{Bar}_{\mathcal{X}}(B) \leftrightarrow \forall\gamma \in \mathcal{X}\exists n[\overline{\gamma}n \in B]$.

For each β, we define: $\mathrm{Fan}(\beta) \leftrightarrow \big(\mathrm{Spr}(\beta) \wedge \forall s\exists n\forall m > n[\beta(s*\langle m\rangle) \neq 0]\big)$. If $\mathrm{Fan}(\beta)$, one says: β *is a fan-law*. $\mathcal{F} \subseteq \omega^\omega$ is a *fan* if and only if $\exists\beta[\mathrm{Fan}(\beta) \ \wedge \ \mathcal{F} = \mathcal{F}_\beta]$.

The Fan Theorem:

> For every fan $\mathcal{F} \subseteq \omega^\omega$ and every $B \subseteq \omega$, if $\mathrm{Bar}_{\mathcal{F}}(B)$ then $\exists s[D_s \subseteq B \ \wedge \ \mathrm{Bar}_{\mathcal{F}}(D_s)]$.

The *restricted* Fan Theorem:

> **FT**: For each fan $\mathcal{F} \subseteq \omega^\omega$ and every δ, if $\mathrm{Bar}_{\mathcal{F}}(D_\delta)$ then $\exists n\left[\mathrm{Bar}_{\mathcal{F}}(D_{\overline{\delta}n})\right]$.

1.1.8 Stumps

Axiom on the existence of the set of stumps:

STP: $\mathcal{STP}$ is a subset of 2^ω such that:[3]

(i) $1^* := \underline{1} \in \mathcal{STP}$;

(ii) for all σ in 2^ω, if

(a) $\sigma(0) = 0$, and

(b) for all n, $\sigma^n \in \mathcal{STP}$,

then $\sigma \in \mathcal{STP}$; and

(iii) for all $\mathcal{Q} \subseteq \mathcal{STP}$, if

(a) $1^* \in \mathcal{Q}$, and

(b) for all σ in $\mathcal{STP}$, if $\sigma(0) = 0$ and, for all n, $\sigma^n \in \mathcal{Q}$, then $\sigma \in \mathcal{Q}$;

then $\mathcal{STP} = \mathcal{Q}$.

The elements of $\mathcal{STP}$ are called *stumps*.

For each β in ω^ω, we define β^* in 2^ω by: for all s, $\beta^*(s) = 1$ if $\beta(s) = 0$ and $\beta^*(s) = 0$ if $\beta(s) \neq 0$.

1^* is/codes *the empty stump*. For each σ in $\mathcal{STP}$, $\sigma = 1^*$ if and only if $\sigma(0) = 1$.

For each $\sigma \neq 1^*$ in $\mathcal{STP}$, for each n, σ^n is a stump, *the n–th immediate substump of* σ. (Also, for each n, $(1^*)^n = 1^*$ is a stump.)

We define relations $<, \leq$ on $\mathcal{STP}$ by simultaneous transfinite induction: for all σ, τ in $\mathcal{STP}$,

(i) $\sigma \leq \tau \leftrightarrow \big(\sigma \neq 1^* \rightarrow \forall n[\sigma^n < \tau]\big)$, and

(ii) $\sigma < \tau \leftrightarrow \big(\tau \neq 1^* \ \wedge \ \exists n[\sigma \leq \tau^n]\big)$.

Using the axiom **STP** one proves the following.

Principle of Induction on $\mathcal{STP}$:

For all $\mathcal{Q} \subseteq \mathcal{STP}$, if $\forall \sigma \in \mathcal{STP}[\forall \tau \in \mathcal{STP}[\tau < \sigma \rightarrow \tau \in \mathcal{Q}] \rightarrow \sigma \in \mathcal{Q}]$ then $\mathcal{STP} = \mathcal{Q}$.

One may prove:[4] for all σ, τ in $\mathcal{STP}$, $\sigma \leq \tau$ if and only if $\sigma \leq^* \tau$.

For all α, we let $S^*(\alpha)$ be the element β of ω^ω such that $\beta(0) = 0$ and $\forall n[\beta^n = \alpha]$. $S^*(\alpha)$ is called the *successor of* α.

Note that $\forall \alpha \in \mathcal{STP}[S^*(\alpha) \in \mathcal{STP}]$.

[3] There is a small difference between the set $\mathcal{STP}$ as it is introduced here and the sets called **Stp** in Veldman [35, 36], respectively.

[4] The relation $\leq^*$ has been defined at the end of Section 1.1.2.

1.1.9 Bar Induction

Brouwer's *Thesis on bars in ω^ω*:

> **BT**: For each $B \subseteq \omega$, if $\mathrm{Bar}_{\omega^\omega}(B)$, then $\exists \sigma \in \mathcal{STP}[\mathrm{Bar}_{\omega^\omega}(B \cap T_\sigma)]$.

Recall, from Section 1.1.2, that $T_\sigma = \{s \mid \forall t \sqsubset s[\sigma(t) = 0]\}$.

$B \subseteq \omega$ is *monotone* if and only if $\forall s \forall n[s \in B \to s * \langle n \rangle \in B]$.

$C \subseteq \omega$ is *inductive* if and only if $\forall s[\forall n[s * \langle n \rangle \in C] \to s \in C]$.

BT proves the following.

Principle of Bar Induction:

> **BI**: For all $B, C \subseteq \omega$, if $\mathrm{Bar}_{\omega^\omega}(B)$, $B \subseteq C$, and C is monotone and inductive, then $0 = \langle\ \rangle \in C$.

Assume $\mathrm{Spr}(\beta)$. We define:

$B \subseteq \omega$ is *monotone within* $\{s \mid \beta(s) = 0\}$ if and only if:

$$\forall s[\big(\beta(s) = 0 \ \wedge \ s \in B\big) \to \forall n[\beta(s * \langle n \rangle) = 0 \to s * \langle n \rangle \in B]]$$

$C \subseteq \omega$ is *inductive within* $\{s \mid \beta(s) = 0\}$ if and only if:

$$\forall s[\big(\beta(s) = 0 \ \wedge \ \forall n[\beta(s * \langle n \rangle) = 0 \to s * \langle n \rangle \in C]\big) \to s \in C]$$

BI admits the following extension:

> **BI**, extended to spreads: For all β such that $\mathrm{Spr}(\beta)$ and $\beta(0) = 0$, for all $B, C \subseteq \omega$, if $\mathrm{Bar}_{\mathcal{F}_\beta}(B)$, $B \subseteq C$, and C is monotone and inductive within $\{s \mid \beta(s) = 0\}$, then $0 = \langle\ \rangle \in C$.

Using **BI** and calling to aid the canonical retraction ρ of ω^ω onto $\mathcal{F}_\beta$, one easily proves this extended form of **BI** from **BI** itself.

1.1.10 The creating subject

The *Brouwer–Kripke axiom*, also called: *Kripke's scheme*[5] is the following statement:

> **KS**: Given any *definite* mathematical proposition P, one may build α such that $P \leftrightarrow \exists n[\alpha(n) \neq 0]$.

[5] Kripke's scheme plays a role in the proof of Theorem 2.11 and it is mentioned in Section 5.

The idea underlying the axiom is that, once P is given, I may, identifying myself with the creating subject, start thinking upon it, and the truth of P will consist in my finding a proof of P, at some point of time. Time is supposed to be divided into stages that are numbered by natural numbers. For each n, $\alpha(n) \neq 0$ if and only if, at stage n, I possess a proof of P.

This is a rather wild idea, actually too wild, if we allow P to be a statement about an object that is itself unfinished, like an infinite sequence $\beta = \beta(0), \beta(1), \dots$ of natural numbers I am creating step by step, freely choosing each one of its values. At any stage, only finitely many values will have been determined, and the statement: $\forall n[\beta(n) = 0]$, provided it has not been violated already, is unprovable at any stage, although possibly true *'in the end'*.

We therefore require P to be *definite*:[6] P should not be about unfinished objects. In a formal context, one forbids that the formula corresponding to the proposition contain a free variable over elements of Baire space.

If one do not take this precaution, **KS** leads to a contradiction with $\mathbf{AC}_{1,1}$, as was first observed by J. Myhill, see Myhill [23]:

Assume $\forall\beta\exists\alpha[\beta = \underline{0} \leftrightarrow \exists n[\alpha(n) \neq 0]]$. Applying $\mathbf{AC}_{1,1}$, find $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\beta[\beta = \underline{0} \leftrightarrow \exists n[(\varphi|\beta)(n) \neq 0]]$. Then find n such that $(\varphi|\underline{0})(n) \neq 0$. Finally, find m such that $\forall\beta[\overline{\underline{0}}m \sqsubset \beta \to (\varphi|\beta)(n) = (\varphi|\underline{0})(n)]$ and conclude that $\forall\beta[\overline{\underline{0}}m \sqsubset \beta \to \beta = \underline{0}]$, a contradiction.

Myhill wanted to give up $\mathbf{AC}_{1,1}$ because of this argument. Johan de Iongh proposed the restriction of **KS** to definite propositions; see Gielen–de Swart–Veldman [11, § 3].

Theorem 1.1 (Consequences of **KS**)

(i) *If $X \subseteq \omega$ is definite, then $\exists\delta[X = E_\delta]$, ie X is enumerable.*

(ii) *If $\mathcal{X} \subseteq \omega^\omega$ is definite, then $\exists\delta[E_\delta = \{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\}]$, ie $\mathcal{X}$ is semi-located.*

Proof (i) Let $X \subseteq \omega$ be definite. By **KS**, $\forall n\exists\alpha[n \in X \leftrightarrow \exists m[\alpha(m) \neq 0]]$. Using $\mathbf{AC}_{0,1}$, find α such that $\forall n[n \in X \leftrightarrow \exists m[\alpha^n(m) \neq 0]]$. Now define δ such that $\delta(0) = 0$ and, for all n, m, if $\alpha^n(m) \neq 0$ then $\delta(\langle n\rangle * m) = n + 1$; if not, then $\delta(\langle n\rangle * m) = 0$, and note that $X = E_\delta$.

(ii) Let $\mathcal{X} \subseteq \omega^\omega$ be definite. The set $\{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\}$ also is definite, and one may apply (i). □

[6] The term *'definite'* will also be applied to (other) mathematical objects. The infinite sequence $\underline{0}$, for instance, deserves to be called definite.

1.1.11 Semi-classical principles

The Limited Principle of Omniscience:

$$\mathbf{LPO}\colon\ \forall\alpha[\exists n[\alpha(n)\neq 0]\ \vee\ \forall n[\alpha(n)=0]].$$

The Lesser Limited Principle of Omniscience:[7]

$$\mathbf{LLPO}\colon\ \forall\alpha[\forall m[2m\neq \mu p[\alpha(p)\neq 0]\ \vee\ \forall m[2m+1\neq \mu p[\alpha(p)\neq 0]].$$

Note that $\mathbf{LPO}\rightarrow\mathbf{LLPO}$: Let α be given. Define β such that $\forall n[\beta(n)\neq 0\leftrightarrow 2n+1=\mu p[\alpha(p)\neq 0]]$. If $\exists n[\beta(n)\neq 0]]$, then $\forall m[2m\neq \mu p[\alpha(p)\neq 0]$, and if $\forall n[\beta(n)=0]$, then $\forall m[2m+1\neq \mu p[\alpha(p)\neq 0]]$.

LLPO and **BCP** together give a contradiction: assuming both, find p such that $\forall\alpha[\overline{0}p\sqsubset\alpha\rightarrow\forall m[2m\neq\mu p[\alpha(p)\neq 0]]$ or $\forall\alpha[\overline{0}p\sqsubset\alpha\rightarrow\forall m[2m{+}1\neq\mu p[\alpha(p)\neq 0]]$. The sequences $\overline{0}(2p)*\underline{1}$ and $\overline{0}(2p+1)*\underline{1}$ show that both alternatives are false.

Markov's Principle,

$$\mathbf{MP}\colon\ \forall\alpha[\neg\neg\exists n[\alpha(n)\neq 0]\rightarrow\exists n[\alpha(n)\neq 0]]$$

has been defended by Markov for algorithmically computable α only.

1.2 Descriptive set theory

Information on classical descriptive set theory may be found in Lusin [17], Moschovakis [22], Kechris [14] and Srivastava [27]. Some results on the borderline of classical and intuitionistic descriptive set theory may be found in Moschovakis [19] and [21].

1.2.1 Some basic notions

For all $\mathcal{X},\mathcal{Y}\subseteq\omega^\omega$, for all $\varphi\colon\ \omega^\omega\rightarrow\omega^\omega$, we define: φ *reduces* $\mathcal{X}$ *to* $\mathcal{Y}$ if and only if $\forall\alpha[\alpha\in\mathcal{X}\leftrightarrow\varphi|\alpha\in\mathcal{Y}]$. We define: $\mathcal{X}$ *reduces to* $\mathcal{Y}$, $\mathcal{X}\preceq\mathcal{Y}$, if and only if there exists $\varphi\colon\ \omega^\omega\rightarrow\omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{Y}$. For all $\mathcal{X}_0,\mathcal{X}_1,\mathcal{Y}_0,\mathcal{Y}_1\subseteq\omega^\omega$, we define: $(\mathcal{X}_0,\mathcal{X}_1)$ *simultaneously reduces to* $(\mathcal{Y}_0,\mathcal{Y}_1)$, $(\mathcal{X}_0,\mathcal{X}_1)\preceq(\mathcal{Y}_0,\mathcal{Y}_1)$, if and only if there exists $\varphi\colon\ \omega^\omega\rightarrow\omega^\omega$ reducing $\mathcal{X}_0$ to $\mathcal{Y}_0$ and also $\mathcal{X}_1$ to $\mathcal{Y}_1$.

Let $\mathfrak{K}$ be a class of subsets of ω^ω.

[7] **LPO** and **LLPO** were introduced by E. Bishop, as special cases of the principle of the excluded middle $X\vee\neg X$. If one reads well-known theorems constructively, many of them turn out to be equivalent to one of these 'principles'. From a constructive point of view, these 'principles' are, of course, totally wrong.

Assume $\mathcal{X} \subseteq \omega^\omega$. We often say *'$\mathcal{X}$ is $\mathfrak{K}$'* for *'the set $\mathcal{X}$ belongs to the class $\mathfrak{K}$'* .

We define $\mathcal{X} \subseteq \omega^\omega$ is $\mathfrak{K}$*–complete* if and only if $\mathfrak{K}$ is the class of all $\mathcal{Y} \subseteq \omega^\omega$ reducing to $\mathcal{X}$, and we define $\mathcal{X} \subseteq \omega^\omega$ is $\mathfrak{K}$*–universal* if and only if $\mathfrak{K}$ is the class of all sets of the form $\mathcal{X} \upharpoonright \alpha$, for some α in ω^ω.

Note that if $\mathcal{X}$ is $\mathfrak{K}$–universal then $\mathcal{X}$ is $\mathfrak{K}$–complete.

1.2.2 Open sets and closed sets

$\mathbf{\Sigma}^0_1 := \{\mathcal{G}_\beta \mid \beta \in \omega^\omega\}$ and $\mathbf{\Pi}^0_1 := \{\mathcal{F}_\beta \mid \beta \in \omega^\omega\}$.

$\mathcal{E}_1 := \{\alpha \mid \exists n[\alpha(n) \neq 0]\} = \{\alpha \mid \alpha \# \underline{0}\}$ and $\mathcal{A}_1 := \{\alpha \mid \forall n[\alpha(n) = 0]\} = \{\underline{0}\}$. $\mathcal{E}_1$ is $\mathbf{\Sigma}^0_1$–complete and $\mathcal{A}_1$ is $\mathbf{\Pi}^0_1$–complete.

$\mathcal{US}_1 := \{\alpha \mid \alpha_{II} \in \mathcal{G}_{\alpha_I}\} = \{\alpha \mid \exists n[\alpha_I(\overline{\alpha_{II}}n) \neq 0]\}$ and $\mathcal{UP}_1 := \{\alpha \mid \alpha_{II} \in \mathcal{F}_{\alpha_I}\} = \{\alpha \mid \forall n[\alpha_I(\overline{\alpha_{II}}n) = 0]\}$. $\mathcal{US}_1$ is $\mathbf{\Sigma}^0_1$–universal and $\mathcal{UP}_1$ is $\mathbf{\Pi}^0_1$–universal.

1.2.3 Borel sets of finite rank

For each $m > 0$, for each β, we define $\mathcal{G}^m_\beta, \mathcal{F}^m_\beta \subseteq \omega^\omega$ by induction. $\mathcal{G}^1_\beta := \mathcal{G}_\beta$ and $\mathcal{F}^1_\beta := \mathcal{F}_\beta$; and, for each $m > 0$, $\mathcal{G}^{m+1}_\beta = \bigcup_n \mathcal{F}^m_{\beta^n}$ and $\mathcal{F}^{m+1}_\beta = \bigcap_n \mathcal{G}^m_{\beta^n}$.

For each $m > 0$, for each β, the pair of sets $(\mathcal{G}^m_\beta, \mathcal{F}^m_\beta)$ is called a *complementary pair of (positively) Borel sets of rank m*.

For each $m > 0$, $\mathbf{\Sigma}^0_m := \{\mathcal{G}^m_\beta \mid \beta \in \omega^\omega\}$ and $\mathbf{\Pi}^0_m := \{\mathcal{F}^m_\beta \mid \beta \in \omega^\omega\}$.

For each $m > 0$, we define $\mathcal{E}_m, \mathcal{A}_m \subseteq \omega^\omega$ by induction. $\mathcal{E}_1, \mathcal{A}_1$ were defined in Section 1.2.2. For each $m > 0$, $\mathcal{E}_{m+1} := \{\alpha \mid \exists n[\alpha^n \in \mathcal{A}_m]\}$ and $\mathcal{A}_{m+1} := \{\alpha \mid \forall n[\alpha^n \in \mathcal{E}_m]\}$.

For each $m > 0$:

$\mathcal{E}_m$ is $\mathbf{\Sigma}^0_m$–complete and $\mathcal{A}_m$ is $\mathbf{\Pi}^0_m$–complete.
$(\mathcal{E}_m, \mathcal{A}_m)$ is a complementary pair of rank m.
$\mathcal{US}_m := \{\alpha \mid \alpha_{II} \in \mathcal{G}^m_{\alpha_I}\}$ and $\mathcal{UP}_m := \{\alpha \mid \alpha_{II} \in \mathcal{F}^m_{\alpha_I}\}$.
$\mathcal{US}_m$ is $\mathbf{\Sigma}^0_m$–universal and $\mathcal{UP}_m$ is $\mathbf{\Pi}^0_m$–universal.
$(\mathcal{US}_m, \mathcal{UP}_m)$ is a complementary pair of rank m.

1.2.4 Borel sets in general

The set $\mathcal{HRS}$ of the *hereditarily repetitive stumps* is defined inductively: for each stump σ, $\sigma \in \mathcal{HRS} \leftrightarrow \big(\sigma(0) = 0 \rightarrow \forall n[\sigma^n \in \mathcal{HRS} \ \wedge \ \forall m \exists n > m[\sigma^n = \sigma^m]\big)$.

For each σ in $\mathcal{HRS}$, for each β, we define $\mathcal{G}^\sigma_\beta, \mathcal{F}^\sigma_\beta \subseteq \omega^\omega$ by induction: if $\sigma = 1^*$, then $\mathcal{G}^\sigma_\beta = \mathcal{G}_\beta$ and $\mathcal{F}^\sigma_\beta = \mathcal{F}_\beta$; and, if $\sigma \neq 1^*$, then $\mathcal{G}^\sigma_\beta := \bigcup_n \mathcal{F}^{\sigma^n}_{\beta^n}$ and $\mathcal{F}^\sigma_\beta := \bigcap_n \mathcal{G}^{\sigma^n}_{\beta^n}$.

Note that for each σ in $\mathcal{HRS}$, for each β, $\mathcal{G}^\sigma_\beta \ \# \ \mathcal{F}^\sigma_\beta$. The pair of sets $(\mathcal{G}^\sigma_\beta, \mathcal{F}^\sigma_\beta)$ is called a *complementary pair of (positively) Borel sets of rank* σ.

For each σ in $\mathcal{HRS}$, $\mathbf{\Sigma}^0_\sigma := \{\mathcal{G}^\sigma_\beta \mid \beta \in \omega^\omega\}$ and $\mathbf{\Pi}^0_\sigma := \{\mathcal{F}^\sigma_\beta \mid \beta \in \omega^\omega\}$.

For each σ in $\mathcal{HRS}$, we define $\mathcal{E}_\sigma, \mathcal{A}_\sigma \subseteq \omega^\omega$ by induction: if $\sigma = 1^*$, then $\mathcal{E}_\sigma := \mathcal{E}_1$ and $\mathcal{A}_\sigma := \mathcal{A}_1$; and, if $\sigma \neq 1^*$, then $\mathcal{E}_\sigma := \{\alpha \mid \exists n[\alpha^n \in \mathcal{A}_{\sigma^n}]\}$ and $\mathcal{A}_\sigma := \{\alpha \mid \forall n[\alpha^n \in \mathcal{E}_{\sigma^n}]\}$. For each σ in $\mathcal{HRS}$, $\mathcal{E}_\sigma$ is $\mathbf{\Sigma}^0_\sigma$–complete and $\mathcal{A}_\sigma$ is $\mathbf{\Pi}^0_\sigma$–complete and $(\mathcal{E}_\sigma, \mathcal{A}_\sigma)$ is a complementary pair of rank σ.

For each σ in $\mathcal{HRS}$, $\mathcal{US}_\sigma := \{\alpha \mid \alpha_{II} \in \mathcal{G}^\sigma_{\alpha_I}\}$ and $\mathcal{UP}_\sigma := \{\alpha \mid \alpha_{II} \in \mathcal{F}^\sigma_{\alpha_I}\}$. For each σ in $\mathcal{HRS}$, $\mathcal{US}_\sigma$ is $\mathbf{\Sigma}^0_\sigma$–universal, $\mathcal{UP}_\sigma$ is $\mathbf{\Pi}^0_\sigma$–universal and $(\mathcal{US}_\sigma, \mathcal{UP}_\sigma)$ is a complementary pair of rank σ.

The function $S^* : \omega^\omega \rightarrow \omega^\omega$ has been defined in Section 1.1.8. Note that $\forall \sigma \in \mathcal{HRS}[S^*(\sigma) \in \mathcal{HRS}]$.

Define $1^* := \underline{1}$ and, for all m, $(m+1)^* = S^*(m^*)$. Note that for all $m > 0$, $\mathbf{\Sigma}^0_m = \mathbf{\Sigma}^0_{m^*}$ and $\mathcal{E}_m = \mathcal{E}_{m^*}$ and $\mathbf{\Pi}^0_m = \mathbf{\Pi}^0_{m^*}$ and $\mathcal{A}_m = \mathcal{A}_{m^*}$, ...

$\mathfrak{Borel} := \{\mathcal{G}^\sigma_\beta \mid \sigma \in \mathcal{HRS}, \beta \in \omega^\omega\}$.

The following is proven in Veldman [35, Theorems 4.9, 7.9, 7.10].

Theorem 1.2 (Borel Hierarchy Theorem)

(i) *For all* σ, τ *in* $\mathcal{HRS}$*, if* $\sigma < \tau$*, then* $\mathcal{E}_\sigma, \mathcal{A}_\sigma$ *reduce to both* $\mathcal{E}_\tau$ *and* $\mathcal{A}_\tau$*.*

(ii) (Not using **BCP**): *For all* σ *in* $\mathcal{HRS}$*:*

$$\forall \varphi \colon \omega^\omega \rightarrow \omega^\omega \exists \alpha[(\alpha \in \mathcal{E}_\sigma \leftrightarrow \varphi|\alpha \in \mathcal{E}_\sigma) \ \wedge \ (\alpha \in \mathcal{A}_\sigma \leftrightarrow \varphi|\alpha \in \mathcal{A}_\sigma)]$$

(iii) (Using **BCP**)*: For all* σ *in* $\mathcal{HRS}$*:*
$\forall \varphi \colon \omega^\omega \rightarrow \omega^\omega[\varphi|\mathcal{E}_\sigma \subseteq \mathcal{A}_\sigma \rightarrow \exists \alpha[\alpha \in \mathcal{A}_\sigma \ \wedge \ \varphi|\alpha \in \mathcal{A}_\sigma]]$ *and*
$\forall \varphi \colon \omega^\omega \rightarrow \omega^\omega[\varphi|\mathcal{A}_\sigma \subseteq \mathcal{E}_\sigma \rightarrow \exists \alpha[\alpha \in \mathcal{E}_\sigma \ \wedge \ \varphi|\alpha \in \mathcal{E}_\sigma]]$*; or, equivalently,*
for all $\mathcal{X}$ *in* $\mathbf{\Pi}^0_\sigma$*, if* $\mathcal{E}_\sigma \subseteq \mathcal{X}$ *then* $\exists \alpha \in \mathcal{A}_\sigma[\alpha \in \mathcal{X}]$*, and*
for all $\mathcal{X}$ *in* $\mathbf{\Sigma}^0_\sigma$*, if* $\mathcal{A}_\sigma \subseteq \mathcal{X}$*, then* $\exists \alpha \in \mathcal{E}_\sigma[\alpha \in \mathcal{X}]$

Theorem 1.2(iii) implies that $\mathcal{E}_\sigma$ positively fails to be $\mathbf{\Pi}^0_\sigma$ and $\mathcal{A}_\sigma$ positively fails to be $\mathbf{\Sigma}^0_\sigma$. For the intuitionistic mathematician, Theorem 1.2(ii) does *not* establish the hierarchy, as, for almost every σ in $\mathcal{HRS}$, he is unable to prove: $\neg \exists \alpha[\alpha \notin \mathcal{E}_\sigma \ \wedge \ \alpha \notin \mathcal{A}_\sigma]$.

1.2.5 On disjunction

For every infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots$ of subsets of ω^ω, we define:

$$\mathbb{D}_n(\mathcal{X}_n) := \{\alpha \mid \exists n[\alpha^n \in \mathcal{X}_n]\} \text{ and } \mathbb{C}_n(\mathcal{X}_n) := \{\alpha \mid \forall n[\alpha^n \in \mathcal{X}_n]\}$$

$\mathbb{D}_n(\mathcal{X}_n), \mathbb{C}_n(\mathcal{X}_n)$ are the *disjunction* and the *conjunction* of the infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots$, respectively.

Note that, for each σ in $\mathcal{HRS}$, if $\sigma \neq 1^*$, then $\mathcal{E}_\sigma = \mathbb{D}_n(\mathcal{A}_{\sigma^n})$ and $\mathcal{A}_\sigma = \mathbb{C}_n(\mathcal{E}_{\sigma^n})$.

For all $\mathcal{X}_0, \mathcal{X}_1 \subseteq \omega^\omega$, we define:

$$\mathbb{D}(\mathcal{X}_0, \mathcal{X}_1) := \{\alpha \mid \exists i < 2[\alpha^i \in \mathcal{X}_i]\} \text{ and } \mathbb{D}^2(\mathcal{X}_0) := \mathbb{D}(\mathcal{X}_0, \mathcal{X}_0)$$

$\mathbb{D}(\mathcal{X}_0, \mathcal{X}_1)$ is called the *disjunction of* $\mathcal{X}_0$ *and* $\mathcal{X}_1$.

Note that $\mathcal{Z} \subseteq \omega^\omega$ reduces to $\mathbb{D}(\mathcal{X}_0, \mathcal{X}_1)$ if and only if there exist $\mathcal{Z}_0, \mathcal{Z}_1$ such that $\mathcal{Z} = \mathcal{Z}_0 \cup \mathcal{Z}_1$ and $\forall i < 2[\mathcal{Z}_i \preceq \mathcal{X}_i]$.

The following result is not difficult but very important.

Theorem 1.3 $\neg\big(\overline{\mathbb{D}^2(\mathcal{A}_1)} \subseteq \mathbb{D}^2(\mathcal{A}_1)\big)$.

Proof Assume $\overline{\mathbb{D}^2(\mathcal{A}_1)} \subseteq \mathbb{D}^2(\mathcal{A}_1) = \{\alpha \mid \alpha^0 = \underline{0} \ \vee \ \alpha^1 = \underline{0}\}$. Note that $\overline{\mathbb{D}^2(\mathcal{A}_1)}$ is a spread containing $\underline{0}$. Applying **BCP**, find m such that either $\forall \alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\overline{0}m \sqsubset \alpha \to \alpha^0 = \underline{0}]$; or $\forall \alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\overline{0}m \sqsubset \alpha \to \alpha^1 = \underline{0}]$. Both alternatives are false. □

Theorem 1.3 shows that the union of two $\mathbf{\Pi}^0_1$–sets is not always $\mathbf{\Pi}^0_1$: $\mathbb{D}(\mathcal{A}_1, \mathcal{A}_1)$ does not reduce to $\mathcal{A}_1$. This result admits of a vast extension.

Assume that $\sigma \in \mathcal{HRS}$. Define, as in Veldman [35, page 39]:

$$\sigma \textit{ is weakly comparative } \leftrightarrow \big(\sigma(0) = 0 \to \forall m \forall n \exists p[\sigma^m \leq \sigma^p \ \wedge \ \sigma^n \leq \sigma^p]\big)$$

The following result is [35, Theorem 8.8].

Theorem 1.4 (The persisting difficulty of disjunction) *For each σ in $\mathcal{HRS}$, if σ is weakly comparative, then $\mathbb{D}(\mathcal{A}_1, \mathcal{A}_\sigma)$ does not reduce to $\mathcal{A}_{S^*(\sigma)}$.*

1.2.6 Projective sets

For each $\mathcal{X} \subseteq \omega^\omega$, $Ex(\mathcal{X}) := \{\alpha \mid \exists\beta[\ulcorner\alpha,\beta\urcorner \in \mathcal{X}]\} = \{\alpha_I \mid \alpha \in \mathcal{X}\}$ and $Un(\mathcal{X}) := \{\alpha \mid \forall\beta[\ulcorner\alpha,\beta\urcorner \in \mathcal{X}]\}$. $Ex(\mathcal{X})$ is called the *projection* of $\mathcal{X}$, and $Un(\mathcal{X})$ is called the *co-projection* of $\mathcal{X}$.

For each β, $\mathcal{EF}_\beta := Ex(\mathcal{F}_\beta)$ and $\mathcal{UG}_\beta := Un(\mathcal{G}_\beta)$.

$\mathbf{\Sigma}^1_1 := \{\mathcal{EF}_\beta \mid \beta \in \omega^\omega\}$ is the class of the *analytic* sets and $\mathbf{\Pi}^1_1 := \{\mathcal{UG}_\beta \mid \beta \in \omega^\omega\}$ is the class of the *co-analytic* sets. $\mathbf{\Sigma}^1_1$ thus consists of the projections of the closed subsets of ω^ω and $\mathbf{\Pi}^1_1$ consists of the co-projections of the open subsets of ω^ω.

For each β, $\big(\mathcal{EF}_\beta, \mathcal{UG}_\beta\big)$ is called a *complementary* $\big(\mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1\big)$*–pair.*

$\mathcal{US}^1_1 := \{\alpha \mid \alpha_{II} \in \mathcal{EF}_{\alpha_I}\}$ and $\mathcal{UP}^1_1 := \{\alpha \mid \alpha_{II} \in \mathcal{UG}_{\alpha_I}\}$. We shall prove that $\mathcal{US}^1_1$ is $\mathbf{\Sigma}^1_1$–universal; see Theorem 2.1(i). We shall prove that $\mathcal{UP}^1_1$ is $\mathbf{\Pi}^1_1$–universal; see Theorem 4.1(i).

$\mathcal{E}^1_1 := \{\alpha \mid \exists\gamma\forall n[\alpha(\overline{\gamma}n) = 0]\}$ and $\mathcal{A}^1_1 := \{\alpha \mid \forall\gamma\exists n[\alpha(\overline{\gamma}n) \neq 0]\}$. We shall prove that $\mathcal{E}^1_1$ is $\mathbf{\Sigma}^1_1$–complete, see Theorem 2.1(ii). We shall prove that $\mathcal{A}^1_1$ is $\mathbf{\Pi}^1_1$–complete, see Theorem 4.1(ii).

For certain purposes, the class $\mathbf{\Sigma}^1_1$ is too wide. We therefore introduce the class $\mathbf{\Sigma}^{1,*}_1 := \{\mathcal{EF}_\beta \mid \mathrm{Spr}(\beta)\}$ of the *strictly analytic* sets. $\mathbf{\Sigma}^{1,*}_1$ consists of the projections of the subsets of ω^ω that are both closed and located.

For certain purposes, the class $\mathbf{\Pi}^1_1$ is too narrow. We therefore introduce the class:

$$\mathbf{\Pi}^{1+}_1 := \{\mathrm{Un}(\mathcal{X}) \mid \mathcal{X} \in \mathfrak{Borel}\}$$

$\mathbf{\Pi}^{1+}_1$ is the class of the *broadly co-analytic* sets.

For each β, $\mathcal{UEF}_\beta := \mathrm{Un}(\mathcal{EF}_\beta)$ and $\mathcal{EUG}_\beta := Ex(\mathcal{UG}_\beta)$.

$\mathbf{\Pi}^1_2 := \{\mathcal{UEF}_\beta \mid \beta \in \omega^\omega\}$ and $\mathbf{\Sigma}^1_2 := \{\mathcal{EUG}_\beta \mid \beta \in \omega^\omega\}$.

For each β, $(\mathcal{EUG}_\beta,\ \mathcal{UEF}_\beta)$ is a *complementary* $(\mathbf{\Sigma}^1_2,\ \mathbf{\Pi}^1_2)$*–pair.*

$\mathcal{E}^1_2 := \{\alpha \mid \exists\delta\forall\gamma\forall n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) = 0]\}$ and $\mathcal{A}^1_2 := \{\alpha \mid \forall\delta\exists\gamma\exists n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) \neq 0]\}$. We shall prove that $\mathcal{E}^1_2$ is $\mathbf{\Sigma}^1_2$–complete, and that $\mathcal{A}^1_2$ is $\mathbf{\Pi}^1_2$–complete; see Theorem 7.1(ii).

$\mathcal{US}^1_2 := \{\alpha \mid \alpha_{II} \in \mathcal{EUG}_{\alpha_I}\}$ and $\mathcal{UP}^1_1 := \{\alpha \mid \alpha_{II} \in \mathcal{UEF}_{\alpha_I}\}$. We shall prove that $\mathcal{US}^1_2$ is $\mathbf{\Sigma}^1_2$–universal, and that $\mathcal{UP}^1_2$ is $\mathbf{\Pi}^1_2$–universal; see Theorem 7.1(i).

1.2.7 Perhaps

For every $\mathcal{X} \subseteq \omega^\omega$, $\mathsf{Perhaps}(\mathcal{X}) := \{\alpha \mid \exists \beta \in \mathcal{X}[\alpha \# \beta \rightarrow \alpha \in \mathcal{X}]\}$.

If $\mathcal{X}$ is inhabited, then $\mathcal{X} \subseteq \mathsf{Perhaps}(\mathcal{X})$.

$\mathcal{X} \subseteq \omega^\omega$ is *perhapsive* if and only if $\mathcal{X} = \mathsf{Perhaps}(\mathcal{X})$.

In Waaldijk [41], perhapsive subsets of ω^ω are called *weakly stable*. [41] is the birthplace of the notion of 'perhapsity'. The notion has been studied further in Veldman [31, 33, 36].

Theorem 1.5

(i) *For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, if $\mathcal{X} \preceq \mathcal{Y}$ and $\mathcal{Y}$ is perhapsive, then $\mathcal{X}$ is perhapsive.*

(ii) *$\mathbb{D}^2(\mathcal{A}_1)$ and $\mathcal{E}_2$ are not perhapsive.*

(iii) *$\mathcal{A}_2$ is perhapsive and $\neg\big(\mathbb{D}^2(\mathcal{A}_1) \preceq \mathcal{A}_2\big)$.*

(iv) *$\mathcal{A}_1^1$ is perhapsive and $\neg\big(\mathbb{D}^2(\mathcal{A}_1) \preceq \mathcal{A}_1^1\big)$.*

Proof (i) Let $\mathcal{X}, \mathcal{Y}, \varphi$ be given such that $\varphi \colon \omega^\omega \rightarrow \omega^\omega$ reduces $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{Y}$ is perhapsive. Let α, β be given such that $\beta \in \mathcal{X}$ and $\alpha \# \beta \rightarrow \alpha \in \mathcal{X}$. Note that $\varphi|\beta \in \mathcal{Y}$; and, if $\varphi|\alpha \# \varphi|\beta$, then $\alpha \# \beta$, $\alpha \in \mathcal{X}$, and $\varphi|\alpha \in \mathcal{Y}$. As $\mathcal{Y}$ is perhapsive, we conclude that $\varphi|\alpha \in \mathcal{Y}$ and $\alpha \in \mathcal{X}$.

We thus see $\forall \alpha[\exists \beta \in \mathcal{X}[\alpha \# \beta \rightarrow \alpha \in \mathcal{X}] \rightarrow \alpha \in \mathcal{X}]$, ie $\mathcal{X}$ is perhapsive.

(ii) Let α in $\overline{\mathbb{D}^2(\mathcal{A}_1)}$ be given. Define α_0 such that $(\alpha_0)^0 = \underline{0}$ and $\forall j[\neg \exists n[j = \langle 0 \rangle * n] \rightarrow \alpha_0(j) = \alpha(j)]$. Note that $\alpha_0 \in \mathbb{D}^2(\mathcal{A}_1)$ and, if $\alpha \# \alpha_0$, then $\alpha^1 = \underline{0}$ and $\alpha \in \mathbb{D}^2(\mathcal{A}_1)$. We thus see that $\forall \alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\alpha \in \mathsf{Perhaps}\big(\mathbb{D}^2(\mathcal{A}_1)\big)]$. Using Theorem 1.3, we conclude that $\mathbb{D}^2(\mathcal{A}_1) \neq \mathsf{Perhaps}\big(\mathbb{D}^2(\mathcal{A}_1)\big)$, ie $\mathbb{D}^2(\mathcal{A}_1)$ is not perhapsive.

As $\mathbb{D}^2(\mathcal{A}_1)$ is $\mathbf{\Sigma}^0_2$ and reduces to $\mathcal{E}_2$, also $\mathcal{E}_2$ is not perhapsive, by (i).

(iii) Let α, β be given such that $\beta \in \mathcal{A}_2$ and $\alpha \# \beta \rightarrow \alpha \in \mathcal{A}_2$. Let m be given. Find n such that $\beta^m(n) \neq 0$. Either $\alpha^m(n) = \beta^m(n) \neq 0$; or $\alpha \# \beta$, $\alpha \in \mathcal{A}_2$, and $\exists p[\alpha^m(p) \neq 0]$. We thus see that $\forall m \exists p[\alpha^m(p) \neq 0]$, ie $\alpha \in \mathcal{A}_2$. Conclude that $\forall \alpha[\exists \beta \in \mathcal{A}_2[\alpha \# \beta \rightarrow \alpha \in \mathcal{A}_2] \rightarrow \alpha \in \mathcal{A}_2]$, ie $\mathcal{A}_2$ is perhapsive.

It follows that $\mathbb{D}^2(\mathcal{A}_1)$ does not reduce to $\mathcal{A}_2$, by (ii) and (i).

(iv) Let α, β be given such that $\beta \in \mathcal{A}_1^1$ and $\alpha \# \beta \rightarrow \alpha \in \mathcal{A}_1^1$. Let γ be given. Find n such that $\beta(\overline{\gamma}n) \neq 0$. Either $\alpha(\overline{\gamma}n) = \beta(\overline{\gamma}n) \neq 0$; or $\alpha \# \beta$, $\alpha \in \mathcal{A}_1^1$, and $\exists p[\alpha(\overline{\gamma}p) \neq 0]$. We thus see that $\forall \gamma \exists p[\alpha(\overline{\gamma}p) \neq 0]$, ie $\alpha \in \mathcal{A}_1^1$. Conclude that $\forall \alpha[\exists \beta \in \mathcal{A}_1^1[\alpha \# \beta \rightarrow \alpha \in \mathcal{A}_1^1] \rightarrow \alpha \in \mathcal{A}_1^1]$, ie $\mathcal{A}_1^1$ is perhapsive.

It follows that $\mathbb{D}^2(\mathcal{A}_1)$ does not reduce to $\mathcal{A}_1^1$, by (ii) and (i).

Note that, as $\mathcal{A}_2 \preceq \mathcal{A}_1^1$, (iii) in fact follows from (iv). $\square$

1.3 The main results of this paper

Apart from this introductory section, the paper contains sections numbered 2 to 7.

In Section 2, we first establish some properties of the class $\mathbf{\Sigma}^1_1$.

We then prove that the set

$$\mathcal{IF} := \{\alpha \mid \exists \beta \in (T_\alpha)^\omega \forall n[\beta(n+1) <_{KB} \beta(n)]\}$$

ie the set of all α such that the tree $T_\alpha := \{s \mid \forall t \sqsubset s[\alpha(t) = 0]\}$ is (positively) *ill-founded with respect to the Kleene-Brouwer-ordering* $<_{KB}$, is $\mathbf{\Sigma}^1_1$ but not $\mathbf{\Sigma}^1_1$–complete. We also prove that the set

$$\mathcal{UNC} := \{\beta \mid \forall \alpha \exists \gamma \in \mathcal{F}_\beta \forall n[\gamma \# \alpha^n]\}$$

of codes of the *positively uncountable* closed subsets of ω^ω is $\mathbf{\Sigma}^1_1$–complete, and that the same holds for the set

$$\mathsf{Share}^*(\mathcal{INF}) := \{\beta \mid \mathrm{Spr}(\beta) \ \wedge \ \exists \alpha \in \mathcal{F}_\beta \forall m \exists n > m[\alpha(n) \neq 0]\}$$

of codes of the spreads that contain an element α such that $D_\alpha = \{n \mid \alpha(n) \neq 0\}$ is an infinite subset of ω.

The final subsection of Section 2 is devoted to the class $\mathbf{\Sigma}^{1*}_1$ of the *strictly analytic* subsets of ω^ω. $\mathbf{\Sigma}^{1*}_1$ is a proper subclass of $\mathbf{\Sigma}^1_1$ and is lacking some of the useful closure properties of $\mathbf{\Sigma}^1_1$.

In Section 3, we give intuitionistic proofs of the Separation Theorems due to Lusin and Novikov. Novikov's Theorem is the stronger one and says that, given any infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \dots$ of $\mathbf{\Sigma}^{1*}_1$ subsets of ω^ω such that $\#_n(\mathcal{X}_n)$, (that is, in a constructively strong sense: $\bigcap_n(\mathcal{X}_n) = \emptyset$), one may find an infinite sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of Borel subsets of ω^ω such that $\forall n[\mathcal{X}_n \subseteq \mathcal{B}_n]$ and $\#_n(\mathcal{B}_n)$. The proofs use Brouwer's Thesis on bars in ω^ω.

We give an intuitionistic proof of Lusin's result that the range of a strongly one-to-one function from a spread into ω^ω is (positively) Borel. It is shown that the positively Borel set $\mathbb{D}^2(\mathcal{A}_1) := \{\alpha \mid \alpha^0 = \underline{0} \ \vee \ \alpha^1 = \underline{0}\}$ positively fails to be the range of a strongly one-to-one function from a spread into ω^ω.

In Section 4, we establish some properties of the class $\mathbf{\Pi}^1_1$ of the co-analytic subsets of ω^ω. We prove that the set

$$\mathcal{WF} := \{\alpha \mid \forall \beta \in (T_\alpha)^\omega \exists n[\beta(n) \leq_{KB} \beta(n+1)]\}$$

ie the set of all α such that the tree T_α is *well-founded with respect to* $<_{KB}$, coincides with $\mathcal{A}_1^1$ and thus is $\mathbf{\Pi}_1^1$–complete. The proof uses Brouwer's Thesis on bars in ω^ω. We also show that the set

$$\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN}) := \{\beta \mid \mathrm{Spr}(\beta) \ \wedge \ \forall\alpha \in \mathcal{F}_\beta \forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]\}$$

consisting of the codes of all spreads all of whose elements α have the property that D_α is an *almost-finite* subset of ω, is $\mathbf{\Pi}_1^1$–complete. We then prove that the set

$$\mathcal{E}_1^1! := \{\alpha \mid \exists\gamma[\forall n[\alpha(\overline{\gamma}n) = 0] \ \wedge \forall\delta[\delta \# \gamma \to \exists n[\alpha(\overline{\delta}n) \neq 0]]\}$$

consisting of those α that admit exactly one path γ is not $\mathbf{\Pi}_1^1$ although, in classical descriptive set theory, $\mathcal{E}_1^1!$ is $\mathbf{\Pi}_1^1$–complete. It remains true that every $\mathbf{\Pi}_1^1$ set reduces to $\mathcal{E}_1^1!$.

In Section 5, we prove that there exist $\mathbf{\Sigma}_1^1$ sets that positively fail to be $\mathbf{\Pi}_1^1$ and $\mathbf{\Pi}_1^1$ sets that positively fail to be $\mathbf{\Sigma}_1^{1*}$. We use Kripke's scheme **KS** in order to prove that there are $\mathbf{\Pi}_1^1$ sets that are not $\mathbf{\Sigma}_1^1$. We also see that some $\mathbf{\Sigma}_1^1$ sets positively fail to be (positively) Borel and that some $\mathbf{\Pi}_1^1$ sets are not (positively) Borel. Using Brouwer's Thesis on bars in ω^ω, we prove one half of Souslin's Theorem: $\mathbf{\Sigma}_1^{1*} \cap \mathbf{\Pi}_1^1 \subseteq \mathfrak{Borel}$. The converse statement fails intuitionistically.

In Section 6, we study the set

$$\mathcal{ALMOST}^*\mathcal{COUNT} := \{\beta \mid \mathrm{Spr}(\beta) \ \wedge \ \exists\delta\forall\gamma \in \mathcal{F}_\beta \forall\alpha\exists n[\overline{\gamma}\alpha(n) = \overline{\delta^n}\alpha(n)]\}$$

of codes of *almost-countable spreads*. This set is $\mathbf{\Sigma}_2^1$ and probably not $\mathbf{\Pi}_1^1$, although we have no proof of the latter fact. We prove, again using Brouwer's Thesis on bars in ω^ω, that the almost-countable spreads are just the spreads that are *reducible* in Cantor's sense and that they form a hierarchy in various senses, the so-called *Cantor–Bendixson hierarchy*.

In Section 7, we study the class $\mathbf{\Pi}_2^1$ of the co-projections of analytic sets and the class $\mathbf{\Sigma}_2^1$ of the projections of co-analytic sets. We prove that the Second Axiom of Continuous Choice, $\mathbf{AC}_{1,1}$, implies: $\mathbf{\Pi}_2^1 \subseteq \mathbf{\Sigma}_2^1$ and thus causes *the collapse of the (positive) projective hierarchy*. We draw a parallel with arithmetic, where Church's Thesis causes the collapse of the (positive) arithmetical hierarchy.

2 Analytic sets

2.1 The class $\mathbf{\Sigma}_1^1$

Some relevant definitions may be found in Section 1.2.6.

Definition 1 *$\mathcal{X} \subseteq \omega^\omega$ is* analytic *or $\mathbf{\Sigma}^1_1$ if and only if there exists β such that $\mathcal{X} = \mathcal{EF}_\beta := Ex(\mathcal{F}_\beta) = \{\alpha \mid \exists\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{F}_\beta]\}$.*

$\mathcal{X} \subseteq \omega^\omega$ thus is analytic if $\mathcal{X}$ is the projection of a closed subset of ω^ω.

Definition 2 *A* Souslin system *is a mapping $s \mapsto \mathcal{P}_s$ that associates to every s a subset $\mathcal{P}_s$ of ω^ω. The* Souslin operation *applied to such a system produces the set $\mathbb{A}_s\mathcal{P}_s := \bigcup_\alpha \bigcap_n \mathcal{P}_{\overline{\alpha}n}$.*

The next theorem shows that the class $\mathbf{\Sigma}^1_1$ behaves nicely. The class is closed under the operations of countable union and countable intersection, and contains all (positively) Borel subsets of ω^ω. Every set reducing to an analytic set is itself analytic. The class $\mathbf{\Sigma}^1_1$ is also closed under projection and under the Souslin operation.

Theorem 2.1

(i) *$\mathcal{US}^1_1 := \{\alpha \mid \alpha_{II} \in \mathcal{EF}_{\alpha_I}\}$ is $\mathbf{\Sigma}^1_1$–universal.*

(ii) *$\mathcal{E}^1_1 := \{\alpha \mid \exists\gamma\forall n[\alpha(\overline{\gamma}n) = 0]\}$ is $\mathbf{\Sigma}^1_1$–complete.*

(iii) *For every infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \dots$ in $\mathbf{\Sigma}^1_1$, $\bigcup_n \mathcal{X}_n \in \mathbf{\Sigma}^1_1$ and $\bigcap_n \mathcal{X}_n \in \mathbf{\Sigma}^1_1$, ie $\forall\beta\exists\gamma\exists\delta[\bigcup_n \mathcal{EF}_{\beta^n} = \mathcal{EF}_\gamma \ \wedge\ \bigcap_n \mathcal{EF}_{\beta^n} = \mathcal{EF}_\delta]$.*

(iv) *$\mathfrak{Borel} \subseteq \mathbf{\Sigma}^1_1$, ie $\forall\sigma \in \mathcal{HRS}\forall\beta\exists\gamma\exists\delta[\mathcal{G}^\sigma_\beta = \mathcal{EF}_\gamma \ \wedge\ \mathcal{F}^\sigma_\beta = \mathcal{EF}_\delta]$.*

(v) *For all $\mathcal{X} \subseteq \omega^\omega$, if $\mathcal{X} \in \mathbf{\Sigma}^1_1$, then $Ex(\mathcal{X}) \in \mathbf{\Sigma}^1_1$, ie $\forall\beta\exists\gamma[Ex(\mathcal{EF}_\beta) = \mathcal{EF}_\gamma]$.*

(vi) *For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, if $\mathcal{X} \preceq \mathcal{Y} \in \mathbf{\Sigma}^1_1$ then $\mathcal{X} \in \mathbf{\Sigma}^1_1$, ie $\forall\beta\forall\varphi\colon \omega^\omega \to \omega^\omega\exists\gamma[\{\alpha \mid \varphi|\alpha \in \mathcal{EF}_\beta\} = \mathcal{EF}_\gamma]$.*

(vii) *For each β, $\mathbb{A}_s\mathcal{EF}_{\beta^s} \in \mathbf{\Sigma}^1_1$.*

Proof (i) For each α, $\alpha \in \mathcal{US}^1_1 \leftrightarrow \alpha_{II} \in \mathcal{EF}_{\alpha_I} \leftrightarrow \exists\gamma[\ulcorner\alpha_{II},\gamma\urcorner \in \mathcal{F}_{\alpha_I}] \leftrightarrow \exists\gamma\forall n[\alpha_I(\overline{\ulcorner\alpha_{II},\gamma\urcorner}n) = 0]$. Define β such that, for all n, for all a, c in ω^n, $\beta(\ulcorner a, c\urcorner)) \neq 0$ if and only if, for some $m < n$, $\overline{\ulcorner a_{II}, c\urcorner}m < n$ and $a_I(\overline{\ulcorner a_{II}, c\urcorner}m) \neq 0$. Then, for each α, $\alpha \in \mathcal{EF}_\beta$ if and only if $\exists\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{F}_\beta]$ if and only if $\exists\gamma\forall n[\beta(\overline{\ulcorner\alpha,\gamma\urcorner}n) = 0]$ if and only if $\exists\gamma\forall n[\alpha_I(\overline{\ulcorner\alpha_{II},\gamma\urcorner}n) = 0]$ if and only if $\alpha_{II} \in \mathcal{EF}_{\alpha_I}$ if and only if $\alpha \in \mathcal{US}^1_1$. Conclude that $\mathcal{US}^1_1 = \mathcal{EF}_\beta \in \mathbf{\Sigma}^1_1$.

Also, for each ε, $\mathcal{EF}_\varepsilon = \mathcal{US}^1_1 \upharpoonright \varepsilon$. Conclude that $\mathcal{US}^1_1$ is $\mathbf{\Sigma}^1_1$–universal.

(ii) For each α, $\alpha \in \mathcal{E}^1_1 \leftrightarrow \exists\gamma\forall n[\alpha(\overline{\gamma}n) = 0]$. Define $\mathcal{F} := \{\alpha \mid \forall n[\alpha_I(\overline{\alpha_{II}}n) = 0]\}$ and note $\mathcal{E}^1_1 = Ex(\mathcal{F})$. Define β such that $\forall a[\beta(a) = 0 \leftrightarrow \forall n[\overline{a_{II}}n < \text{length}(a_I) \to a_I(\overline{a_{II}}n) = 0]]$ and note that $\mathcal{F} = \mathcal{F}_\beta$. We thus see that $\mathcal{E}^1_1 \in \mathbf{\Sigma}^1_1$.

Let ε be given. Note that $\forall\alpha[\alpha \in \mathcal{EF}_\varepsilon \leftrightarrow \exists\gamma\forall n[\varepsilon(\overline{\ulcorner\alpha,\gamma\urcorner}n) = 0]]$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall k\forall c \in \omega^k[(\varphi|\alpha)(c) = \varepsilon(\ulcorner\overline{\alpha}k, c\urcorner)]$. Note that φ reduces $\mathcal{EF}_\varepsilon$ to $\mathcal{E}^1_1$. Conclude that $\mathcal{E}^1_1$ is $\mathbf{\Sigma}^1_1$–complete.

(iii) Let $\mathcal{X}_0, \mathcal{X}_1, \ldots$ be an infinite sequence of analytic subsets of ω^ω. Using $\mathbf{AC}_{0,1}$, find β such that $\forall n[\mathcal{X}_n = \mathcal{EF}_{\beta^n}]$. Note that for all α, $\alpha \in \bigcup_n \mathcal{X}_n \leftrightarrow \exists n \exists \gamma[\ulcorner \alpha, \gamma \urcorner \in \mathcal{F}_{\beta^n}] \leftrightarrow \exists \gamma[\ulcorner \alpha, \gamma \circ S \urcorner \in \mathcal{F}_{\beta^{\gamma(0)}}]$. Define $\mathcal{Z}_0 := \{\ulcorner \alpha, \gamma \urcorner \mid \forall k[\beta^{\gamma(0)}(\overline{\ulcorner \alpha, \gamma \circ S \urcorner} k) = 0]\}$, and note that $\mathcal{Z}_0 \in \mathbf{\Pi}^0_1$ and $\bigcup_n \mathcal{X}_n = Ex(\mathcal{Z}_0) \in \mathbf{\Sigma}^1_1$.

Note, using $\mathbf{AC}_{0,1}$, that for all α:

$$\alpha \in \bigcap_n \mathcal{X}_n \leftrightarrow \forall n \exists \gamma[\ulcorner \alpha, \gamma \urcorner \in \mathcal{F}_{\beta^n}] \leftrightarrow \exists \gamma \forall n[\ulcorner \alpha, \gamma^n \urcorner \in \mathcal{F}_{\beta^n}]$$

Define $\mathcal{Z}_1 := \{\ulcorner \alpha, \gamma \urcorner \mid \forall n \forall m[\beta^n(\overline{\ulcorner \alpha, \gamma^n \urcorner} m) = 0]\}$, and note that $\mathcal{Z}_1 \in \mathbf{\Pi}^0_1$ and $\bigcap_n \mathcal{X}_n = Ex(\mathcal{Z}_1) \in \mathbf{\Sigma}^1_1$.

(iv) follows from (iii) by induction on the class of positively Borel sets.

(v) Let β be given. Note that for every α:

$$\begin{aligned}\alpha \in Ex(\mathcal{EF}_\beta) &\leftrightarrow \exists \gamma[\ulcorner \alpha, \gamma \urcorner \in \mathcal{EF}_\beta] \leftrightarrow \exists \gamma \exists \delta[\ulcorner \ulcorner \alpha, \gamma \urcorner, \delta \urcorner \in \mathcal{F}_\beta] \\ &\leftrightarrow \exists \gamma[\ulcorner \ulcorner \alpha, \gamma_I \urcorner, \gamma_{II} \urcorner \in \mathcal{F}_\beta]\end{aligned}$$

Define $\mathcal{Z} := \{\ulcorner \alpha, \gamma \urcorner \mid \forall n[\beta(\overline{\ulcorner \ulcorner \alpha, \gamma_I \urcorner, \gamma_{II} \urcorner} n) = 0]\}$ and note that $\mathcal{Z} \in \mathbf{\Pi}^0_1$ and $Ex(\mathcal{EF}_\beta) = Ex(\mathcal{Z}) \in \mathbf{\Sigma}^1_1$.

(vi) Let $\varphi\colon \omega^\omega \to \omega^\omega$ and β be given. For every α, $\varphi|\alpha \in \mathcal{EF}_\beta \leftrightarrow \exists \gamma[\ulcorner \varphi|\alpha, \gamma \urcorner \in \mathcal{F}_\beta]$. Define $\mathcal{Z} := \{\ulcorner \alpha, \gamma \urcorner \mid \forall n[\beta(\overline{\ulcorner \varphi|\alpha, \gamma \urcorner} n) = 0]\}$ and note that $\mathcal{Z} \in \mathbf{\Pi}^0_1$ and $\{\alpha \mid \varphi|\alpha \in \mathcal{EF}_\beta\} = Ex(\mathcal{Z}) \in \mathbf{\Sigma}^1_1$.

(vii) Let β be given. Note, using $\mathbf{AC}_{0,1}$: for each α:

$$\begin{aligned}\alpha \in \mathbb{A}_s \mathcal{EF}_{\beta^s} &\leftrightarrow \exists \gamma \forall n[\alpha \in \mathcal{EF}_{\beta^{\overline{\gamma} n}}] \leftrightarrow \exists \gamma \forall n \exists \delta[\ulcorner \alpha, \delta \urcorner \mathcal{F}_{\beta^{\overline{\gamma} n}}] \\ &\leftrightarrow \exists \gamma \exists \delta \forall n[\ulcorner \alpha, \delta^n \urcorner \in \mathcal{F}_{\beta^{\overline{\gamma} n}}] \leftrightarrow \exists \gamma \forall n[\ulcorner \alpha, (\gamma_{II})^n \urcorner \in \mathcal{F}_{\beta^{\overline{\gamma_I} n}}]\end{aligned}$$

Define $\mathcal{Z} := \{\ulcorner \alpha, \gamma \urcorner \mid \forall n \forall m[\beta^{\overline{\gamma_I} n}(\overline{\ulcorner \alpha_I, (\gamma_{II})^n \urcorner} m) = 0]\}$ and note that $\mathcal{Z} \in \mathbf{\Pi}^0_1$ and $\mathbb{A}_s \mathcal{EF}_{\beta^s} = Ex(\mathcal{Z}) \in \mathbf{\Sigma}^1_1$. □

2.2 The set $\mathcal{IF}$

Definition 3 *For all s, t in ω one defines: $s <_{KB} t$ if and only if either $t \sqsubset s$ or $\exists i[i < \text{length}(s) \wedge i < \text{length}(t)\ \overline{s} i = \overline{t} i \ \wedge s(i) < t(i)]$.*

$<_{KB}$ is a linear ordering on ω. We define, for all s, t, $\max_{KB}(s,t) := s$ if $t \leq_{KB} s$, and $\max_{KB}(s,t) := t$ otherwise. $<_{KB}$ is called the *Kleene-Brouwer ordering of* ω.

Definition 4 *We define $\mathcal{IF} := \{\alpha \mid \exists \beta \in (T_\alpha)^\omega \forall n[\beta(n+1) <_{KB} \beta(n)]\}$.*

$\mathcal{IF}$ is the set of all α such that the tree $T_\alpha := \{s \mid \forall t \sqsubset s[\alpha(t) = 0]\}$ is (positively) *ill-founded* with respect to the Kleene-Brouwer-ordering $<_{KB}$.

In classical mathematics, $\mathcal{IF} = \mathcal{E}^1_1$, see also Theorem 4.2. In our intuitionistic context, the two sets are different. The reason is that the class of all sets reducing to $\mathcal{IF}$ is not closed under the operation of finite union:

Theorem 2.2

(i) *The set $\mathbb{D}^2(\mathcal{A}_1)$ does not reduce to the set $\mathcal{IF}$: $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{IF}$.*
(ii) *The set $\mathcal{E}^1_1$ is a proper subset of the set $\mathcal{IF}$: $\mathcal{E}^1_1 \subsetneqq \mathcal{IF}$.*
(iii) *The set $\mathcal{IF}$ is $\mathbf{\Sigma}^1_1$ but not $\mathbf{\Sigma}^1_1$–complete.*

Proof Assume that $\varphi\colon \omega^\omega \to \omega^\omega$ reduces $\mathbb{D}^2(\mathcal{A}_1) = \{\alpha \mid \alpha^0 = \underline{0} \ \vee \ \alpha^1 = \underline{0}\}$ to $\mathcal{IF}$. Assume: $\alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}$. Define α_0, α_1 such that $\forall i < 2[(\alpha_i)^i = \underline{0} \ \wedge \ \forall j[\neg\exists n[j = \langle i\rangle * n] \to \alpha_i(j) = \alpha(j)]]$. Note that $\forall i < 2[\alpha_i \in \mathbb{D}^2(\mathcal{A}_1) \ \wedge \ \varphi|\alpha_i \in \mathcal{IF}]$. Find δ_0, δ_1 such that $\forall i < 2\forall n[\delta_i(n) \in T_{\varphi|\alpha_i} \ \wedge \ \delta_i(n+1) <_{KB} \delta_i(n)]$. Define ζ such that, for each n,

(1) if $\forall i < 2\forall j \leq n[\delta_i(j) \in T_{\varphi|\alpha}]$, then $\zeta(n) = \max_{KB}\big(\delta_0(n), \delta_1(n)\big)$, and
(2) for all $i < 2$, if $\exists j \leq n[\delta_i(j) \notin T_{\varphi|\alpha}]$, then $\zeta(n) = \delta_{1-i}(n)$.

This is a good definition: if, for some $i < 2$, for some j, $\delta_i(j) \notin T_{\varphi|\alpha}$, then $\alpha \,\#\, \alpha_i$, and, therefore, $\alpha = \alpha_{1-i}$, and, for each j, $\delta_{1-i}(j) \in T_{\varphi|\alpha}$. Note that $\forall n[\zeta(n) \in T_{\varphi|\alpha} \ \wedge \ \zeta(n+1) <_{KB} \zeta(n)]$, and conclude $\varphi|\alpha \in \mathcal{IF}$, and $\alpha \in \mathbb{D}^2(\mathcal{A}_1)$.

We thus see that $\forall\alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\alpha \in \mathbb{D}^2(\mathcal{A}_1)]$. According to Theorem 1.3, we have a contradiction. Conclude that $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{IF}$.

(ii) Assume that $\alpha \in \mathcal{E}^1_1$. Find γ such that $\forall n[\alpha(\overline{\gamma}n) = 0]$. Note that $\forall n[\overline{\gamma}n \in T_\alpha \ \wedge \ \overline{\gamma}(n+1) <_{KB} \overline{\gamma}n]$ and $\alpha \in \mathcal{IF}$. We thus see that $\mathcal{E}^1_1 \subseteq \mathcal{IF}$.

According to Theorem 2.1, $\mathbb{D}^2(\mathcal{A}_1) \preceq \mathcal{E}^1_1$, but, as we saw in (i), $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{IF}$. Conclude that $\mathcal{E}^1_1 \neq \mathcal{IF}$ and $\mathcal{E}^1_1 \subsetneqq \mathcal{IF}$.

(iii) Define $\mathcal{Z} := \{\ulcorner\alpha, \gamma\urcorner \mid \forall n[\gamma(n) \in T_\alpha \ \wedge \ \gamma(n+1) <_{KB} \gamma(n)]\}$ and note that $\mathcal{Z} \in \mathbf{\Pi}^0_1$ and $\mathcal{IF} = Ex(\mathcal{Z})$. Conclude that $\mathcal{IF}$ is $\mathbf{\Sigma}^1_1$. As, according to (i), the analytic set $\mathbb{D}^2(\mathcal{A}_1)$ does not reduce to $\mathcal{IF}$, $\mathcal{IF}$ is not $\mathbf{\Sigma}^1_1$–complete. □

2.3 The sets $\mathcal{UNC}$, $\mathcal{UNC}'$ and $\mathcal{UNC}''$

Definition 5 *$\mathcal{X} \subseteq \omega^\omega$ is (positively) uncountable if and only if $\forall\alpha\exists\beta \in \mathcal{X}\forall n[\beta \,\#\, \alpha^n]$. $\mathcal{X} \subseteq \omega^\omega$ is weakly (positively) uncountable if and only if $\exists\alpha[\alpha \in \mathcal{X}]$ and $\forall\alpha \in \mathcal{X}^\omega\exists\beta \in \mathcal{X}\forall n[\beta \,\#\, \alpha^n]$.*

Clearly, every uncountable subset of ω^ω is weakly uncountable. For spreads, the two notions coincide:

Theorem 2.3 *If $\mathcal{F} \subseteq \omega^\omega$ is a spread and weakly (positively) uncountable, then $\mathcal{F}$ is (positively) uncountable.*

Proof Let β be given such that $Spr(\beta)$ and $\mathcal{F} := \mathcal{F}_\beta$ is weakly uncountable. Let ρ be the canonical retraction of ω^ω onto $\mathcal{F}$. Note that $\forall \alpha[\rho|\alpha \in \mathcal{F} \ \wedge \ (\alpha \ \# \ \rho|\alpha \rightarrow \exists n[\beta(\overline{\alpha}n) \neq 0])]$. Let α be given. Find δ in $\mathcal{F}$ such that $\forall n[\delta \ \# \ \rho|(\alpha^n)]$.

Let n be given. As the apartness relation # is co-transitive and $\delta \ \# \ \rho|(\alpha^n)$, either $\delta \ \# \ \alpha^n$ or $\alpha^n \ \# \ \rho|(\alpha^n)$. In the latter case, find m such that $\beta(\overline{\alpha^n}m) \neq 0$. Note $\beta(\overline{\delta}m) = 0$ and conclude that $\overline{\alpha^n}m \neq \overline{\delta}m$, and $\delta \ \# \ \alpha^n$. Conclude that $\forall n[\delta \ \# \ \alpha^n]$.

We thus see that $\forall \alpha \exists \delta \in \mathcal{F} \forall n[\delta \ \# \ \alpha^n]$, ie $\mathcal{F}$ is uncountable. □

The following intuitionistic theorem is the same as Gielen–de Swart–Veldman [11, Theorem 2.1], see also Veldman [39, Section 8], and was first proven by W. Gielen. Cantor's (classical) famous *Perfect Set Theorem* states that 2^ω embeds continuously in every uncountable $\mathbf{\Pi}^0_1$ subset of ω^ω. P.S. Alexandrov and F. Hausdorff, independently, extended the result to Borel subsets of ω^ω and M. Souslin showed that it also holds for $\mathbf{\Sigma}^1_1$ subsets of ω^ω. In our intuitionistic context the Theorem holds for *every* subset of ω^ω. This is due to the Second Axiom of Continuous Choice, $\mathbf{AC}_{1,1}$, see Section 1.1.6.

Theorem 2.4 *$\mathcal{X} \subseteq \omega^\omega$ is (positively) uncountable if and only if 2^ω embeds into $\mathcal{X}$.*

Proof (i) First, assume: $\mathcal{X} \subseteq \omega^\omega$ and 2^ω embeds into $\mathcal{X}$. Find $\varphi \colon 2^\omega \rightarrowtail \mathcal{X}$. We now prove that $\mathcal{X}$ is positively uncountable.

Let α be given. Using induction, define δ such that, for each n, $\delta(n) \in Bin$ and $\delta(n) \sqsubset \delta(n+1)$ and $\varphi|\big(\delta(n+1)\big) \perp \alpha^n$, as follows. Define $\delta(0) = 0 = \langle \ \rangle$. Suppose n is given such that $\delta(n)$ has been defined. Find p such that $\varphi|(\delta(n) * \overline{0}p) \perp \varphi|(\delta(n) * \overline{1}p)$. If $\alpha^n \perp \varphi|(\delta(n) * \overline{0}p)$, define $\delta(n+1) := \delta(n) * \overline{0}p$; and, if not, define $\delta(n+1) := \delta(n) * \overline{1}p$. It will be clear that α satisfies the requirements. Now find ε in 2^ω such that $\forall n[\delta(n) \sqsubset \varepsilon]$ and define: $\beta := \varphi|\varepsilon$. Note that $\beta \in \mathcal{X}$ and $\forall n[\alpha^n \ \# \ \varphi|\varepsilon = \beta]$.

We thus see that $\forall \alpha \exists \beta \in \mathcal{X} \forall n[\alpha^n \ \# \ \beta]$, ie $\mathcal{X}$ is (positively) uncountable.

(ii) Next, assume $\mathcal{X} \subseteq \omega^\omega$ is (positively) uncountable. We want to prove that 2^ω embeds into $\mathcal{X}$.

Using the Second Axiom of Continuous Choice $\mathbf{AC}_{1,1}$ (see Section 1.1.6) find $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[\varphi|\alpha \in \mathcal{X} \wedge \forall n[\varphi|\alpha \mathrel{\#} \alpha^n]]$.

We first prove: $\forall s\exists t\exists u[s \sqsubset t \wedge s \sqsubset u \wedge \varphi|t \perp \varphi|u]$. Let s be given. Define ε such that $s \sqsubset \varepsilon \wedge \varepsilon^s = \varphi|(s * \underline{0})$. Note that $\varphi|(s*\underline{0}) = \varepsilon^s \mathrel{\#} \varphi|\varepsilon$. Find m such that $\varphi|(\overline{s*\underline{0}}m) \perp \varphi|\overline{\varepsilon}m$ and define $t := \overline{s*\underline{0}}m$ and $u := \overline{\varepsilon}m$. Clearly, t,u satisfy the requirements.

Now define ζ such that $\zeta(0) = 0$ and, for each s in *Bin*, $\zeta(s*\langle 0\rangle) = u'$ and $\zeta(s*\langle 1\rangle) = u''$, where u is the least v such that $\zeta(s) \sqsubset v' \wedge \zeta(s) \sqsubset v'' \wedge \varphi|v' \perp \varphi|v''$. Note that $\forall s \in \mathrm{Bin}\,\forall t \in \mathit{Bin}[s \sqsubset t \to \zeta(s) \sqsubset \zeta(t)]$. Find $\rho\colon 2^\omega \to \omega^\omega$ such that $\forall\gamma \in 2^\omega\forall n[\zeta(\overline{\gamma}n) \sqsubset \rho|\gamma]$. Find $\psi\colon 2^\omega \to \omega^\omega$ such that $\forall\gamma \in 2^\omega\forall n[\psi|\gamma = \varphi|(\rho|\gamma)]$. Note that $\psi\colon 2^\omega \to \mathcal{X}$. Also note that $\forall s \in \mathrm{Bin}\,\forall t \in \mathit{Bin}[s \perp t \to \varphi|(\zeta(s)) \perp \varphi|(\zeta(t))]$. Conclude that $\psi\colon 2^\omega \rightarrowtail \mathcal{X}$ and 2^ω embeds into $\mathcal{X}$. □

Theorem 2.5

(i) *The set* $\omega^{(2^\omega)}$ *is* $\mathbf{\Sigma}^0_1$*–complete.*
(ii) *The set* $(\omega^\omega)^{(2^\omega)}$ *is* $\mathbf{\Pi}^0_2$*–complete.*
(iii) *The set* $\mathrm{Emb}(2^\omega, \omega^\omega)$ *is* $\mathbf{\Pi}^0_2$*–complete.*

Proof (i) Using the Fan Theorem **FT**, see Section 1.1.7, note that for all φ, $\varphi \in \omega^{(2^\omega)} \leftrightarrow \forall\gamma \in 2^\omega\exists n[\varphi(\overline{\gamma}n) \neq 0] \leftrightarrow \exists m\forall s \in \mathrm{Bin}_m\,\exists n \le m[\varphi(\overline{s}n) \neq 0]$. Conclude that $\omega^{(2^\omega)}$ is $\mathbf{\Sigma}^0_1$.

We now want to prove that the set $\mathcal{E}_1$ reduces to the set $\omega^{(2^\omega)}$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall n\forall s \in \mathrm{Bin}_n[(\varphi|\alpha)(s) = \alpha(n)]$. Note that, for each α, for each n, if $n = \mu p[\alpha(p) \neq 0]$ then $\varphi|\alpha\colon 2^\omega \to \omega$ and $\forall\alpha \in 2^\omega[\varphi(\alpha) = \alpha(n) - 1]$. Clearly, φ reduces $\mathcal{E}_1 = \{\alpha \mid \exists n[\alpha(n) \neq 0]\}$ to $\omega^{(2^\omega)}$. As $\mathcal{E}_1$ is $\mathbf{\Sigma}^0_1$–complete, so is $\omega^{(2^\omega)}$.

(ii) and (iii). We first prove that the two sets $(\omega^\omega)^{(2^\omega)}$ and $\mathrm{Emb}(2^\omega, \omega^\omega)$ both belong to $\mathbf{\Pi}^0_2$.

First note that for all φ, $\varphi \in (\omega^\omega)^{(2^\omega)}$ if and only if $\forall n[\varphi^n \in \omega^{(2^\omega)}]$. Using (i), conclude that $(\omega^\omega)^{(2^\omega)} \in \mathbf{\Pi}^0_2$.

Then note, using the Fan Theorem **FT**: for all φ, $\varphi \in \mathrm{Emb}(2^\omega, \omega^\omega)$ if and only if $\varphi \in (\omega^\omega)^{(2^\omega)}$ and $\forall s \in \mathrm{Bin}\,\forall\alpha \in 2^\omega\forall\beta \in 2^\omega\exists n[\varphi|s*\langle 0\rangle*\overline{\alpha}n \perp \varphi|s*\langle 1\rangle*\overline{\beta}n]$ if and only if $\varphi \in (\omega^\omega)^{(2^\omega)}$ and $\forall s \in \mathrm{Bin}\,\exists n\forall t \in \mathrm{Bin}_n\,\forall u \in \mathrm{Bin}_n[\varphi|s*\langle 0\rangle*t \perp \varphi|s*\langle 1\rangle*u]$. Conclude that $\mathrm{Emb}(2^\omega, \omega^\omega) \in \mathbf{\Pi}^0_2$.

We now prove that the set $\mathcal{A}_2$ reduces to both the set $(\omega^\omega)^{(2^\omega)}$ and the set $\mathrm{Emb}(2^\omega, \omega^\omega)$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that, for all m, for all α, for all s in $2^{<\omega}$, *if* $m < \mathrm{length}(s)$ and

$\exists n < \mathrm{length}(s)[\alpha^m(n) \neq 0]$, then $(\psi|\alpha)^m(s) = s(m)+1$, and, *if not*, then $(\psi|\alpha)^m(s) = 0$. Note that for all α, for all m, (i) if $\alpha^m \in \mathcal{E}_1$ then $(\psi|\alpha)^m : 2^\omega \to \omega$ and, for all β in 2^ω, $(\psi|\alpha)^m(\beta) = \beta(m)$, and (ii) if $\exists n[(\psi|\alpha)^m(\overline{\beta}n) \neq 0]$ then $\alpha^m \in \mathcal{E}_1$. Conclude that for all α, for all m, $\alpha^m \in \mathcal{E}_1$ if and only if $(\psi|\alpha)^m : 2^\omega \to \omega$. Therefore, for all $\alpha, \alpha \in \mathcal{A}_2$ if and only if $\psi|\alpha : 2^\omega \to \omega^\omega$.

We thus see that ψ reduces $\mathcal{A}_2$ to $(\omega^\omega)^{(2^\omega)}$. As $\mathcal{A}_2$ is $\mathbf{\Pi}^0_2$–complete, also $(\omega^\omega)^{(2^\omega)}$ is $\mathbf{\Pi}^0_2$–complete.

Note that for all α, if $\alpha \in \mathcal{A}_2$, then $\psi|\alpha : 2^\omega \to \omega^\omega$ and $\forall\beta \in 2^\omega[(\psi|\alpha)|\beta = \beta]$. Conclude that for all α, $\alpha \in \mathcal{A}_2$ if and only if $\psi|\alpha \in \mathrm{Emb}(2^\omega, \omega^\omega)$. We thus see that ψ reduces $\mathcal{A}_2$ to $\mathrm{Emb}(2^\omega, \omega^\omega)$. As $\mathcal{A}_2$ is $\mathbf{\Pi}^0_2$–complete, also $\mathrm{Emb}(2^\omega, \omega^\omega)$ is $\mathbf{\Pi}^0_2$–complete. $\square$

We will need the next Lemma, Lemma 2.6, in the proof of Theorem 2.7(iii).

Lemma 2.6

(i) *For all finite* $A \subseteq \omega$, *for every* $P \subseteq A$, *for every proposition* Q, *if* $\forall m \in A[m \in P \ \vee \ Q]$, *then* $\forall m \in A[m \in P] \ \vee \ Q$.

(ii) *For all finite sets* $A, B \subseteq \omega$, *for all* $P \subseteq A$, *for all* $Q \subseteq B$, *if* $\forall m \in A \forall n \in B[m \in P \ \vee \ n \in Q]$, *then* $\forall m \in A[m \in P] \ \vee \ \forall n \in B[n \in Q]$.

Proof (i) Use induction on the number of elements of A. If $A = \emptyset$, the statement is true. Now assume the statement has been proven for A, and $q \in \omega \setminus A$. We prove that the statement is true for $A \cup \{q\}$. Assume $P \subseteq A \cup \{q\}$ and $\forall m \in A \cup \{q\}[m \in P \ \vee \ Q]$. Then, by the induction hypothesis, $\forall m \in A[m \in P] \ \vee \ Q$. But also $q \in P \ \vee \ Q$. Conclude that $\forall m \in A \cup \{q\}[m \in P] \ \vee \ Q$.

(ii) Assume that A, B are finite subsets of ω, and $\forall m \in A \forall n \in B[m \in P \ \vee \ n \in Q]$. Using (i), conclude that $\forall n \in B[\forall m \in A[m \in P] \ \vee \ n \in Q]$. Using (i) once more, conclude that $\forall m \in A[m \in P] \ \vee \ \forall n \in B[n \in Q]$. $\square$

Definition 6 *For each* β, *we define:* β is a perfect-spread-law, *Pfspr*(β), *if and only if* $\mathrm{Spr}(\beta)$ *and* $\beta(0) = 0$ *and, for all* s, *if* $\beta(s) = 0$, *then:*

$$\exists t \exists u[s \sqsubset t \ \wedge \ s \sqsubset u \ \wedge \ t \perp u \ \wedge \ \beta(t) = \beta(u) = 0]$$

If Pfspr(β), *then* $\mathcal{F}_\beta = \{\alpha \mid \forall n[\beta(\overline{\alpha}n) = 0]\}$ *is called* a perfect spread.

In intuitionistic real analysis it is not true that the image of the closed interval $[0,1]$ under a continuous function is itself a closed subset of $\mathcal{R}$. One may see this from the failure of the Intermediate Value Theorem[8] and the failure of the theorem that a continuous function from $[0,1]$ to $\mathcal{R}$ always attains its greatest value. The next theorem brings to light related facts. The image of Cantor space 2^ω under a continuous function from 2^ω to ω^ω is always a located subset of ω^ω but not always a closed subset of ω^ω. The latter remains true, however, if the function is *strongly injective*. $\mathcal{F} \subseteq \omega^\omega$ is a spread if and only if $\mathcal{F}$ is both located and closed; see Section 1.2.2.

Theorem 2.7

(i) *Cantor space* 2^ω *embeds into every perfect spread.*
(ii) *For each* $\varphi\colon 2^\omega \to \omega^\omega$, $\varphi|2^\omega$ *is a located subset of* ω^ω.
(iii) *For each* $\varphi\colon 2^\omega \rightarrowtail \omega^\omega$, $\varphi|2^\omega$ *is a perfect spread and a fan.*
(iv) $\neg\forall\varphi \in (\omega^\omega)^{(2^\omega)}\exists\beta[\mathrm{Spr}(\beta) \wedge \varphi|2^\omega = \mathcal{F}_\beta]$.

Proof (i) Let $\mathcal{F} \subseteq \omega^\omega$ be a perfect spread. Find β such that $Pfspr(\beta)$ and $\mathcal{F} = \mathcal{F}_\beta$. Define ζ such that $\zeta(0) = 0$ and, for all s in Bin, $\zeta(s * \langle 0\rangle) := u'$ and $\zeta(s * \langle 1\rangle) = u''$ where u is the least v such that $v' \perp v''$ and $\zeta(s) \sqsubset v'$ and $\zeta(s) \sqsubset v''$ and $\beta(v') = \beta(v'') = 0$. Define $\varphi\colon 2^\omega \to \omega^\omega$ such that $\forall\alpha \in 2^\omega\forall n[\zeta(\overline{\alpha}n) \sqsubset \varphi|\alpha]$. Note that $\forall\alpha \in 2^\omega[\varphi|\alpha \in \mathcal{F}_\beta]$. Also note that for all α,β in 2^ω, if $\alpha \mathrel{\#} \beta$, then, for some n, $\overline{\alpha}n \perp \overline{\beta}n$ and $\zeta(\overline{\alpha}n) \perp \zeta(\overline{\beta}n)$, and $\varphi|\alpha \mathrel{\#} \varphi|\beta$. Conclude that $\varphi\colon 2^\omega \rightarrowtail \mathcal{F}$.

(ii) Let $\varphi\colon 2^\omega \to \omega^\omega$ be given. We define δ as follows. Let s be given. Note that $\forall\alpha \in 2^\omega\exists m[s \sqsubset \varphi|\overline{\alpha}m \ \vee\ s \perp \varphi|\overline{\alpha}m]$. Using **FT**, find m such that $\forall\alpha \in 2^\omega[s \sqsubset \varphi|\overline{\alpha}m \ \vee\ s \perp \varphi|\overline{\alpha}m]$, ie $\forall t \in \mathrm{Bin}_m[s \sqsubset \varphi|t \ \vee\ s \perp \varphi|t]$. Define $\delta(s) := 0$ if $\exists t \in \mathrm{Bin}_m[s \sqsubset \varphi|t]$ and $\delta(s) := 1$ if $\forall t \in \mathrm{Bin}_m[s \perp \varphi|t]$. Conclude that $\forall s[\delta(s) = 0 \leftrightarrow \exists\alpha \in 2^\omega[s \sqsubset \varphi|\alpha]]$ and $\varphi|2^\omega$ is a located subset of ω^ω. Also note that $\mathrm{Fan}(\delta)$ and $\varphi|2^\omega \subseteq \mathcal{F}_\delta$.

(iii) Let $\varphi\colon 2^\omega \rightarrowtail \omega^\omega$ be given. Using (ii), find δ such that $\forall s[\delta(s) = 0 \leftrightarrow \exists\alpha \in 2^\omega[s \sqsubset \varphi|\alpha]]$ and $\mathrm{Fan}(\delta)$ and $\varphi|2^\omega \subseteq \mathcal{F}_\delta$.

We first prove $Pfspr(\delta)$. Let s be given such that $\delta(s) = 0$. Find α in 2^ω such that $s \sqsubset \varphi|\alpha$. Find m such that $s \sqsubset \varphi|\overline{\alpha}m$. Find n such that $\varphi|(\overline{\alpha}m * \overline{\underline{0}}n) \perp \varphi|(\overline{\alpha}m * \overline{\underline{1}}n)$ and define: $t := \varphi|(\overline{\alpha}m * \overline{\underline{0}}n)$ and $u := \varphi|(\overline{\alpha}m * \overline{\underline{1}}n)$. Note $\delta(t) = \delta(u) = 0$ and $s \sqsubset t$ and $s \sqsubset u$ and $t \perp u$.

Assume $s \in Bin$. Note that $\forall\alpha \in 2^\omega[\varphi|(s * \langle 0\rangle * \alpha_I) \mathrel{\#} \varphi|(s * \langle 1\rangle * \alpha_{II})]$ and $\forall\alpha \in 2^\omega\exists m[\varphi|\big(s * \langle 0\rangle * \overline{\alpha_I}m\big) \perp \varphi|\big(s * \langle 1\rangle * \overline{\alpha_{II}}m\big)]$ and, using **FT**, $\exists m\forall\alpha \in$

[8] See [40].

$2^\omega[\varphi|\big(s*\langle 0\rangle*\overline{\alpha_I}m\big) \perp \varphi|\big(s*\langle 1\rangle*\overline{\alpha_{II}}m\big)]$, ie $\exists m\forall a \in \mathrm{Bin}_m \forall b \in \mathrm{Bin}_m[\varphi|(s*\langle 0\rangle*a) \perp \varphi|(s*\langle 1\rangle*b)]$. Define ζ such that, for each s in Bin, $\zeta(s)$ is the least m such that $\forall a \in \mathrm{Bin}_m \forall b \in \mathrm{Bin}_m[\varphi|(s*\langle 0\rangle*a) \perp \varphi|(s*\langle 1\rangle*b)]$.

We now prove $\mathcal{F}_\delta \subseteq \varphi|2^\omega$. Let $\gamma \in \mathcal{F}_\delta$ be given. Assume that $s \in Bin$. Note that $\forall a \in \mathrm{Bin}_{\zeta(s)} \forall b \in \mathrm{Bin}_{\zeta(s)}[\varphi|(s*\langle 0\rangle*a) \perp \gamma \ \vee \ \gamma \perp \varphi|(s*\langle 1\rangle*b)]$. Conclude, using Lemma 2.6 that $\forall a \in \mathrm{Bin}_{\zeta(s)}[\varphi|(s*\langle 0\rangle*a) \perp \gamma] \ \vee \ \forall a \in \mathrm{Bin}_{\zeta(s)}[\gamma \perp \varphi|(s*\langle 1\rangle*a)]$.

Define η in 2^ω such that $\forall s \in Bin[\eta(s) = 1 \leftrightarrow \forall a \in \mathrm{Bin}_{\zeta(s)}[\varphi|(s*\langle 0\rangle*a) \perp \gamma]]$. Define α in 2^ω such that $\forall n[\alpha(n) = \eta(\overline{\alpha}n)]$. Note that, for all β in 2^ω, for all n, if $n = \mu p[\alpha(p) \neq \beta(p)$, then $\varphi|\beta \perp \gamma$, ie, for all β in 2^ω, if $\beta \perp \alpha$, then $\varphi|\beta \perp \gamma$.

We now prove $\varphi|\alpha = \gamma$. Assume that $\varphi|\alpha \perp \gamma$. Find n such that $\varphi|\overline{\alpha}n \perp \gamma$. Define $m = n + \zeta(\overline{\alpha}n)$ and note that $\forall d \in Bim_m[d \perp \overline{\alpha}n \to \varphi|d \perp \gamma]$. Conclude that $\forall d \in \mathrm{Bin}_m[\varphi|d \perp \gamma]$. Note that $\forall d \in \mathrm{Bin}_m[length(\varphi|d) \leq m]$. Conclude that $\delta(\overline{\gamma}m) \neq 0$, which is a contradiction. We thus see that $\neg(\varphi|\alpha \perp \gamma)$ and $\varphi|\alpha = \gamma$.

Conclude that $\forall\gamma \in \mathcal{F}_\delta \exists\alpha \in 2^\omega[\varphi|\alpha = \gamma]$ and $\varphi|2^\omega = \mathcal{F}_\delta$.

(iv) Assume that $\forall\varphi \in (\omega^\omega)^{(2^\omega)}\exists\beta[\mathrm{Spr}(\beta) \wedge \varphi|2^\omega = \mathcal{F}_\beta]$. Using Brouwer's Continuity Principle **BCP** (see Section 1.1.6) we prove that this assumption leads to a contradiction as it implies **LPO**, see Section 1.1.11.

Let α be given. We intend to prove $\alpha = \underline{0} \ \vee \ \alpha \# \underline{0}$.

Define $\varphi\colon 2^\omega \to \omega^\omega$ such that $\forall\gamma \in 2^\omega[\varphi|(\langle 0\rangle*\gamma) = \alpha \ \wedge \ \varphi|(\langle 1\rangle*\gamma) = \underline{0}]$. Note that $\varphi|2^\omega = \{\alpha, \underline{0}\}$. Find β such that $\mathrm{Spr}(\beta)$ and $\{\alpha, \underline{0}\} = \mathcal{F}_\beta$. Note that $\forall s[\beta(s) = 0 \leftrightarrow (s \sqsubset \alpha \ \vee \ s \sqsubset \underline{0})]$. Note that $\forall\gamma \in \mathcal{F}_\beta[\gamma = \alpha \ \vee \ \gamma = \underline{0}]$. Applying **BCP**, find m such that either $\forall\gamma \in \mathcal{F}_\beta[\overline{\underline{0}}m \sqsubset \gamma \to \gamma = \underline{0}]$, and $\overline{\underline{0}}m \perp \alpha \ \vee \ \alpha = \underline{0}$; or $\forall\gamma \in \mathcal{F}_\beta[\overline{\underline{0}}m \sqsubset \gamma \to \gamma = \alpha]$, and $\alpha = \underline{0}$. Conclude that $\alpha = \underline{0} \ \vee \ \alpha \# \underline{0}$.

We thus see that $\forall\alpha[\alpha = \underline{0} \ \vee \ \alpha \# \underline{0}]$, ie **LPO**, a contradiction. □

Definition 7 *We introduce three subsets of* ω^ω*:*

$$\begin{aligned}\mathcal{UNC} &:= \{\beta \mid \forall\alpha\exists\gamma \in \mathcal{F}_\beta \forall n[\gamma \# \alpha^n]\}\\ \mathcal{UNC}' &:= \{\beta \in \mathcal{UNC} \mid \mathrm{Spr}(\beta)\}\\ \mathcal{UNC}'' &:= \{\beta \mid \forall\alpha\exists\gamma \in \mathcal{EF}_\beta \forall n[\gamma \# \alpha^n]\}\end{aligned}$$

$\mathcal{UNC}$, $\mathcal{UNC}'$ and $\mathcal{UNC}''$ are the sets of the codes of (positively) uncountable *closed sets*, (positively) uncountable *located closed sets* and (positively) uncountable *analytic sets*, respectively.

The classical result corresponding to the following theorem is due to W. Hurewicz, see Kechris [14, Theorem 27.5]. The proof in [14] is very different from ours and not constructive.

Theorem 2.8 *$\mathcal{UNC}$, $\mathcal{UNC}'$ and $\mathcal{UNC}''$ are $\mathbf{\Sigma}^1_1$–complete.*

Proof We first prove that $\mathcal{UNC}$ is $\mathbf{\Sigma}^1_1$.

Using Theorem 2.4, note that, for each β, $\beta \in \mathcal{UNC}$ if and only if there exists $\varphi\colon 2^\omega \rightarrowtail \mathcal{F}_\beta$. Now define $\mathcal{A} := \{\ulcorner\beta,\varphi\urcorner \mid \varphi\colon 2^\omega \rightarrowtail \omega^\omega \ \wedge\ \forall s \in 2^{<\omega}\forall t[t \sqsubseteq \varphi|s \rightarrow \beta(t) = 0]\}$. Then $\mathcal{UNC} = Ex(\mathcal{A})$. Note, using Theorem 2.5, $\mathcal{A} \in \mathbf{\Pi}^0_2$. Conclude, using Theorem 2.1, that $\mathcal{UNC} \in \mathbf{\Sigma}^1_1$.

We now prove that $\mathcal{UNC}$ is $\mathbf{\Sigma}^1_1$–complete. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that, for all α, for all s, $(\varphi|\alpha)(s) = 0$ if and only if there exists u such that $\forall t \sqsubseteq u[\alpha(t) = 0]$ and $\text{length}(u) = \text{length}(s)$ and $\forall i < \text{length}(s)[s(i) = 2u(i) + 1 \ \vee\ s(i) = 2u(i) + 2]$. We prove that φ reduces $\mathcal{E}^1_1$ to $\mathcal{UNC}$.

First, assume that $\alpha \in \mathcal{E}^1_1$. Find γ such that $\forall n[\alpha(\overline{\gamma}n) = 0]$. Define β such that, for all s, $\beta(s) = 0$ if and only if $\forall i < \text{length}(s)[s(i) = 2\gamma(i) + 1 \ \vee\ s(i) = 2\gamma(i) + 2]$. Note that $\mathit{Pfspr}(\beta)$ and $\mathcal{F}_\beta \subseteq \mathcal{F}_{\varphi|\alpha}$. Conclude, using Theorems 2.5(i) and 2.4, that $\varphi|\alpha \in \mathcal{UNC}$.

Now let α be given such that $\varphi|\alpha \in \mathcal{UNC}$. Using Theorem 2.5, find β such that $\mathit{Pfspr}(\beta)$ and $\mathcal{F}_\beta \subseteq \mathcal{F}_{\varphi|\alpha}$. Find δ in $\mathcal{F}_\beta$. Find γ such that $\forall n[\delta(n) = 2\gamma(n)+1 \vee \delta(n) = 2\gamma(n) + 2]$. Conclude that $\forall n[\alpha(\overline{\gamma}n) = 0]$ and $\alpha \in \mathcal{E}^1_1$. We thus see that $\mathcal{E}^1_1$ reduces to $\mathcal{UNC}$. As $\mathcal{E}^1_1$ is $\mathbf{\Sigma}^1_1$–complete (see Theorem 2.1) so is $\mathcal{UNC}$.

We now consider $\mathcal{UNC}'$. Define $\mathcal{A}' := \{\ulcorner\beta,\varphi\urcorner \in \mathcal{A} \mid \text{Spr}(\beta)\}$. Note that $\mathcal{A}' \in \mathbf{\Pi}^0_2$ and $\mathcal{UNC}' = Ex(\mathcal{A}')$. Conclude that $\mathcal{UNC}' \in \mathbf{\Sigma}^1_1$. We now want to prove that $\mathcal{UNC}'$ is $\mathbf{\Sigma}^1_1$–complete. We would like to use again the function φ we used in the previous paragraph, but, unfortunately, not: for every α, $\varphi|\alpha$ is a spread-law. We therefore define $\psi\colon \omega^\omega \to \omega^\omega$ such that, for all α, for all s, $(\psi|\alpha)(s) = 0$ if and only if there exist k, t such that $(\varphi|\alpha)(t) = 0$ and $s = t * \overline{0}k$. Observe that, for every α, $\psi|\alpha$ is a spread-law and $\mathcal{F}_{\varphi|\alpha} \subseteq \mathcal{F}_{\psi|\alpha}$. We prove that ψ reduces $\mathcal{E}^1_1$ to $\mathcal{UNC}'$.

First, assume that $\alpha \in \mathcal{E}^1_1$. Then $\mathcal{F}_{\varphi|\alpha} \in \mathcal{UNC}$. Note that $\mathcal{F}_{\varphi|\alpha} \subseteq \mathcal{F}_{\psi|\alpha}$, so also $\mathcal{F}_{\psi|\alpha}$ is (positively) uncountable, and, as $\psi|\alpha$ is a spread-law, $\psi|\alpha \in \mathcal{UNC}'$.

Now let α be given such that $\psi|\alpha \in \mathcal{UNC}'$. Find β such that $\mathit{Pfspr}(\beta)$ and $\mathcal{F}_\beta \subseteq \mathcal{F}_{\psi|\alpha}$. Note that for all s, if $\exists\gamma \in \mathcal{F}_\beta[s \sqsubset \gamma]$, then $\forall i < \text{length}(s)[s(i) > 0]]$, and $(\varphi|\alpha)(s) = 0$. Conclude that $\mathcal{F}_\beta \subseteq \mathcal{F}_{\varphi|\alpha}$ and $\alpha \in \mathcal{E}^1_1$.

We now consider $\mathcal{UNC}''$. Define $\mathcal{A}'' := \{\ulcorner\beta, \varphi\urcorner \mid \varphi\colon 2^\omega \rightarrowtail \mathcal{EF}_\beta\}$. Note, using the Second Axiom of Continuous Choice $\mathbf{AC}_{1,1}$ (see Section 1.1.6), for every β, for every φ, $\varphi\colon 2^\omega \to \mathcal{EF}_\beta$ if and only if $\exists\psi\colon 2^\omega \to \omega^\omega \forall\gamma \in 2^\omega[\ulcorner\varphi|\gamma, \psi|\gamma\urcorner \in \mathcal{F}_\beta]$. Define $\mathcal{A}^* := \{\ulcorner\beta, \varphi\urcorner \mid \varphi_I\colon 2^\omega \rightarrowtail \omega^\omega \wedge \varphi_{II}\colon 2^\omega \to \omega^\omega \wedge \forall\gamma \in 2^\omega[\ulcorner\varphi_I|\gamma, \varphi_{II}|\gamma\urcorner \in \mathcal{F}_\beta]\}$. Note that $\mathcal{UNC}'' = Ex(\mathcal{A}'') = Ex(\mathcal{A}^*)$, and, using Theorem 2.5, $\mathcal{A}^* \in \mathbf{\Pi}^0_2$. Conclude that $\mathcal{UNC}'' \in \mathbf{\Sigma}^1_1$.

In order to see that $\mathcal{UNC}''$ is $\mathbf{\Sigma}^1_1$–complete, we remind ourselves of the fact: $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^1_1$. Define $\tau\colon \omega^\omega \to \omega^\omega$ such that $\forall\beta\forall s[(\tau|\beta)(s) = \beta(s_I)]$ and note that $\forall\beta[\mathcal{EF}_{\tau|\beta} = \mathcal{F}_\beta]$. Conclude that τ reduces $\mathcal{UNC}$ to $\mathcal{UNC}''$, and, as $\mathcal{UNC}$ is $\mathbf{\Sigma}^1_1$–complete, so is $\mathcal{UNC}''$. □

2.4 Share($\mathcal{INF}$) and Share($\mathcal{INF} \cap 2^\omega$)

The following definition occurs already in Veldman [31].

Definition 8 *For each $\mathcal{X} \subseteq \omega^\omega$, we define* Share$(\mathcal{X}) := \{\beta \mid \mathrm{Spr}(\beta) \wedge \exists\gamma \in \mathcal{F}_\beta[\gamma \in \mathcal{X}]\}$.

If $\beta \in$ Share$(\mathcal{X})$, one says: '*The spread $\mathcal{F}_\beta$ shares an element with the set $\mathcal{X}$*'.

Definition 9 $\mathcal{INF} := \{\alpha \mid \forall m\exists n > m[\alpha(n) \neq 0]\}$.

If $\alpha \in \mathcal{INF}$, then $D_\alpha := \{n \mid \alpha(n) \neq 0\}$ is a decidable and infinite subset of ω.

The next result corresponds to a well-known fact in classical descriptive set theory, see Kechris [14, page 209, Exercise 27] or [27, page 137, Exercise 4.2.3].

Theorem 2.9 *Share($\mathcal{INF}$) and Share($\mathcal{INF} \cap 2^\omega$) are $\mathbf{\Sigma}^1_1$–complete.*

Proof We first observe that these two sets are indeed $\mathbf{\Sigma}^1_1$. Note that $\{\beta \mid \mathrm{Spr}(\beta)\}$ is $\mathbf{\Pi}^0_2$. For each β, $\beta \in$ Share($\mathcal{INF}$) if and only if Spr(β) and $\exists\alpha\exists\zeta \in [\omega]^\omega\forall n[\beta(\overline{\alpha}n) = 0 \wedge \alpha \circ \zeta(n) \neq 0]$. Conclude, using Theorem 2.1, Share($\mathcal{INF}$) is $\mathbf{\Sigma}^1_1$.

For each β, $\beta \in$ Share($\mathcal{INF} \cap 2^\omega$) if and only if Spr$(\beta)$ and $\exists\alpha \in 2^\omega\exists\zeta \in [\omega]^\omega\forall n[\beta(\overline{\alpha}n) = 0 \wedge \alpha \circ \zeta(n) \neq 0]$. Conclude that Share($\mathcal{INF} \cap 2^\omega$) is $\mathbf{\Sigma}^1_1$.

We now prove that Share($\mathcal{INF}$) and Share($\mathcal{INF} \cap 2^\omega$) are $\mathbf{\Sigma}^1_1$–complete. First define δ such that $\delta(0) = 0$ and $\forall s\forall n[\delta(s * \langle n\rangle) = \delta(s) * \underline{\overline{0}}n * \langle 1\rangle]$. Then define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\varphi|\alpha)(s) = 0 \leftrightarrow \exists n\exists t[s = \delta(t) * \underline{\overline{0}}n \wedge \forall u \sqsubseteq t[\alpha(u) = 0]]]$. Note that, for each α, $\mathrm{Spr}(\varphi|\alpha)$, ie $\varphi|\alpha$ is a spread-law, and $\mathcal{F}_{\varphi|\alpha} \subseteq 2^\omega$. We show that φ reduces $\mathcal{E}_1^1$ to both $\mathsf{Share}(\mathcal{INF} \cap 2^\omega)$ and $\mathsf{Share}(\mathcal{INF})$.

First, assume that $\alpha \in \mathcal{E}_1^1$. Find γ such that $\forall n[\alpha(\overline{\gamma}n) = 0]$. Note that $\forall n \forall t[t \sqsubseteq \delta(\overline{\gamma}n) \to (\varphi|\alpha)(t) = 0]$. Find ε in 2^ω such that $\forall n[\delta(\overline{\gamma}n) \sqsubset \varepsilon]$. Note that $\varepsilon \in \mathcal{F}_{\varphi|\alpha}$ and, as $\forall n[\varepsilon\big(n + \sum_{i=0}^{i=n}\gamma(i)\big) = 1]$, also $\varepsilon \in \mathcal{INF}$. Conclude that $\varphi|\alpha \in \mathsf{Share}(\mathcal{INF} \cap 2^\omega) \subseteq \mathsf{Share}(\mathcal{INF})$.

Now assume that $\varphi|\alpha \in \mathsf{Share}(\mathcal{INF})$. Find ε in $\mathcal{INF} \cap \mathcal{F}_{\varphi|\alpha}$. Define γ such that $\gamma(0) := \mu i[\varepsilon(i) \neq 0]$ and $\forall n[\gamma(n+1) = \mu i[\varepsilon\big(\gamma(n) + i + 1\big) \neq 0]$. Note that $\forall n[\delta(\overline{\gamma}n) \sqsubset \varepsilon]$ and $\forall n[\alpha(\overline{\gamma}n) = 0]$ and $\alpha \in \mathcal{E}_1^1$. We thus see that φ reduces $\mathcal{E}_1^1$ to both $\mathsf{Share}(\mathcal{INF} \cap 2^\omega)$ and $\mathsf{Share}(\mathcal{INF})$. It follows that these sets, like $\mathcal{E}_1^1$, are $\mathbf{\Sigma}_1^1$–complete. □

2.5 Strictly analytic subsets of ω^ω

Definition 10 *$\mathcal{X} \subseteq \omega^\omega$ is* strictly analytic *or $\mathbf{\Sigma}_1^{1*}$ if and only if there exists β such that $\mathrm{Spr}(\beta)$ and $\mathcal{X} = \mathcal{EF}_\beta := Ex(\mathcal{F}_\beta) = \{\alpha \mid \exists\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{F}_\beta]\}$.*

$\mathcal{X} \subseteq \omega^\omega$ thus is strictly analytic if it its the projection of a closed *and located* subset of ω^ω; see Section 1.1.4.

Recall that $\mathcal{X} \subseteq \omega^\omega$ is *located* if and only if $\exists\alpha[\{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\} = D_\alpha]$, ie the set $\{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\}$ is a *decidable* subset of ω, and $\mathcal{X} \subseteq \omega^\omega$ is *semi-located* if and only if $\exists\alpha[\{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\} = E_\alpha]$, ie the set $\{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\}$ is an *enumerable* subset of ω.

Also recall that, for every infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \dots$ of subsets of ω^ω, $\mathbb{D}_n(\mathcal{X}_n) = \{\gamma \mid \exists n[\gamma^n \in \mathcal{X}_n]\}$ and $\mathbb{C}_n(\mathcal{X}_n) = \{\gamma \mid \forall n[\gamma^n \in \mathcal{X}_n]\}$; see Section 1.2.5.

The following theorem shows that $\mathbf{\Sigma}_1^{1*}$ is a proper subclass of $\mathbf{\Sigma}_1^1$ and behaves less nicely.

Note that, as a consequence of the first item of the theorem, every strictly analytic subset of ω^ω is either empty or inhabited.

Theorem 2.10

(i) *For every $\mathcal{X} \subseteq \omega^\omega$, $\mathcal{X} \in \mathbf{\Sigma}_1^{1*} \leftrightarrow (\mathcal{X} = \emptyset \ \vee \ \exists\varphi\colon \omega^\omega \to \omega^\omega[\mathcal{X} = \varphi|\omega^\omega])$.*
(ii) *For every $\mathcal{X} \subseteq \omega^\omega$, if $\mathcal{X} \in \mathbf{\Sigma}_1^{1*}$, then $\mathcal{X}$ is semi-located.*
(iii) *For every $\mathcal{X} \subseteq \omega^\omega$, if $\mathcal{X}$ is inhabited and semi-located, then $\overline{\mathcal{X}} \in \mathbf{\Sigma}_1^{1*}$.*

(iv) *Not every inhabited and closed subset of ω^ω is semi-located, ie* $\neg\forall\beta[\exists\gamma[\gamma \in \mathcal{F}_\beta] \to \mathcal{F}_\beta$ *is semi-located* $]$.

(v) *Every spread is strictly analytic but not every closed subset of ω^ω is strictly analytic, ie* $\forall\beta[\mathrm{Spr}(\beta) \to \mathcal{F}_\beta \in \mathbf{\Sigma}^{1*}_1]$ *but* $\neg\forall\beta[\mathcal{F}_\beta \in \mathbf{\Sigma}^{1*}_1]$, *ie* $\neg(\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^{1*}_1)$.

(vi) *Semi-located and closed subsets of ω^ω are not always located subsets of ω^ω, ie* $\neg\forall\beta[\mathcal{F}_\beta$ *is semi-located* $\to \mathcal{F}_\beta$ *is located* $]$.

(vii) *The closure of an open subset of ω^ω is not always a closed subset of ω^ω, ie* $\neg\forall\beta\exists\gamma[\mathcal{F}_\gamma = \overline{\mathcal{G}_\beta}]$.

(viii) $\mathbf{\Sigma}^{1*}_1$ *is closed under the operation of (finite) union but* $\mathbf{\Sigma}^{1*}_1$ *is not closed under the operation of (finite) intersection, because:* $\neg\forall\beta[\{\beta\} \cap \{\underline{0}\} \in \mathbf{\Sigma}^{1*}_1]$ *and* $\neg\forall\beta[\{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\} \in \mathbf{\Sigma}^{1*}_1]$.

(ix) $\mathbf{\Sigma}^{1*}_1$ *is not closed under the operation of countable union, because:* $\neg\forall\alpha[\bigcup_n\{\beta \mid \beta = \underline{0} \wedge \alpha(n) \neq 0\} \in \mathbf{\Sigma}^{1*}_1]$.

(x) *For every infinite sequence* $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \dots$ *of strictly analytic and* inhabited *subsets of ω^ω, the sets* $\bigcup_n \mathcal{X}_n$, $\mathbb{D}_n(\mathcal{X}_n)$ *and* $\mathbb{C}_n(\mathcal{X}_n)$ *are strictly analytic.*

(xi) *For every strictly analytic* $\mathcal{X} \subseteq \omega^\omega$, $Ex(\mathcal{X})$ *is strictly analytic.*

Proof (i) First, assume that $\mathcal{X} \in \mathbf{\Sigma}^{1*}_1$. Find β such that $\mathrm{Spr}(\beta)$ and $\mathcal{X} = Ex(\mathcal{F}_\beta)$. There are two cases: $\beta(0) \neq 0$ and $\beta(0) = 0$. In the first case: $\mathcal{X} = \mathcal{F}_\beta = \emptyset$. In the second case, let $\rho : \omega^\omega \to \mathcal{F}_\beta$ be the canonical retraction[9] of ω^ω onto $\mathcal{F}_\beta$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[\varphi|\alpha = (\rho|\alpha)_I]$ and note that $\mathcal{X} = \varphi|\omega^\omega$.

Conversely, let $\mathcal{X} \subseteq \omega^\omega$ and $\varphi\colon \omega^\omega \to \omega^\omega$ be given such that $\mathcal{X} = \varphi|\omega^\omega$. Define β in 2^ω such that $\beta(\langle\,\rangle) = 0$ and, for each $n > 0$, for each s in ω^n, $\beta(s) = 0$ if and only if $\forall i < n-1[s_{II}(i) \sqsubset s_{II}(i+1)]]$ and $\forall i < n[\overline{s_I}(i+1) \sqsubseteq \varphi|\big(s_{II}(i)\big)]$. Note that $\mathrm{Spr}(\beta)$, $\mathcal{Y} = \mathcal{F}_\beta$ and $\varphi|\omega^\omega = \mathcal{Y}$.

(ii) Assume that $\mathcal{X} \in \mathbf{\Sigma}^{1*}_1$; that is, by (i) either $\mathcal{X} = \emptyset$ or $\exists\varphi\colon \omega^\omega \to \omega^\omega[\mathcal{X} = \varphi|\omega^\omega]$. Note that $\emptyset$ is semi-located. Now assume that $\mathcal{X}$ is inhabited. Find $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{X} = \varphi|\omega^\omega$. Note that $\forall s[\exists\gamma[s \sqsubset \varphi|\gamma] \leftrightarrow \exists t[s \sqsubset \varphi|t]]$. Define δ such that $\forall n[(n_I \sqsubseteq \varphi|n_{II} \to \delta(n) = n_I + 1) \ \wedge\ (\neg(n_I \sqsubseteq \varphi|n_{II}) \to \delta(n) = 0)]$. Note that $E_\delta = \{s \mid \exists\gamma[s \sqsubset \varphi|\gamma]\}$ and conclude that $\mathcal{X} = \varphi|\omega^\omega$ is semi-located.

(iii) Assume that $\mathcal{X} \subseteq \omega^\omega$ is inhabited and semi-located. Find δ such that $E_\delta = \{s \mid \exists\gamma \in \mathcal{X}[s \sqsubset \gamma]\}$. Note that $\exists n[\delta(n) = \langle\,\rangle + 1 = 1]$ and $\forall s \in E_\delta \exists n \exists p[\delta(n) = s * \langle p\rangle + 1]$. Define ε such that $\varepsilon(0) = 0$ and, for all s, n, *if* $\exists p[\delta(n) = \varepsilon(s) * \langle p\rangle + 1]$, then $\varepsilon(s * \langle n\rangle) = \delta(n) - 1$, and, *if not*, then $\varepsilon(s * \langle n\rangle) = \delta(m) - 1$, where $m = \mu q[\exists p[\delta(q) = \varepsilon(s) * \langle p\rangle + 1]]$. Now define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall n[\varepsilon(\overline{\alpha}n) \sqsubset \varphi|\alpha]$ and note that $\overline{\mathcal{X}} = \varphi|\omega^\omega$.

[9] see Section 1.1.5

(iv) Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\varphi|\alpha)(s) = 0 \leftrightarrow \big(s \sqsubset \underline{0} \vee (s \sqsubset \underline{1} \wedge \overline{0}s \sqsubset \alpha)\big)]$. Note that $\forall\alpha\forall\gamma[\gamma \in \mathcal{F}_{\varphi|\alpha} \leftrightarrow \big(\gamma = \underline{0} \vee (\gamma = \underline{1} \wedge \alpha = \underline{0})\big)]$. Assume that $\forall\alpha[\mathcal{F}_{\varphi|\alpha}$ *is semi-located*]. Using $\mathbf{AC}_{1,1}$, find $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[E_{\psi|\alpha} = \{s \mid \exists\gamma \in \mathcal{F}_{\varphi|\alpha}[s \sqsubset \gamma]\}]$. Note that $\langle 1\rangle \in E_{\psi|\underline{0}}$. Find p such that $(\psi|\underline{0})(p) = \langle 1\rangle + 1$. Find q such that $\psi^p(\overline{0}q) = \langle 1\rangle + 2$ and $\forall i < q[\psi^p(\overline{0}i) = 0]$. Note that $\forall\alpha[\overline{0}q \sqsubset \alpha \to \langle 1\rangle \in E_{\psi|\alpha}]$. Conclude that $\forall\alpha[\overline{0}q \sqsubset \alpha \to \alpha = \underline{0}]$, a contradiction.

(v) Let β be given such that $\mathrm{Spr}(\beta)$. Define γ such that $\forall s[\gamma(s) = 0 \leftrightarrow \beta(s_I) = 0]$. Note that $\mathrm{Spr}(\gamma)$ and $\mathcal{F}_\beta = Ex(\mathcal{F}_\gamma)$. Conclude that $\mathcal{F}_\beta \in \mathbf{\Sigma}^{1*}_1$.

Assume that $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^{1*}_1$. Then, according to (ii), $\forall\beta[\mathcal{F}_\beta$ is semi-located]. This conclusion contradicts (iv).

(vi) Assume that $\forall\beta[\mathcal{F}_\beta$ is semi-located $\to \mathcal{F}_\beta$ is located]. Let α be given. Define β such that $\forall s[\beta(s) = 0 \leftrightarrow \big(length(s) \geq 1 \to \alpha \circ s(0) \neq 0\big)]$. Note that $\mathcal{F}_\beta = \{\gamma \mid \alpha \circ \gamma(0) \neq 0]\}$. Define δ such that for each n, *if* either $\mathrm{length}(n_I) \geq 1$ and $\alpha \circ n_I(0) \neq 0$, or $n_I = 0 = \langle\ \rangle$ and $\alpha(n_{II}) \neq 0$; then: $\delta(n) = n_I + 1$ and, if not, then $\delta(n) = 0$. Note that $E_\delta = \{s \mid \exists\gamma \in \mathcal{F}_\beta[s \sqsubset \gamma]\}$. Conclude that $\mathcal{F}_\beta$ is semi-located. Using the above assumption, conclude that $\mathcal{F}_\beta$ is located. Find ε such that $E_\delta = D_\varepsilon$. Note that if $\varepsilon(0) = 0$, then $0 \notin D_\varepsilon = E_\delta$ and $\forall n[\alpha(n) = 0]$ and, if $\varepsilon(0) \neq 0$, then $0 \in D_\varepsilon = E_\delta$ and $\exists n[\alpha(n) \neq 0]$. Conclude that $\forall n[\alpha(n) = 0] \vee \exists n[\alpha(n) \neq 0]$. We thus see that our assumption implies **LPO** and is contradictory; see Section 1.1.11.

(vii) Assume that $\forall\beta\exists\gamma[\mathcal{F}_\gamma = \overline{\mathcal{G}_\beta}]$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\varphi|\alpha)(s) = 0 \leftrightarrow (s \perp \underline{0} \wedge \overline{\alpha}s \perp \underline{0})]$. Note that $\mathcal{G}_{\varphi|\underline{0}} = \emptyset$, and, for every α, if $\alpha \# \underline{0}$, then $\mathcal{G}_{\varphi|\alpha} = \{\delta \mid \delta \# \underline{0}\}$. By our assumption, $\forall\alpha\exists\gamma[\mathcal{F}_\gamma = \overline{\mathcal{G}_{\varphi|\alpha}}]$. Using $\mathbf{AC}_{1,1}$, find $\rho : \omega^\omega \to \omega^\omega$ such that $\forall\alpha[\mathcal{F}_{\rho|\alpha} = \overline{\mathcal{G}_{\varphi|\alpha}}]$. Note that $\mathcal{F}_{\rho|\underline{0}} = \emptyset$, and, for every α, if $\alpha \# \underline{0}$, then $\mathcal{F}_{\rho|\alpha} = \omega^\omega$. Assume that we find n such that $(\rho|\underline{0})(\overline{0}n) \neq 0$. Determine p such that $\forall\alpha[\overline{0}p \sqsubset \alpha \to (\rho|\alpha)(\overline{0}n) \neq 0]$. Conclude that $\forall\alpha[\overline{0}p \sqsubset \alpha \to \underline{0} \notin \mathcal{F}_{\rho|\alpha}]$, a contradiction. Conclude that $\forall n[(\rho|\underline{0})(\overline{0}n) = 0]$ and $\underline{0} \in \mathcal{F}_{\rho|\underline{0}}$, a contradiction.

(viii) Assume $\mathcal{X}_0, \mathcal{X}_1 \subseteq \omega^\omega$ are strictly analytic. It suffices to consider the case that both $\mathcal{X}_0, \mathcal{X}_1$ are inhabited. Find φ such that $\forall i < 2[\varphi^i\colon \omega^\omega \to \omega^\omega \wedge \mathcal{X}_i = \varphi^i|\omega^\omega]$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall n[\psi|(\langle 0\rangle * \alpha) = \varphi^0|\alpha \wedge \psi|(\langle n+1\rangle * \alpha) = \varphi^1|\alpha]$ and note that $\mathcal{X}_0 \cup \mathcal{X}_1 = \psi|\omega^\omega$.

Assume that $\forall\beta[\{\beta\} \cap \{\underline{0}\} \in \mathbf{\Sigma}^{1*}_1]$. Using (i), conclude that $\forall\beta[\{\beta\} \cap \{\underline{0}\} = \emptyset \vee \exists\gamma[\gamma \in \{\beta\} \cap \underline{0}\}]]$, and $\forall\beta[\beta \neq \underline{0} \vee \beta = \underline{0}]$. Using **BCP**, find p such that either: $\forall\beta[\overline{0}p \sqsubset \beta \to \beta \neq \underline{0}]$ or $\forall\beta[\overline{0}p \sqsubset \beta \to \beta = \underline{0}]$. Both alternatives are false, so we obtain a contradiction.

Now assume that $\forall\beta[\{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\} \in \mathbf{\Sigma}^{1*}_1]$. According to (ii), for each β, $\{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\}$ is semi-located, ie $\exists\delta[E_\delta = \{s \mid \exists\gamma \in \{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\}[s \sqsubset \gamma]\}]$. Using $\mathbf{AC}_{1,1}$, find $\varphi\colon \omega^\omega \to \omega^\omega$ such that, for each β, $E_{\varphi|\beta} = \{s \mid \exists\gamma \in \{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\}[s \sqsubset \gamma]\}$. Note that $\langle 0\rangle \in E_{\varphi|\underline{0}}$ and find p such that $(\varphi|\underline{0})(p) = \langle 0\rangle + 1$. Find m such that $\forall\beta[\overline{0}m \sqsubset \beta \to (\varphi|\beta)(p) = (\varphi|\underline{0})(p)]$. Conclude that $\forall\beta[\overline{0}m \sqsubset \beta \to \langle 0\rangle \in E_{\varphi|\beta}]$ and $\forall\beta[\overline{0}m \sqsubset \beta \to \underline{0} \in \{\beta, \underline{1}\} \cap \{\underline{0}, \underline{1}\}]$, ie $\forall\beta[\overline{0}m \sqsubset \beta \to \beta = \underline{0}]$, a contradiction.

(ix) Assume that $\forall\alpha[\bigcup_n\{\beta \mid \beta = \underline{0} \ \wedge\ \alpha(n) \neq 0\} \in \mathbf{\Sigma}^{1*}_1$. Then, according to (i), $\forall\alpha[\bigcup_n\{\beta \mid \beta = \underline{0} \ \wedge\ \alpha(n) \neq 0\} = \emptyset \ \vee\ \exists\gamma[\gamma \in \bigcup_n\{\beta \mid \beta = \underline{0} \ \wedge\ \alpha(n) \neq 0\}]]$, and $\forall\alpha[\forall n[\alpha(n) = 0] \ \vee\ \exists n[\alpha(n) \neq 0]]$, ie **LPO**, a contradiction; see Section 1.1.11.

(x) Let $\mathcal{X}_0, \mathcal{X}_1, \ldots$ be an infinite sequence of inhabited strictly analytic subsets of ω^ω. Using (i) and $\mathbf{AC}_{0,1}$, find φ such that $\forall n[\varphi^n : \omega^\omega \to \omega^\omega \ \wedge\ \mathcal{X}_n = \varphi^n|\omega^\omega]$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that, for all n, for all α, $\psi|(\langle n\rangle * \alpha) = \varphi^n|\alpha$ and note that $\bigcup_n \mathcal{X}_n = \psi|\omega^\omega$ is strictly analytic. Define $\rho\colon \omega^\omega \to \omega^\omega$ such that, for all n, for all α, $\big(\rho|(\langle n\rangle * \alpha)\big)^n = \varphi^n|(\alpha^n)$ and, for all $i \neq n$, $\big(\rho|(\langle n\rangle * \alpha)\big)^i = \alpha^i$ and note that $\mathbb{D}_n\mathcal{X}_n = \rho|\omega^\omega$ is strictly analytic. Define $\tau\colon \omega^\omega \to \omega^\omega$ such that, for all n, for all α, $(\tau|\alpha)^n = \varphi^n|(\alpha^n)$ and conclude that $\mathbb{C}_n\mathcal{X}_n = \tau|\omega^\omega$ is strictly analytic.

(xi) Assume $\mathcal{X} \subseteq \omega^\omega$ is strictly analytic. Then, according to (i), one may decide that $\mathcal{X} = \emptyset$ or $\mathcal{X}$ is inhabited. Note that $Ex(\emptyset) = \emptyset$ is strictly analytic. If $\mathcal{X}$ is inhabited, find $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{X} = \varphi|\omega^\omega$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[\psi|\alpha = (\varphi|\alpha)_I]$ and note that $Ex(\mathcal{X}) = \psi|\omega^\omega$ is strictly analytic. □

Using Theorem 2.10(x), one may prove that for every σ in $\mathcal{HRS}$, $\mathcal{E}_\sigma$ and $\mathcal{A}_\sigma$ are strictly analytic. The sets $\mathcal{E}_\sigma, \mathcal{A}_\sigma$, are the leading sets of the intuitionistic Borel hierarchy; see Section 1.2.4.

We conclude our discussion of strictly analytic subsets of ω^ω by observing that *Kripke's scheme* **KS**, see Section 1.1.10, makes the gap between analytic and strictly analytic subsets of ω^ω somewhat smaller.

Theorem 2.11 *(Using* **KS***:)*

(i) *Every* inhabited *and* definite *closed subset of ω^ω is strictly analytic.*

(ii) *Every* inhabited *and* definite *analytic subset of ω^ω is strictly analytic.*

Proof (i) Assume $\mathcal{F} \subseteq \omega^\omega$ is inhabited, definite and closed. According to Theorem 1.1 in Section 1.1.10, $\mathcal{F}$ is semi-located. According to Theorem 2.10(iii), $\mathcal{F}$ is strictly analytic.

(ii) Assume $\mathcal{X} \subseteq \omega^\omega$ is inhabited, definite and analytic. Find $\mathcal{F}$ in $\mathbf{\Pi}^0_1$ such that $\mathcal{X} = Ex(\mathcal{F})$. Note that $\mathcal{F}$ is inhabited. We assume that also $\mathcal{F}$ is definite. According to (i), $\mathcal{F}$ is strictly analytic. According to Theorem 2.10(xi), also $\mathcal{X} = Ex(\mathcal{F})$ is strictly analytic. □

John Burgess, in [7], also studies strictly analytic subsets of ω^ω, or, as he calls them, using a term of of Brouwer's and following Gielen–de Swart–Veldman [11], "*dressed spreads*". Avoiding $\mathbf{AC}_{1,1}$ but not restricting application of the Brouwer-Kripke scheme to definite propositions, he concludes that every inhabited analytic subset of ω^ω is strictly analytic. The argument given here for Theorem 2.11(ii) is essentially his.

3 Separation theorems

3.1 Results by Lusin and Novikov

Definition 11 *Let $\mathcal{X}, \mathcal{Y}$ be subsets of ω^ω. We define: the pair $(\mathcal{X}, \mathcal{Y})$ is* positively disjoint, *notation $\mathcal{X} \# \mathcal{Y}$, if and only if, for all α in $\mathcal{X}$, for all β in $\mathcal{Y}$, $\alpha \# \beta$.*[10]

We also define: the pair $(\mathcal{X}, \mathcal{Y})$ is Borel-separable, *notation $\mathcal{X} \#^{\mathfrak{Borel}} \mathcal{Y}$, if and only if there exist (positively) Borel sets $\mathcal{A}, \mathcal{B}$ such that $\mathcal{X} \subseteq \mathcal{A}$, $\mathcal{Y} \subseteq \mathcal{B}$ and $\mathcal{A} \# \mathcal{B}$.*

Lemma 3.1 *Let $\mathcal{Y}, \mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots$ be an infinite sequence of subsets of ω^ω. If, for each n, $\mathcal{Y} \#^{\mathfrak{Borel}} \mathcal{X}_n$, then $\mathcal{Y} \#^{\mathfrak{Borel}} \bigcup_n \mathcal{X}_n$.*

Proof Assume that for each n, $\mathcal{Y} \#^{\mathfrak{Borel}} \mathcal{X}_n$. Find,[11] for each n, Borel sets $\mathcal{A}_n, \mathcal{B}_n$ such that $\mathcal{Y} \subseteq \mathcal{A}_n$ and $\mathcal{X}_n \subseteq \mathcal{B}_n$ and $\mathcal{A}_n \# \mathcal{B}_n$. Define $\mathcal{A} := \bigcap_n \mathcal{A}_n$ and $\mathcal{B} := \bigcup_n \mathcal{B}_n$. Note that $\mathcal{A}, \mathcal{B}$ are Borel and $\mathcal{Y} \subseteq \mathcal{A}$ and $\bigcup_n \mathcal{X}_n \subseteq \mathcal{B}$ and $\mathcal{A} \# \mathcal{B}$. Conclude that $\mathcal{Y} \#^{\mathfrak{Borel}} \bigcup_n \mathcal{X}_n$. □

A version of the next theorem occurs in Veldman [30, Theorem 18.4.1, page 163]. A related result is proven in Aczel [1].

Theorem 3.2 (Lusin's Separation Theorem) *Let $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$ be strictly analytic. If $\mathcal{X} \# \mathcal{Y}$, then $\mathcal{X} \#^{\mathfrak{Borel}} \mathcal{Y}$.*

Proof Let $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$ be strictly analytic. Assume that $\mathcal{X} \# \mathcal{Y}$.

If $\mathcal{X} = \emptyset$, we define $\mathcal{A} := \emptyset$ and $\mathcal{B} := \omega^\omega$, and are done. If $\mathcal{Y} = \emptyset$, we define $\mathcal{A} := \omega^\omega$ and $\mathcal{B} := \emptyset$, and are done.

[10] $\alpha \perp \beta \leftrightarrow \alpha \# \beta \leftrightarrow \exists n[\alpha(n) \neq \beta(n)]$, see Section 1.1.2.

[11] We are silently applying the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, as Borel sets should be thought as given by means of their codes, see Section 1.2.4. We do so at other occasions too, without further warning.

We thus may assume that $\mathcal{X}, \mathcal{Y}$ are inhabited. Find $\varphi, \psi\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{X} = \varphi|\omega^\omega$ and $\mathcal{Y} = \psi|\omega^\omega$. Define $B := \{s \mid \varphi|s^0 \perp \psi|s^1\}$. We first prove that B is a bar in ω^ω.

Let α be given. Find n such that $\overline{\varphi|\alpha^0}n \perp \overline{\psi|\alpha^1}n$. Then find m such that $\overline{\varphi|\alpha^0}n \sqsubseteq \varphi|\overline{\alpha^0}m$ and $\overline{\psi|\alpha^1}n \sqsubseteq \psi|\overline{\alpha^1}m$. Find p such that $\overline{\alpha^0}m \sqsubseteq (\overline{\alpha}p)^0$ and $\overline{\alpha^1}m \sqsubseteq (\overline{\alpha}p)^1$ and note that $\overline{\alpha}p \in B$. We thus see that $\forall\alpha\exists p[\overline{\alpha}p \in B]$, ie B is a bar in ω^ω.

Now define $C := \{s \mid \varphi|(\omega^\omega \cap s^0) \;\#^{\mathfrak{Borel}}\; \psi|(\omega^\omega \cap s^1)\}$. We first prove that $B \subseteq C$. Let s in B be given. Then $\varphi|s^0 \perp \psi|s^1$. Define $\mathcal{A} := \omega^\omega \cap (\varphi|s^0)$ and $\mathcal{B} := \omega^\omega \cap (\psi|s^0)$. Note that $(\mathcal{A}, \mathcal{B})$ is a (positively) disjoint pair of basic open sets and $\varphi|(\omega^\omega \cap s^0) \subseteq \mathcal{A}$ and $\psi|(\omega^\omega \cap s^1) \subseteq \mathcal{B}$. Conclude that $s \in C$. We thus see that $\forall s \in B[s \in C]$, ie $B \subseteq C$.

Note that C is monotone: for each s, for each n, $s^0 \sqsubseteq (s * \langle n\rangle)^0$ and $s^1 \sqsubseteq (s * \langle n\rangle)^1$, and, therefore, if $s \in C$, also, $s * \langle n\rangle \in C$.

We finally prove that C is inductive. Let s be given such that $\forall n[s * \langle n\rangle \in C]$. We want to prove: $s \in C$. Consider $k := \mathrm{length}(s)$ and distinguish three cases.

Case (a). $\neg\exists i < 2\exists t[k = \langle i\rangle * t]$. Then, for each n, $(s * \langle n\rangle)^0 = s^0$ and $(s * \langle n\rangle)^1 = s^1$. Note that $s * \langle 0\rangle \in C$, and, therefore, also $s \in C$.

Case (b). $\exists t[k = \langle 0\rangle * t]$. Then, for all n, $(s * \langle n\rangle)^0 = s^0 * \langle n\rangle$ and $(s * \langle n\rangle)^1 = s^1$. Conclude that for all n, $\varphi|(\omega^\omega \cap s^0 * \langle n\rangle) \;\#^{\mathfrak{Borel}}\; \psi|(\omega^\omega \cap s^1)$. Note that $\varphi|(\omega^\omega \cap s^0) = \bigcup_n \varphi|(\omega^\omega \cap s^0 * \langle n\rangle)$. Conclude, using Lemma 3.1, $\varphi|(\omega^\omega \cap s^0) \;\#^{\mathfrak{Borel}}\; \psi|(\omega^\omega \cap s^1)$, ie $s \in C$.

Case (c). $\exists t[k = \langle 1\rangle * t]$. Then, for all n, $(s * \langle n\rangle)^0 = s^0$ and $(s * \langle n\rangle)^1 = s^1 * \langle n\rangle$. Conclude that for all n, $\varphi|(\omega^\omega \cap s^0) \;\#^{\mathfrak{Borel}}\; \psi|(\omega^\omega \cap s^1 * \langle n\rangle)$. Note that $\psi|(\omega^\omega \cap s^1) = \bigcup_n \psi|(\omega^\omega \cap s^1 * \langle n\rangle)$. Conclude, using Lemma 3.1, $\varphi|(\omega^\omega \cap s^0) \;\#^{\mathfrak{Borel}}\; \psi|(\omega^\omega \cap s^1)$, ie $s \in C$.

Using the Principle of Bar Induction **BI**, see Section 1.1.9, we conclude that $\langle\;\rangle \in C$, ie $\varphi|\omega^\omega \;\#^{\mathfrak{Borel}}\; \psi|\omega^\omega$. $\square$

Definition 12 *Let $\mathcal{X}_0, \mathcal{X}_1, \ldots$ be an infinite sequence of subsets of ω^ω. We define: the infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \ldots$* positively refuses to have a common point, or is ω–separate, *notation $\#_n\mathcal{X}_n$, if and only if, for every α, if $\forall n[\alpha^n \in \mathcal{X}_n]$, then $\exists i\exists j[\alpha^i \perp \alpha^j]$.*

We also define: the infinite sequence $\mathcal{X}_0, \mathcal{X}_1, \ldots$ is ω–Borel separable, *notation $\#_n^{\mathfrak{Borel}}\mathcal{X}_n$, if and only if there exists an infinite sequence $\mathcal{B}_0, \mathcal{B}_1\ \ldots$ of (positively) Borel sets such that $\forall n[\mathcal{X}_n \subseteq \mathcal{B}_n]$ and $\#_n\mathcal{B}_n$.*

Lemma 3.3 *Let $\mathcal{Y}_0, \mathcal{Y}_1, \dots$ and $\mathcal{X}_0, \mathcal{X}_1, \dots$ be infinite sequences of subsets of ω^ω.*

If, for each n, the infinite sequence $\mathcal{Y}_n, \mathcal{X}_0, \mathcal{X}_1, \dots$ is ω–Borel separable, then also the infinite sequence $\bigcup_n \mathcal{Y}_n, \mathcal{X}_0, \mathcal{X}_1, \dots$ is ω–Borel-separable.

Proof Assume that for each n, the infinite sequence $\mathcal{Y}_n, \mathcal{X}_0, \mathcal{X}_1, \dots$ is ω–Borel separable, Find, for each n, an infinite and ω–separate sequence $\mathcal{B}_n, \mathcal{C}_{n,0}, \mathcal{C}_{n,1}, \dots$ of (positively) Borel sets such that $\mathcal{Y} \subseteq \mathcal{B}_n$ and, for each i, $\mathcal{X}_i \subseteq \mathcal{C}_{n,i}$. Define $\mathcal{B} := \bigcup_n \mathcal{B}_n$ and, for each i, $\mathcal{C}_i := \bigcap_n \mathcal{C}_{n,i}$. Note that $\mathcal{B}$ is Borel, and for each i, $\mathcal{C}_i$ is Borel and $\bigcup_n \mathcal{Y}_n \subseteq \mathcal{B}$ and, for each i, $\mathcal{X}_i \subseteq \mathcal{C}_i$ and the infinite sequence $\mathcal{B}, \mathcal{C}_0, \mathcal{C}_1, \dots$ is ω–separate. Conclude that the infinite sequence $\bigcup_n \mathcal{Y}_n, \mathcal{X}_0, \mathcal{X}_1, \dots$ is ω–Borel-separable. □

Theorem 3.4 (Novikov's Separation Theorem) *Let $\mathcal{X}_0, \mathcal{X}_1, \dots$ be an infinite sequence of* inhabited *strictly analytic subsets of ω^ω. If $\#_n(\mathcal{X}_n)$, then $\#_n^{\mathfrak{Borel}}(\mathcal{X}_n)$.*

Proof Let $\mathcal{X}_0, \mathcal{X}_1, \dots$ be an infinite sequence of *inhabited* strictly analytic subsets of ω^ω such that $\#_n(\mathcal{X}_n)$. Using $\mathbf{AC}_{0,1}$, find φ such that $\forall n[\varphi^n : \omega^\omega \to \omega^\omega \wedge \mathcal{X}_n = \varphi^n|\omega^\omega]$. Define $B := \{s \mid \exists i \exists j[\varphi^i|s^i \perp \varphi^j|s^j]\}$. We first prove that B is a bar in ω^ω.

Let α be given. Find i, j, n such that $\overline{\varphi^i|\alpha^i}n \perp \overline{\varphi^j|\alpha^j}n$. Then find m such that $\overline{\varphi^i|\alpha^i}n \sqsubseteq \varphi^i|(\overline{\alpha^i}m)$ and $\overline{\varphi^j|\alpha^j}n \sqsubseteq \varphi^j|(\overline{\alpha^j}m)$. Find p such that $\overline{\alpha^i}m \sqsubseteq (\overline{\alpha}p)^i$ and $\overline{\alpha^j}m \sqsubseteq (\overline{\alpha}p)^j$ and note that $\overline{\alpha}p \in B$. We thus see that $\forall \alpha \exists p[\overline{\alpha}p \in B]$.

Define $C := \{s \mid \#_n^{\mathfrak{Borel}} \varphi^n|(\omega^\omega \cap s^n)\}$. Note that, for each p, $\langle p \rangle \in C$ if and only if $\langle\ \rangle \in C$, as, for each p, $\langle p \rangle^0 = \langle p \rangle^1 = \langle\ \rangle$; see Section 1.1.1. We prove that $B \subseteq C$. Let s in B be given. Find i, j such that $\varphi^i|s^i \perp \varphi^j|s^j$. Define an infinite sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of subsets of ω^ω such that $\mathcal{B}_i = \omega^\omega \cap \varphi^i|s_i$ and $\mathcal{B}_j = \omega^\omega \cap \varphi^j|s_j$, and, for all k, if $k \neq i$ and $k \neq j$, then $\mathcal{B}_k = \omega^\omega$. Note that, for all n, $\mathcal{B}_n$ is Borel and $\varphi^n|(\omega^\omega \cap s^n) \subseteq \mathcal{B}_n$. Also note that $\#_n \mathcal{B}_n$. Conclude $s \in C$. We thus see that $\forall s \in B[s \in C]$, ie $B \subseteq C$.

Note that C is monotone as, for all s, t, for all $\psi : \omega^\omega \to \omega^\omega$, if $s \sqsubseteq t$, then $\psi|(\omega^\omega \cap t) \subseteq \psi|(\omega^\omega \cap s)$.

We finally prove that C is inductive. Let s be given such that $\forall n[s * \langle n \rangle \in C]$. We want to prove: $s \in C$. Consider $k := \text{length}(s)$.

Case (a). $k = 0$. Then $s = \langle\ \rangle$ and $s * \langle 0 \rangle = \langle 0 \rangle$ and $s * \langle 0 \rangle \in C$ and, therefore, $s \in C$.

Case (b). $k \neq 0$. Find i such that $k = \langle i \rangle * t$. Note that for each n, $(s * \langle n \rangle)^i = s^i * \langle n \rangle$, and, for all $j \neq i$. $(s * \langle n \rangle)^j = s^j$. Conclude that for each n, the infinite sequence of sets

$$\varphi_0|(\omega^\omega \cap s_0), \varphi_1|(\omega^\omega \cap s_1), \dots \varphi_{i-1}|(\omega^\omega \cap s_{i-1}), \varphi_i|(\omega^\omega \cap s_i * \langle n \rangle), \varphi_{i+1}|(\omega^\omega \cap s_{i+1}), \dots$$

is ω–Borel-separable. Note that $\varphi_i|(\omega^\omega \cap s_i) = \bigcup_n \varphi_i|(\omega^\omega \cap s_i * \langle n\rangle)$. Conclude, using Lemma 3.3 that the infinite sequence of sets:

$$\varphi_0|(\omega^\omega \cap s_0), \varphi_1|(\omega^\omega \cap s_1), \ldots \varphi_{i-1}|(\omega^\omega \cap s_{i-1}), \varphi_i|(\omega^\omega \cap s_i), \varphi_{i+1}|(\omega^\omega \cap s_{i+1}), \ldots$$

is ω–Borel-separable, ie $s \in C$.

Using the Principle of Bar Induction **BI**, we conclude that $\langle\,\rangle \in C$, ie $\#_n^{\mathfrak{Borel}} \varphi_n|\omega^\omega$. $\square$

3.2 Lusin's representation Theorem

Definition 13 *We define:* $\mathcal{X} \subseteq \omega^\omega$ *is* regular in Lusin's sense *if and if there exists a spread* $\mathcal{F} \subseteq \omega^\omega$ *and a strongly injective function* $\varphi\colon\ \mathcal{F} \rightarrowtail \omega^\omega$ *such that* $\varphi|\mathcal{F} = \mathcal{X}$.

Theorem 3.5 (One half of Lusin's Regular Representation Theorem) *For all* $\mathcal{X} \subseteq \omega^\omega$, *if* $\mathcal{X}$ *is regular in Lusin's sense, then* $\mathcal{X}$ *is positively Borel.*

Proof Let β, φ be given such that $\mathrm{Spr}(\beta)$ and $\varphi\colon\ \mathcal{F}_\beta \rightarrowtail \omega^\omega$. Note that for all s, t, if $\beta(s) = \beta(t) = 0$ and $s <_{lex} t$, then $s \perp t$ and $\varphi|(\mathcal{F}_\beta \cap s) \# \varphi|(\mathcal{F}_\beta \cap t)$. Using Theorem 3.2, find for all s, t such that $\beta(s) = \beta(t) = 0$ and $s <_{lex} t$ a positively disjoint pair $(\mathcal{B}_{s,t,0}, \mathcal{B}_{s,t,1})$ of Borel sets such that $\varphi|(\mathcal{F}_\beta \cap s) \subseteq \mathcal{B}_{s,t,0}$ and $\varphi|(\mathcal{F}_\beta \cap t) \subseteq \mathcal{B}_{s,t,1}$. Define, for each s such that $\beta(s) = 0$:

$$\mathcal{D}_s := \bigcap_{\beta(t)=0, s<_{lex}t} \mathcal{B}_{s,t,0} \ \cap \bigcap_{\beta(t)=0, t<_{lex}s} \mathcal{B}_{t,s,1}$$

Note that for all s, if $\beta(s) = 0$, then $\mathcal{D}_s$ is (positively) Borel and $\varphi|(\mathcal{F}_\beta \cap s) \subseteq \mathcal{D}_s$. Also note that for all s, t, if $\beta(s) = \beta(t) = 0$ and $s <_{lex} t$, then $\mathcal{D}_s \# \mathcal{D}_t$. Note that $\forall \gamma \in \mathcal{F} \forall n[\varphi|\gamma \in \mathcal{D}_{\overline{\gamma}n}]$ and $\forall \alpha \forall s[\big(\beta(s) = 0 \ \wedge\ \alpha \in \mathcal{D}_s\big) \to \varphi|s \sqsubset \alpha]$.

Now define, for each n,

$$\mathcal{H}_n = \bigcup \{\mathcal{D}_s \mid \beta(s) = 0 \ \wedge\ s \in \omega^n\}$$

and note that $\forall n[\varphi|\mathcal{F}_\beta \subseteq \mathcal{H}_n]$. We thus see that $\varphi|\mathcal{F}_\beta \subseteq \bigcap_n \mathcal{H}_n$ and now prove $\bigcap_n \mathcal{H}_n \subseteq \varphi|\mathcal{F}_\beta$.

Assume that $\alpha \in \bigcap_n \mathcal{H}_n$. Find δ such that, for each n, $\delta(n) \in \omega^n$, $\beta\big(\delta(n)\big) = 0$, and $\alpha \in \mathcal{D}_{\delta(n)}$. Note that for each n, $\alpha \in D_{\delta(n)} \cap D_{\delta(n+1)}$, so $\neg\big(\delta(n) \perp \delta(n+1)\big)$ and $\delta(n) \sqsubset \delta(n+1)$. Note that for each n, $\alpha \in D_{\delta(n)}$, and therefore $\varphi|\big(\delta(n)\big) \sqsubset \alpha$. Find γ such that $\forall n[\delta(n) \sqsubset \gamma]$. Note that $\gamma \in \mathcal{F}_\beta$ and $\varphi|\gamma = \alpha$ and $\alpha \in \varphi|\mathcal{F}_\beta$.

We thus see that $\varphi|\mathcal{F}_\beta = \bigcap_n \mathcal{H}_n$ is (positively) Borel. $\square$

Theorem 3.5 shows that if $\mathcal{X} \subseteq \omega^\omega$ is regular in Lusin's sense, then $\mathcal{X}$ is (positively) Borel. The converse, a famous result in classical descriptive set theory, can not be true intuitionistically, as every $\mathcal{X} \subseteq \omega^\omega$ that is regular in Lusin's sense is strictly analytic; and, as we know from Theorem 2.10(v), it is not even true that every closed $\mathcal{X} \subseteq \omega^\omega$ is strictly analytic. The next result shows that the converse of Theorem 3.5 is also not true for strictly analytic sets.

Theorem 3.6

(i) *Let $\mathcal{F} \subseteq \omega^\omega$ be a spread and let $\varphi\colon\ \mathcal{F} \twoheadrightarrow \mathbb{D}^2(\mathcal{A}_1) = \{\gamma \mid \gamma^0 = \underline{0} \ \vee\ \gamma^1 = \underline{0}\}$ be surjective. There exist α, γ in $\mathcal{F}$ such that $\alpha \mathrel{\#} \gamma$ and $\varphi|\alpha = \varphi|\gamma = \underline{0}$.*

(ii) *$\mathbb{D}^2(\mathcal{A}_1) = \{\gamma \mid \gamma^0 = \underline{0} \ \vee\ \gamma^1 = \underline{0}\}$ is strictly analytic and not regular in Lusin's sense.*

(iii) *$\mathcal{A}_1$, $\mathcal{E}_1$, $\mathcal{A}_2$ are regular in Lusin's sense and $\mathcal{E}_2$ is not.*

Proof (i) Define, for both $i < 2$, $\mathcal{P}_i := \{\gamma \mid \gamma^i = \underline{0}\}$. Note that $\mathbb{D}^2(\mathcal{A}_1) = \mathcal{P}_0 \cup \mathcal{P}_1$ and $\mathcal{P}_0, \mathcal{P}_1$ are spreads. Assume that $\mathrm{Spr}(\beta)$ and $\varphi\colon\ \mathcal{F}_\beta \twoheadrightarrow \mathbb{D}^2(\mathcal{A}_1) = \{\gamma \mid \gamma^0 = \underline{0} \ \vee\ \gamma^1 = \underline{0}\}$ is surjective. Find α in $\mathcal{F}_\beta$ such that $\varphi|\alpha = \underline{0}$. Note that $\forall \gamma \in \mathcal{F}_\beta \exists i < 2[(\varphi|\gamma)^i = \underline{0}]$. Applying Brouwer's Continuity Principle **BCP**, find m and $i < 2$ such that $\forall \gamma \in \mathcal{F}_\beta \cap \overline{\alpha}m[(\varphi|\gamma)^i = \underline{0}]$. Again applying **BCP**, find n, s such that $s \in \omega^m$ and $\beta(s) = 0$ and $\forall \delta \in \mathcal{P}_{1-i} \cap \overline{\underline{0}}n \exists \gamma \in \mathcal{F}_\beta \cap s[\varphi|\gamma = \delta]$.

Now distinguish two cases.

Case (a). $s \sqsubset \alpha$.

Define δ in $\mathcal{P}_{1-i} \cap \overline{\underline{0}}n$ such that $\delta^i \mathrel{\#} \underline{0}$. Find γ in $\mathcal{F}_\beta \cap s$ such that $\varphi|\gamma = \delta$. Conclude that $\overline{\alpha}m \sqsubset \gamma$ and $\delta^i = (\varphi|\gamma)^i = \underline{0}$, a contradiction. Conclude that Case (a) can not occur.

Case (b). $s \perp \alpha$.

Now find γ in $\mathcal{F}_\beta \cap s$ such that $\varphi|\gamma = \underline{0}$ and note that $\alpha \mathrel{\#} \gamma$ and $\varphi|\alpha = \varphi|\gamma = \underline{0}$.

(ii) As we saw in (i), $\mathbb{D}^2(\mathcal{A}_1) = \mathcal{P}_0 \cup \mathcal{P}_1$ and $\mathcal{P}_0, \mathcal{P}_1$ are spreads. Conclude, using Theorem 2.10(v) and (viii), that $\mathbb{D}^2(\mathcal{A}_1)$ is strictly analytic. It also follows from (i) that $\mathbb{D}^2(\mathcal{A}_1)$ is not regular in Lusin's sense.

(iii) Note $\mathcal{A}_1$ is a spread, and every spread is regular in Lusin's sense, for obvious reasons.

Define $\varphi\colon\ \omega^\omega \to \omega^\omega$ such that $\forall \alpha[\varphi|\alpha = \overline{\underline{0}}\alpha(0) * \langle \alpha(1) + 1\rangle * \alpha \circ S \circ S]$ and note that $\varphi\colon\ \omega^\omega \rightarrowtail \omega^\omega$ and $\varphi|\omega^\omega = \mathcal{E}_1$, so $\mathcal{E}_1$ is regular in Lusin's sense. Define $\psi\colon\ \omega^\omega \to \omega^\omega$

such that $\forall\alpha\forall n[(\psi|\alpha)^n = \varphi|(\alpha^n)]$ and note that $\psi\colon\ \omega^\omega \rightarrowtail \omega^\omega$ and $\psi|\omega^\omega = \mathcal{A}_2$, so $\mathcal{A}_2$ is regular in Lusin's sense. Assume that $\mathcal{F} \subseteq \omega^\omega$ is a spread, and $\varphi\colon\ \mathcal{F} \twoheadrightarrow \mathcal{E}_2$ is surjective.

Slightly adapting the argument given in (i), the reader may find α, γ in $\mathcal{F}$ such that $\alpha \# \gamma$ and $\varphi|\alpha = \varphi|\gamma = \underline{0}$. Conclude that $\mathcal{E}_2$ is *not* regular in Lusin's sense. □

Theorem 3.6 shows that it is not so easy, for a strictly analytic (positively) Borel set, to be regular in Lusin's sense. The set $\mathcal{E}_2!$, to be discussed in the next section, see Theorem 6.4, is an example of a set that is positively Borel and strictly analytic and also regular in Lusin's sense, but, like the set $\mathbb{D}^2(\mathcal{A}_1)$, fails to be co-analytic. It is not true, therefore, that positively Borel sets regular in Lusin's sense must be co-analytic.

Lusin would perhaps have been disappointed that there is no satisfying intuitionistic counterpart to the other half of Lusin's Theorem. He once observed that his representation theorem may help one to believe, in spite of possible qualms about generalized inductive definitions, that, after all, the collection of all positively Borel subsets of ω^ω is a *well-defined set*, see Lusin [17, pages 38–39] and Suslin [26].

4 Co-analytic sets

4.1 The class $\mathbf{\Pi}^1_1$

Some relevant definitions may be found in Section 1.2.6.

Definition 14 *$\mathcal{X} \subseteq \omega^\omega$ is* co-analytic *or $\mathbf{\Pi}^1_1$ if and only if there exists β such that $\mathcal{X} = \mathcal{U}\mathcal{G}_\beta := \mathrm{Un}(\mathcal{G}_\beta) = \{\alpha \mid \forall\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{G}_\beta]\}$.*

$\mathcal{X} \subseteq \omega^\omega$ thus is co-analytic if $\mathcal{X}$ is the co-projection of an open subset of ω^ω.

The next theorem shows that the class $\mathbf{\Pi}^1_1$ behaves not so nicely as the class $\mathbf{\Sigma}^1_1$. The class $\mathbf{\Pi}^1_1$ is closed under the operation of countable intersection but not under the operation of finite union. Most (positively) Borel subsets of ω^ω are not co-analytic. Fortunately, every set reducing to a co-analytic set is itself co-analytic. The class $\mathbf{\Pi}^1_1$ is also closed under co-projection.

Theorem 4.1

(i) *$\mathcal{UP}^1_1 := \{\alpha \mid \alpha_{II} \in \mathcal{U}\mathcal{G}_{\alpha_I}\}$ is $\mathbf{\Pi}^1_1$–universal.*
(ii) *$\mathcal{A}^1_1 := \{\alpha \mid \forall\gamma\exists n[\alpha(\overline{\gamma}n) \neq 0]\}$ is $\mathbf{\Pi}^1_1$–complete.*

(iii) *For every infinite sequence* $\mathcal{X}_0, \mathcal{X}_1, \ldots$ *in* $\mathbf{\Pi}^1_1$, $\bigcap_n \mathcal{X}_n \in \mathbf{\Pi}^1_1$, *ie* $\forall\beta\exists\gamma[\bigcap_n \mathcal{UG}_{\beta^n} = \mathcal{UG}_\gamma]$.
(iv) $\mathbb{D}^2(\mathcal{A}_1) \notin \mathbf{\Pi}^1_1$.
(v) $\mathbf{\Pi}^0_2 \subseteq \mathbf{\Pi}^1_1$ *and* $\mathbf{\Sigma}^0_2 \nsubseteq \mathbf{\Pi}^1_1$.
(vi) *For all* $\mathcal{X} \subseteq \omega^\omega$, *if* $\mathcal{X} \in \mathbf{\Pi}^1_1$, *then* $\mathrm{Un}(\mathcal{X}) \in \mathbf{\Pi}^1_1$, *ie* $\forall\beta\exists\gamma[\mathrm{Un}(\mathcal{UG}_\beta) = \mathcal{UG}_\gamma]$.
(vii) *For all* $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, *if* $\mathcal{X} \preceq \mathcal{Y} \in \mathbf{\Pi}^1_1$, *then* $\mathcal{X} \in \mathbf{\Pi}^1_1$, *ie* $\forall\beta\forall\varphi : \omega^\omega \to \omega^\omega\exists\gamma[\{\alpha \mid \varphi|\alpha \in \mathcal{UG}_\beta\} = \mathcal{UG}_\gamma]$.

Proof (i) For each α, $\alpha \in \mathcal{UP}^1_1 \leftrightarrow \alpha_{II} \in \mathcal{UG}_{\alpha_I} \leftrightarrow \forall\gamma[\ulcorner\alpha_{II},\gamma\urcorner \in \mathcal{G}_{\alpha_I}] \leftrightarrow \forall\gamma\exists n[\alpha_I(\overline{\ulcorner\alpha_{II},\gamma\urcorner}n) \neq 0]$. Define β such that, for all n, for all a, c in ω^n, $\beta(\ulcorner a, c\urcorner)) \neq 0$ if and only if, for some $m < n$, $\overline{\ulcorner a_{II}, c\urcorner}m < n$ and $a_I(\overline{\ulcorner a_{II}, c\urcorner}m) \neq 0$. Then, for each α, $\alpha \in \mathcal{UG}_\beta$ if and only if $\forall\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{G}_\beta]$ if and only if $\forall\gamma\exists n[\beta(\overline{\ulcorner\alpha,\gamma\urcorner}n) \neq 0]$ if and only if $\forall\gamma\exists n[\alpha_I(\overline{\ulcorner\alpha_{II},\gamma\urcorner}n) \neq 0]$ if and only if $\alpha_{II} \in \mathcal{UG}_{\alpha_I}$ if and only if $\alpha \in \mathcal{UP}^1_1$. Conclude that $\mathcal{UP}^1_1 = \mathcal{UG}_\beta \in \mathbf{\Pi}^1_1$.

Also: for each ε, $\mathcal{UG}_\varepsilon = \mathcal{UP}^1_1 \upharpoonright \varepsilon$. Conclude that $\mathcal{UP}^1_1$ is $\mathbf{\Pi}^1_1$–universal.

(ii) For each α, $\alpha \in \mathcal{A}^1_1 \leftrightarrow \forall\gamma\exists n[\alpha(\overline{\gamma}n) \neq 0]$. Define $\mathcal{G} := \{\alpha \mid \exists n[\alpha_I(\overline{\alpha_{II}}n) \neq 0]\}$ and note $\mathcal{A}^1_1 = \mathrm{Un}(\mathcal{G})$. Define β such that $\forall a[\beta(a) \neq 0 \leftrightarrow \exists n[\overline{a_{II}}n < \mathrm{length}(a_I) \wedge a_I(\overline{a_{II}}n) \neq 0]]$ and note that $\mathcal{G} = \mathcal{G}_\beta$. We thus see that $\mathcal{E}^1_1 \in \mathbf{\Pi}^1_1$.

Let ε be given. Note that $\forall\alpha[\alpha \in \mathcal{UG}_\varepsilon \leftrightarrow \forall\gamma\exists n[\varepsilon(\overline{\ulcorner\alpha,\gamma\urcorner}n) \neq 0]]$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall k\forall c \in \omega^k[(\varphi|\alpha)(c) = \varepsilon(\ulcorner\overline{\alpha}k, c\urcorner)]$. Note that φ reduces $\mathcal{UG}_\varepsilon$ to $\mathcal{A}^1_1$. Conclude that $\mathcal{A}^1_1$ is $\mathbf{\Pi}^1_1$–complete.

(iii) Let $\mathcal{X}_0, \mathcal{X}_1, \ldots$ be an infinite sequence of co-analytic subsets of ω^ω. Using $\mathbf{AC}_{0,1}$, find β such that $\forall n[\mathcal{X}_n = \mathcal{UG}_{\beta^n}]$. Define $\mathcal{V}_0 := \{\alpha \mid \exists m[\beta^{\alpha_{II}(0)}(\overline{\ulcorner\alpha_I, \alpha_{II} \circ S\urcorner}m) \neq 0]\}$. Then: $\mathcal{V}_0 \in \mathbf{\Sigma}^0_1$ and, for all α, $\alpha \in \bigcap_n \mathcal{X}_n \leftrightarrow \forall n\forall\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{G}_{\beta^n}] \leftrightarrow \alpha \in \mathrm{Un}(\mathcal{V}_0)$. Conclude that $\bigcap_n \mathcal{X}_n \in \mathbf{\Pi}^1_1$.

(iv) Use (ii) and Theorem 1.5(iv).

(v) Assume that $\mathcal{G} \in \mathbf{\Sigma}^0_1$. Define $\mathcal{V} := \{\alpha \mid \alpha_I \in \mathcal{G}\}$. Then $\mathcal{V} \in \mathbf{\Sigma}^0_1$ and $\mathcal{G} = \mathrm{Un}(\mathcal{V}) \in \mathbf{\Pi}^1_1$. Conclude that $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Pi}^1_1$ and, using (iii), $\mathbf{\Pi}^0_2 \subseteq \mathbf{\Pi}^1_1$. Note that $\mathbb{D}^2(\mathcal{A}_1) \in \mathbf{\Sigma}^0_2$ and conclude that, using (iv), $\neg(\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Pi}^1_1)$.

(vi) Let β be given. Note that for every α, $\alpha \in Un(\mathcal{UG}_\beta) \leftrightarrow \forall\gamma[\ulcorner\alpha,\gamma\urcorner \in \mathcal{UG}_\beta] \leftrightarrow \forall\gamma\forall\delta[\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner \in \mathcal{G}_\beta] \leftrightarrow \forall\gamma[\ulcorner\ulcorner\alpha,\gamma_I\urcorner,\gamma_{II}\urcorner \in \mathcal{G}_\beta]$.

Define $\mathcal{Z} := \{\ulcorner\alpha,\gamma\urcorner \mid \exists n[\beta(\overline{\ulcorner\ulcorner\alpha,\gamma_I\urcorner,\gamma_{II}\urcorner}n) \neq 0]\}$ and note that $\mathcal{Z} \in \mathbf{\Sigma}^0_1$ and $Un(\mathcal{UG}_\beta) = Un(\mathcal{Z}) \in \mathbf{\Pi}^1_1$.

(vii) Assume that $\mathcal{X} \in \mathbf{\Pi}^1_1$ and $\varphi\colon \omega^\omega \to \omega^\omega$ and define: $\mathcal{Y} := \{\alpha \mid \varphi|\alpha \in \mathcal{X}\}$. Find $\mathcal{G}$ in $\mathbf{\Sigma}^0_1$ such that $\mathcal{X} = \mathrm{Un}(\mathcal{G})$. Then, for every α, $\alpha \in \mathcal{Y} \leftrightarrow \varphi|\alpha \in \mathcal{X} \leftrightarrow$

$\forall\beta[\ulcorner\varphi|\alpha,\beta\urcorner \in \mathcal{G}]$. Define $\mathcal{V} := \{\alpha \mid \ulcorner\varphi|\alpha_I, \alpha_{II}\urcorner \in \mathcal{G}\}$. Conclude that $\mathcal{V} \in \mathbf{\Sigma}^0_1$ and $\mathcal{Y} = \mathrm{Un}(\mathcal{V}) \in \mathbf{\Pi}^1_1$. $\square$

4.2 The set $\mathcal{WF}$

Definition 15 *We define* $\mathcal{WF} := \{\alpha \mid \forall\beta \in (T_\alpha)^\omega \exists n[\beta(n) <_{KB} \beta(n+1)]\}$.

$\mathcal{WF}$ is the set of all α such that the tree $T_\alpha := \{s \mid \forall t \sqsubset s[\alpha(t) = 0]\}$ is *well-founded* with respect to the Kleene-Brouwer-ordering $<_{KB}$; see Definition 3 in Section 2.2.

The following theorem is a counterpart to Theorem 2.2. Note that Theorem 2.2 is the statement that $\mathcal{E}^1_1$ does *not* coincide with $\mathcal{IF}$. Note that both $(\mathcal{E}^1_1, \mathcal{A}^1_1)$ and $(\mathcal{IF}, \mathcal{WF})$ are complementary $(\mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1)$–pairs; see Section 1.2.6.

Theorem 4.2 $\mathcal{WF} = \mathcal{A}^1_1$.

Proof We first prove that $\mathcal{WF}$ is a subset of $\mathcal{A}^1_1$.

Assume that $\alpha \in \mathcal{WF}$. Let γ be given. Define β such that $\beta(0) = \langle\,\rangle$ and, for each n, if $\overline{\gamma}(n+1) \in T_\alpha$, then $\beta(n+1) = \overline{\gamma}(n+1)$, and, if not, then $\beta(n+1) = \beta(n)$. Note $\forall n[\beta(n) \in T_\alpha]$ and find n such that $\beta(n) \leq_{KB} \beta(n+1)$. Conclude that $\beta(n+1) \neq \overline{\gamma}(n+1)$ and $\exists i \leq n[\alpha(\overline{\gamma}i) \neq 0]$. We thus see that $\forall\gamma\exists i[\alpha(\overline{\gamma}i) \neq 0]$, ie $\alpha \in \mathcal{A}^1_1$. Conclude that $\mathcal{WF} \subseteq \mathcal{A}^1_1$.

We now prove that $\mathcal{A}^1_1$ is a subset of $\mathcal{WF}$. This proof is more difficult and we have to use the principle of Bar Induction **BI**; see Section 1.1.9.

Assume that $\alpha \in \mathcal{A}^1_1$. Define $B := \omega \setminus T_\alpha = \{s \mid \exists t \sqsubset s[\alpha(t) \neq 0]\}$ and note that B is a bar in ω^ω. Define $C := \{s \mid \forall\beta \in (T_\alpha)^\omega[\forall i[s \sqsubseteq \beta(i)] \to \exists j[\beta(j) \leq_{KB} \beta(j+1)]]\}$ and note that $B \subseteq C$, as, for each s in B, for each u such that $s \sqsubseteq u$, $u \notin T_\alpha$. Also note that C is monotone, ie $\forall s\forall m[s \in C \to s * \langle m\rangle \in C]$.

We now will prove that C is inductive. Let s be given such that $\forall m[s * \langle m\rangle \in C]$. We want to prove that $s \in C$. Define, for each m, $P(m) := \forall\beta \in (T_\alpha)^\omega[\big(\forall i[s \sqsubseteq \beta(i)] \wedge s * \langle m\rangle \sqsubseteq \beta(0)\big) \to \exists j[\beta(j) \leq_{KB} \beta(j+1)]]$. Before proving '$s \in C$', we first prove the auxiliary statement $\forall m[P(m)]$. We use induction. Let m be given such that $\forall k < m[P(k)]$. Let β in $(T_\alpha)^\omega$ be given such that $\forall i[s \sqsubseteq \beta(i)]$ and $s * \langle m\rangle \sqsubseteq \beta(0)$. We intend to prove $\exists j[\beta(j) \leq_{KB} \beta(j+1)]$. Define β^* such that $\beta^*(0) = \beta(0)$ and, for each n, if $\forall i \leq n+1[s * \langle m\rangle \sqsubseteq \beta(i)]$, then $\beta^*(n+1) = \beta(n+1)$; and, if not, then $\beta^*(n+1) = \beta^*(n)$. Note that $\forall n[s * \langle m\rangle \sqsubseteq \beta^*(n)]$ and $s * \langle m\rangle \in C$, and find j such that $\beta^*(j) \leq_{KB} \beta^*(j+1)$.

If $\beta^*(j) = \beta(j)$ and $\beta^*(j+1) = \beta(j+1)$, conclude that $\beta(j) \leq_{KB} \beta(j+1)$; we are done. If not, define $i_0 := \mu i[\neg(s * \langle m\rangle \sqsubseteq \beta(i))]$ and distinguish two cases.

Case (a). $\beta(i_0) = s$. Note that $i_0 > 0$ and $\beta(i_0 - 1) \leq_{KB} \beta(i_0)$; we are done.

Case (b). $s \sqsubset \beta(i_0)$. Find k such that $s * \langle k\rangle \sqsubseteq \beta(i_0)$. Note that $k \neq m$ and distinguish two cases.

Case (b1). $m < k$. Note that $i_0 > 0$ and $\beta(0) <_{KB} \beta(i_0)$ and $\exists j < i_0[\beta(j) <_{KB} \beta(j+1)]$; we are done.

Case (b2). $k < m$. Define $\beta^\dagger$ such that $\forall n[\beta^\dagger(n) = \beta(i_0+n)]$. Note that $s*\langle k\rangle \sqsubseteq \beta^\dagger(0)$ and apply $P(k)$. Find l such that $\beta^\dagger(l) \leq_{KB} \beta^\dagger(l+1)$ and, therefore $\beta(i_0 + l) \leq_{KB} \beta(i_0 + l + 1)$; again, we are done.

We conclude that $P(m)$. This completes the proof of the auxiliary statement $\forall m[P(m)]$.

We now are ready to prove that $s \in C$. Let β in $(T_\alpha)^\omega$ be given such that $\forall i[s \sqsubseteq \beta(i)]$. Consider $\beta(0)$ and $\beta(1)$. Either we find m such that either $s * \langle m\rangle \sqsubset \beta(0)]$ or $s * \langle m\rangle \sqsubseteq \beta(1)$, and, considering β or $\beta \circ S$ and using $P(m)$, we conclude that $\exists j[\beta(j) \leq_{KB} \beta(j+1)]$; or $\beta(0) = \beta(1) = s$ and $\beta(0) \leq_{KB} \beta(1)$. Conclude that $\forall \beta \in (T_\alpha)^\omega[\forall i[s \sqsubseteq \beta(i)] \rightarrow \exists j[\beta(j) \leq_{KB} \beta(j+1)]]$, ie $s \in C$.

Using **BI**, we conclude that $\langle\ \rangle \in C$, ie $\forall \beta \in (T_\alpha)^\omega \exists j[\beta(j) \leq_{KB} \beta(j+1)]$, ie $\alpha \in \mathcal{WF}$. We thus see that $\mathcal{A}^1_1 \subseteq \mathcal{WF}$ and $\mathcal{A}^1_1 = \mathcal{WF}$. □

The statement $\mathcal{A}^1_1 = \mathcal{WF}$ is, in the formal context of Basic Intuitionistic Mathematics BIM, an equivalent of $\mathbf{OI}(2^\omega)$, the Principle of Open induction on Cantor space 2^ω, see Veldman [37].

4.3 $\mathsf{Sink}^*(\mathcal{FIN})$ and $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN})$

Definition 16 *We define:* $\mathcal{FIN} := \{\alpha \mid \exists m \forall n > m[\alpha(n) = 0]\}$.

$\mathcal{FIN}$ is the set of all α such that $D_\alpha := \{n \mid \alpha(n) \neq 0\}$ is a *finite* subset of ω.

For items (i) and (iii) of the next theorem, see also Veldman [33, Theorem 3.3.(iii) and (v)].

Theorem 4.3

(i) $\mathbb{D}^2(\mathcal{A}_1) \npreceq \mathcal{FIN}$.

(ii) $\mathcal{FIN}$ *is* $\mathbf{\Sigma}^0_2$ *but not* $\mathbf{\Sigma}^0_2$*–complete.*

(iii) $\mathbb{D}^2(\mathcal{A}_1)$ *is not* $\mathbf{\Pi}^1_1$.
(iv) $\mathcal{FIN}$ *is not* $\mathbf{\Pi}^1_1$.

Proof (i) Assume that $\varphi\colon \omega^\omega \to \omega^\omega$ reduces $\mathbb{D}^2(\mathcal{A}_1) = \{\alpha \mid \alpha^0 = \underline{0} \vee \alpha^1 = \underline{0}\}$ to $\mathcal{FIN}$. We prove that φ maps the closure $\overline{\mathbb{D}^2(\mathcal{A}_1)}$ of $\mathbb{D}^2(\mathcal{A}_1)$ into $\mathcal{FIN}$ and thus obtain a contradiction. Let α in $\overline{\mathbb{D}^2(\mathcal{A}_1)}$ be given. Define α_0, α_1 such that $\forall i < 2[(\alpha_i)^i = \underline{0} \wedge \forall j[\neg\exists n[j = \langle i\rangle * n] \to \alpha_i(j) = \alpha(j)]]$. Note that $\forall i < 2[\alpha_i \in \mathbb{D}^2(\mathcal{A}_1)]$ and $\neg(\alpha \# \alpha_0 \wedge \alpha \# \alpha_1)$. Find m_0, m_1 such that $\forall i < 2\forall n > m_i[(\varphi|\alpha_i)(n) = 0]$. Define $m = \max(m_0, m_1)$. Suppose $n > m$ and $(\varphi|\alpha)(n) \neq 0$. Then $\alpha \# \alpha_0$ and $\alpha \# \alpha_1$, a contradiction. Conclude that $\forall n > m[(\varphi|\alpha)(n) = 0]$ and $\varphi|\alpha \in \mathcal{FIN}$ and, therefore, $\alpha \in \mathbb{D}^2(\mathcal{A}_1)$. We thus see that $\overline{\mathbb{D}^2(\mathcal{A}_1)} \subseteq \mathbb{D}^2(\mathcal{A}_1)$ and, according to Theorem 1.3 in Section 1.2.5, obtain a contradiction. Conclude that $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{FIN}$.

(ii) $\mathcal{FIN} = \bigcup_m \{\alpha \mid \forall n > m[\alpha(n) = 0]\}$ clearly is $\mathbf{\Sigma}^0_2$, but, as $\mathbb{D}^2(\mathcal{A}_1)$ is $\mathbf{\Sigma}^0_2$ and, according to (i), $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{FIN}$, $\mathcal{FIN}$ is not $\mathbf{\Sigma}^0_2$–complete.

(iii) See Theorem 1.5(iv).

(iv) Assume that $\varphi\colon \omega^\omega \to \omega^\omega$ reduces $\mathcal{FIN}$ to $\mathcal{A}^1_1$. Define $\mathcal{T} := \{\alpha \in 2^\omega \mid \forall m \forall n[\alpha(m) = \alpha(n) = 1 \to m = n]\}$. $\mathcal{T}$ is the set of all infinite binary sequences that assume the value 1 at most one time. Note that $\mathcal{T}$ is a spread, and $\underline{0} \in \mathcal{T}$, and $\forall \alpha \in \mathcal{T}[\alpha \# \underline{0} \to \alpha \in \mathcal{FIN})]$, and $\mathcal{T} \subseteq \mathsf{Perhaps}(\mathcal{FIN})$. Assume that $\mathcal{T} \subseteq \mathcal{FIN}$, ie $\forall \alpha \in \mathcal{T} \exists m \forall n > m[\alpha(n) = 0]$. Applying Brouwer's Continuity Principle **BCP** (see Section 1.1.6) find p, m such that $\forall \alpha \in \mathcal{T}[\overline{0}p \sqsubset \alpha \to \forall n > m[\alpha(n) = 0]]$. We now have a contradiction: define $q := \max(m+1, p)$ and consider $\alpha := \overline{0}q * \langle 1\rangle * \underline{0}$.

Conclude that $\neg(\mathcal{T} \subseteq \mathcal{FIN})$ and $\mathcal{FIN}$ is not perhapsive, and that $\mathcal{FIN}$ does not reduce to $\mathcal{A}^1_1$ and is not $\mathbf{\Pi}^1_1$; see Theorem1.5. □

Definition 17 *We define* $\mathcal{ALMOST}^*\mathcal{FIN} := \{\alpha \mid \forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]\}$.

$\mathcal{ALMOST}^*\mathcal{FIN}$ is the set of all α such that D_α is an *almost-finite* subset of ω.

Lemma 4.4 $\mathcal{ALMOST}^*\mathcal{FIN}$ *is* $\mathbf{\Pi}^1_1$.

Proof We shall prove that, for each α:

$$\forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0] \text{ if and only if } \forall \zeta \exists n[\alpha \circ \zeta(n) = 0] \vee \zeta(n+1) \leq \zeta(n)]$$

The desired conclusion then follows easily.

Let τ be the canonical retraction[12] of ω^ω onto the spread $[\omega]^\omega$. The function τ satisfies the conditions: $\forall\zeta \in [\omega]^\omega[\tau|\zeta = \zeta]$ and $\forall\zeta[\zeta \# \tau|\zeta \to \exists n[\zeta(n+1) \leq \zeta(n)]]$.

Let α be given. First assume $\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]$. Let ζ be given. Find n such that $\alpha \circ (\tau|\zeta)(n) = 0$. Either $(\tau|\zeta)(n) = \zeta(n)$ and $\alpha \circ \zeta(n) = 0$; or, $(\tau|\zeta)(n) \neq \zeta(n)$ and $\exists i \leq n[\zeta(i+1) \leq \zeta(i)]$. We thus see that $\forall\zeta\exists n[\alpha \circ \zeta(n) = 0 \ \vee \ \zeta(n+1) \leq \zeta(n)]$.

Now assume that $\forall\zeta\exists n[\alpha \circ \zeta(n) = 0 \ \vee \ \zeta(n+1) \leq \zeta(n)]$. Let ζ in $[\omega]^\omega$ be given. Find n such that $\alpha \circ \zeta(n) = 0 \ \vee \ \zeta(n+1) \leq \zeta(n)$. Conclude that $\alpha \circ \zeta(n) = 0$. We thus see that $\forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]$. □

The set $\mathcal{ALMOST}^*\mathcal{FIN}$ has been studied in Veldman [33, Section 3]. It has been shown there that $\mathcal{ALMOST}^*\mathcal{FIN}$ is not (positively) Borel, see [33, Section 0.9.2(ii) and Theorem 3.17(iii)]. In particular, $\mathcal{FIN}$ is proper subset[13] of $\mathcal{ALMOST}^*\mathcal{FIN}$. It has also been shown in [33] that $\mathcal{ALMOST}^*\mathcal{FIN}$ is the best $\mathbf{\Pi}^1_1$–approximation of $\mathcal{FIN}$, ie, for every $\mathcal{Z}$ in $\mathbf{\Pi}^1_1$, if $\mathcal{FIN} \subseteq \mathcal{Z}$, then $\mathcal{ALMOST}^*\mathcal{FIN} \subseteq \mathcal{Z}$, see [33, Theorem 3.21(v)]. As one might expect, $\mathcal{ALMOST}^*\mathcal{FIN}$ is not $\mathbf{\Pi}^1_1$–complete, see [33, Theorem 3.24(iii)].

In the following definition we introduce a new word for a well-known concept.

Definition 18 *For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, we define:* $\mathcal{X}$ sinks into $\mathcal{Y}$ *if and only if* $\mathcal{X} \subseteq \mathcal{Y}$. *For each $\mathcal{X} \subseteq \omega^\omega$, we define* $\mathsf{Sink}(\mathcal{X}) := \{\beta \mid \mathcal{F}_\beta \subseteq \mathcal{X}\}$ *and* $\mathsf{Sink}^*(\mathcal{X}) := \{\beta \in \mathsf{Sink}(\mathcal{X}) \mid \mathrm{Spr}(\beta)\}$.

$\mathsf{Sink}(\mathcal{X})$ is the set of the codes of all closed subsets of ω^ω that sink into (ie, are a subset of) $\mathcal{X}$ and $\mathsf{Sink}^*(\mathcal{X})$ is the set of the codes of all spreads, ie all *closed and located* subsets of ω^ω, that sink into (ie are a subset of) $\mathcal{X}$.

We now want to treat some results that, together, are a counterpart[14] to Theorem 2.9. The moral of the story is that, in order to obtain a satisfying such counterpart, one should work with $\mathcal{ALMOST}^*\mathcal{FIN}$ rather than with $\mathcal{FIN}$.

Recall that for all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$: $\mathcal{X} \sim \mathcal{Y}$ ($\mathcal{X}$, $\mathcal{Y}$ *reduce to each other / are Wadge–equivalent*), if and only if both $\mathcal{X} \preceq \mathcal{Y}$ and $\mathcal{Y} \preceq \mathcal{X}$.

Theorem 4.5

[12] see Section 1.1.5

[13] For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, $\mathcal{X}$ is a *proper subset* of $\mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and not $\mathcal{Y} \subseteq \mathcal{X}$, ie the assumption '$\mathcal{Y} \subseteq \mathcal{X}$' leads to a contradiction.

[14] Note that, from a classical point of view, the sets $\mathsf{Share}(\mathcal{INF})$, $\mathsf{Sink}(\mathcal{FIN})$, for instance, are each other's complement.

(i) $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega) \sim \mathcal{FIN}$.
(ii) $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega) \not\preceq \mathcal{A}^1_1$ *and* $\mathsf{Sink}^*(\mathcal{FIN}) \not\preceq \mathcal{A}^1_1$.
(iii) $\mathcal{A}^1_1 \preceq \mathsf{Sink}^*(\mathcal{FIN})$.
(iv) $\mathcal{A}^1_1 \preceq \mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega) \preceq \mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN})$.
(v) $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN}) \preceq \mathcal{A}^1_1$.
(vi) $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN})$ *and* $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$ *are* $\mathbf{\Pi}^1_1$*– complete.*

Proof (i) Assume that $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta \subseteq \mathcal{FIN} \cap 2^\omega$. Conclude that $\forall s[\beta(s) = 0 \rightarrow s \in Bin]$ and $\mathrm{Fan}(\beta)$ and $\forall \gamma \in \mathcal{F}_\beta \exists m \forall n > m[\gamma(n) = 0]$. Applying the First Axiom of Continuous Choice $\mathbf{AC}_{1,0}$, see Section 1.1.3, find $\varphi\colon \mathcal{F}_\beta \rightarrow \omega$ such that $\forall \gamma \in \mathcal{F}_\beta \forall n > \varphi(\gamma)[\gamma(n) = 0]$. Applying the Fan Theorem $\mathbf{FT}$, see Section 1.1.7, find p such that $\forall \gamma \in \mathcal{F}_\beta[\varphi(\gamma) \le p]$. Note that $\forall n > p \forall s \in \mathrm{Bin}_{n+1}[\beta(s) = 0 \rightarrow s(n) = 0]$. Conclude that for each β, $\beta \in \mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega)$ if and only if $\mathrm{Spr}(\beta)$ and $\forall s[\beta(s) = 0 \rightarrow s \in Bin]$ and $\exists p \forall n > p \forall s \in \mathrm{Bin}_{n+1}[\beta(s) = 0 \rightarrow s(n) = 0]]$. Define $\psi\colon \omega^\omega \rightarrow \omega^\omega$ such that, for all β, for all n, $(\psi|\beta)(n) = 0$ if and only if $\forall s \le n[\beta(s) = 0 \rightarrow s \in Bin]$ and $\forall s \in \mathrm{Bin}_{n+1}[\beta(s) = 0 \rightarrow s(n) = 0]$. Note that ψ reduces $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega)$ to $\mathcal{FIN}$. Define $\rho\colon \omega^\omega \rightarrow \omega^\omega$ such that, for all α, for all s, $(\rho|\alpha)(s) = 0$ if and only if $s \in Bin$ and $\forall i < \mathrm{length}(s)[s(i) = 1 \leftrightarrow \alpha(i) \neq 0]$. Note that ρ reduces $\mathcal{FIN}$ to $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega)$. Conclude that $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega) \sim \mathcal{FIN}$.

(ii) Use (i) and Theorem 4.3(iv) and conclude that $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega) \not\preceq \mathcal{A}^1_1$. Define $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ such that, for all β, for all s, $(\varphi|\beta)(s) = 0$ if and only if $(s \in \mathrm{Bin} \wedge \beta(s) = 0) \vee \exists t \le s[t \notin \mathrm{Bin} \wedge \beta(t) = 0]$. Note that φ reduces $\mathsf{Sink}^*(\mathcal{FIN} \cap 2^\omega)$ to $\mathsf{Sink}^*(\mathcal{FIN})$. Conclude that $\mathsf{Sink}^*(\mathcal{FIN}) \not\preceq \mathcal{A}^1_1$.

(iii) Define $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ such that, for all α, for all s, $(\varphi|\alpha)(s) = 0$ if and only if $\exists t \in T_\alpha \exists n[s = (S \circ t) * \overline{\underline{0}}n]$.[15] Note that for all α, $\mathrm{Spr}(\varphi|\alpha)$ and $\forall \gamma \in \mathcal{F}_{\varphi|\alpha} \forall n[\gamma(n) = 0 \rightarrow \gamma(n+1) = 0]$. We now prove that φ reduces $\mathcal{A}^1_1$ to $\mathsf{Sink}^*(\mathcal{FIN})$.

First assume that $\alpha \in \mathcal{A}^1_1$. Also assume that $\gamma \in \mathcal{F}_{\varphi|\alpha}$. Find ε such that, for each n, if $\gamma(n) > 0$, then $\varepsilon(n) + 1 = \gamma(n)$, and, if $\gamma(n) = 0$, then $\varepsilon(n) = 0$. Find m such that $\alpha(\overline{\varepsilon}m) \neq 0$. Then: $\overline{\varepsilon}(m+1) \notin T_\alpha$ and $\overline{\gamma}(m+1) \neq S \circ \overline{\varepsilon}(m+1)$. Find $k \le m$ such that $\gamma(k) = 0$ and note that $\forall n > k[\gamma(n) = 0]$ and $\gamma \in \mathcal{FIN}$. We thus see that $\forall \gamma \in \mathcal{F}_{\varphi|\alpha}[\gamma \in \mathcal{FIN}]$. Conclude that $\mathcal{F}_{\varphi|\alpha} \subseteq \mathcal{FIN}$ and $\varphi|\alpha \in \mathsf{Sink}^*(\mathcal{FIN})$.

Now assume that $\varphi|\alpha \in \mathsf{Sink}^*(\mathcal{FIN})$. Then $\forall \gamma \in \mathcal{F}_{\varphi|\alpha} \exists m \forall n > m[\gamma(n) = 0]$. Let ε be given. Define γ such that, for each n, *if* $\overline{\varepsilon}(n+1) \in T_\alpha$, then $\gamma(n) = \varepsilon(n) + 1$, and, *if not*, then $\gamma(n) = 0$. Note that $\gamma \in \mathcal{F}_{\varphi|\alpha}$ and find m such that $\gamma(m) = 0$. Conclude

[15] Recall: $\mathrm{length}(S \circ t) = \mathrm{length}(t)$ and $\forall i < \mathrm{length}(t)[(S \circ t)(i) = t(i) + 1]$.

that $\overline{\varepsilon}(m+1) \notin T_\alpha$ and $\exists i \le m+1[\alpha(\overline{\varepsilon}i) \neq 0]$. We thus see that $\forall \varepsilon \exists i[\alpha(\overline{\varepsilon}i) \neq 0]$, ie $\alpha \in \mathcal{A}^1_1$. We thus see that $\forall \alpha[\alpha \in \mathcal{A}^1_1 \leftrightarrow \varphi|\alpha \in \mathsf{Sink}^*(\mathcal{FIN})]$, ie φ reduces $\mathcal{A}^1_1$ to $\mathsf{Sink}^*(\mathcal{FIN})$.

(iv) Define δ such that $\delta(0) = 0$ and, for all s, for all n, $\delta(s * \langle n \rangle) = \delta(s) * \underline{\overline{0}}n * \langle 1 \rangle$. Define $\varphi \colon \omega^\omega \to \omega^\omega$ such that for all α, for all s, $(\varphi|\alpha)(s) = 0$ if and only if $\exists t \in T_\alpha \exists n[s = \delta(t) * \underline{\overline{0}}n]$. Note that for all α, $\mathrm{Spr}(\varphi|\alpha)$ and $\mathcal{F}_{\varphi|\alpha} \subseteq 2^\omega$. We prove that φ reduces $\mathcal{A}^1_1$ to $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$.

Assume that $\alpha \in \mathcal{A}^1_1$. Also assume that $\gamma \in \mathcal{F}_{\varphi|\alpha}$ and $\zeta \in [\omega]^\omega$. Define γ' such that $\forall n[\gamma' \circ \zeta(n) = 1]$ and $\forall n[\forall i[n \neq \zeta(i)] \to \gamma'(n) = \gamma(n)]$. Define ε such that $\varepsilon(0) = \mu p[\gamma'(p) = 1]$ and $\forall n[\varepsilon(n+1) = \mu p > \varepsilon(n)[\gamma'(p) = 1]]$. Note that $\varepsilon \in [\omega]^\omega$ and, for all n, $\delta(\overline{\varepsilon}n) \sqsubset \gamma'$. Find n such that $\alpha(\overline{\varepsilon}n) \neq 0$. Note that $\overline{\varepsilon}(n+1) \notin T_\alpha$ and $(\varphi|\alpha)\big(\delta(\overline{\varepsilon}(n+1))\big) \neq 0$. Find m such that $\overline{\gamma'}m = \delta(\overline{\varepsilon}(n+1))$. Note that $(\varphi|\alpha)(\overline{\gamma'}m) \neq 0 = (\varphi|\alpha)(\overline{\gamma}m)$ and conclude that $\overline{\gamma'}m \neq \overline{\gamma}m$. Find $i < m$ such that $\gamma'(i) \neq \gamma(i)$. Determine $j < m$ such that $i = \zeta(j)$ and conclude that $\gamma \circ \zeta(j) = 0$. We thus see that $\forall \gamma \in \mathcal{F}_{\varphi|\alpha} \forall \zeta \in [\omega]^\omega \exists j[\gamma \circ \zeta(j) = 0]$. Conclude that $\mathcal{F}_{\varphi|\alpha} \subseteq \mathcal{ALMOST}^*\mathcal{FIN}$ and $\varphi|\alpha \in \mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$.

Now assume that $\varphi|\alpha \in \mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$. Let γ be given. Find β in 2^ω such that $\forall n[\delta(\overline{\gamma}n) \sqsubset \beta]$. Define ζ such that $\zeta(0) = \gamma(0)$ and $\forall n[\zeta(n+1) = \zeta(n) + \gamma(n+1) + 1]$. Note $\zeta \in [\omega]^\omega$ and $\forall n[\beta \circ \zeta(n) = 1]$. Define β^* such that, for each n, *if* $\overline{\beta}(n+1) \in T_{\varphi|\alpha}$, then $\beta^*(n) = \beta(n)$, and *if not*, then $\beta^*(n) = 0$. Note that $\beta^* \in \mathcal{F}_{\varphi|\alpha} \subseteq \mathcal{ALMOST}^*\mathcal{FIN}$ and find n such that $\beta^* \circ \zeta(n) = 0$. Define $p := \zeta(n) + 1$ and conclude that $\overline{\beta}p \neq \overline{\beta^*}p$ and $\overline{\beta}p \notin T_{\varphi|\alpha}$. Find m such that $\overline{\beta}p \sqsubseteq \delta(\overline{\gamma}m)$ and note that $\overline{\gamma}m \notin T_\alpha$ and $\exists i \le m[\alpha(\overline{\gamma}i) \neq 0]$. We thus see that $\forall \gamma \exists i[\alpha(\overline{\gamma}i) \neq 0]$, ie $\alpha \in \mathcal{A}^1_1$.

Conclude that for each α, $\alpha \in \mathcal{A}^1_1$ if and only if $\varphi|\alpha \in \mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$, ie φ reduces $\mathcal{A}^1_1$ to $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$.

Finally, define $\psi \colon \omega^\omega \to \omega^\omega$ such that, for all β, for all s, $(\psi|\beta)(s) = 0$ if and only if either $\beta(s) = 0 \wedge s \in Bin$ or $\exists t \sqsubseteq s[\beta(t) = 0 \wedge s \notin Bin]$. Note that ψ reduces $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN} \cap 2^\omega)$ to $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN})$.

(v) We first prove a preliminary observation. For all β such that $\mathrm{Spr}(\beta)$, $\forall \alpha \in \mathcal{F}_\beta \forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]$ if and only if $\forall \alpha \forall \zeta \exists n[\alpha \circ \zeta(n) = 0 \vee \zeta(n+1) \le \zeta(n) \vee \beta(\overline{\alpha}n) \neq 0]$. The argument is a small extension of the argument given for Lemma 4.4.

Let β be given such that $\mathrm{Spr}(\beta)$.

First assume $\forall \alpha \in \mathcal{F}_\beta \forall \zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]$. Let ρ, τ be the canonical retractions[16] of ω^ω onto the spreads $\mathcal{F}_\beta$ and $[\omega]^\omega$, respectively. Let α, ζ be given. Find n such that $\big((\rho|\alpha) \circ (\tau|\zeta)\big)(n) = 0$. There are three cases to consider. *Case (a).* $\overline{\tau|\zeta}(n+1) \neq \overline{\zeta}(n+1)$. Then $\exists i[\zeta(i+1) \leq \zeta(i)]$. *Case (b).* $\overline{\tau|\zeta}(n+1) = \overline{\zeta}(n+1)$ and $\overline{\rho|\alpha}(\zeta(n)+1) \neq \overline{\alpha}(\zeta(n)+1)$. Then $\exists i[\beta(\overline{\alpha}i) \neq 0]$. *Case (c).* $\overline{\tau|\zeta}(n+1) = \overline{\zeta}(n+1)$ and $\overline{\rho|\alpha}(\zeta(n)+1) = \overline{\alpha}(\zeta(n)+1)$. Then $\alpha \circ \zeta(n) = 0$. Conclude that $\forall\alpha\forall\zeta\exists n[\alpha \circ \zeta(n) = 0 \ \vee\ \zeta(n+1) \leq \zeta(n) \ \vee\ \beta(\overline{\alpha}n) \neq 0]$.

Now assume $\forall\alpha\forall\zeta\exists n[\alpha \circ \zeta(n) = 0 \ \vee\ \zeta(n+1) \leq \zeta(n) \ \vee\ \beta(\overline{\alpha}n) \neq 0]$. Let α be given in $\mathcal{F}_\beta$ and ζ in $[\omega]^\omega$. Find n such that $\alpha \circ \zeta(n) = 0 \ \vee\ \zeta(n+1) \leq \zeta(n) \ \vee\ \beta(\overline{\alpha}n) \neq 0$ and conclude that $\alpha \circ \zeta(n) = 0$. We thus see that $\forall\alpha \in \mathcal{F}_\beta \forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]$.

Now observe: $\{\beta \mid \mathrm{Spr}(\beta)\}$ belongs to $\mathbf{\Pi}^0_2$. Using our preliminary observation and also Theorem 4.1, conclude that $\mathsf{Sink}^*(\mathcal{ALMOST}^*\mathcal{FIN}) = \{\beta \mid \mathrm{Spr}(\beta) \ \wedge\ \forall\alpha\forall\zeta\exists n[\alpha \circ \zeta(n) = 0 \ \vee\ \zeta(n+1) \leq \zeta(n) \ \vee\ \beta(\overline{\alpha}n) \neq 0]\}$ belongs to $\mathbf{\Pi}^1_1$.

(vi) Use (iv) and (v). $\square$

Theorem 4.5(i) seems to contradict classical results: its proof uses the strongly nonclassical axiom $\mathbf{AC}_{1,0}$. Theorem 4.5(iv) is a counterpart to Theorem 2.9. Both Theorem 4.5(vi) and Theorem 2.9 resemble a classical result due to Hurewicz that plays a key role in the sketch of the proof of a theorem by Solovay and Kaufman in Kechris–Louveau [15]. The Solovay–Kaufman Theorem states that the set of the codes of closed sets of uniqueness and the set of the codes of closed sets of extended uniqueness are $\mathbf{\Pi}^1_1$–complete. Note that we obtained the more *'classical'* results of Theorem 4.5 by replacing $\mathcal{FIN}$ by $\mathcal{ALMOST}^*\mathcal{FIN}$.

4.4 Exactly one path

Definition 19 $\mathcal{E}^1_1! := \{\alpha \mid \exists\gamma[\forall n[\alpha(\overline{\gamma}n) = 0] \ \wedge \forall\delta[\delta \# \gamma \to \exists n[\alpha(\overline{\delta}n) \neq 0]]\}$

$\mathcal{E}^1_1!$ is the set of all α admitting *exactly one path*. In Kechris [14, pages 125–127], there is a fascinating argument, due to Kechris, showing that, in classical descriptive set theory, $\mathcal{E}^1_1!$ is $\mathbf{\Pi}^1_1$–complete. We will see that this result does not go through in our intuitionistic context.

Definition 20 $\mathbb{D}^2!(\mathcal{A}_1) := \{\alpha \mid \exists i < 2[\alpha^i = \underline{0} \ \wedge\ \alpha^{1-i} \# \underline{0}]\}$, *and* $\mathcal{E}_2! := \{\alpha \mid \exists n[\alpha^n = \underline{0} \ \wedge\ \forall m \neq n[\alpha^n \# \underline{0}]]\}$.

[16] See Section 1.1.5

Note that $\mathbb{D}^2!(\mathcal{A}_1)$ is $\mathbf{\Sigma}^0_2$ and $\mathcal{E}_2!$ is $\mathbf{\Sigma}^0_3$.

We will see that the set $\mathcal{E}_2!$ is an example of a subset of ω^ω that is positively Borel and regular in Lusin's sense,[17] see Theorem 4.6, but not $\mathbf{\Pi}^1_1$, see Theorem 4.8.

Theorem 4.6

(i) $\mathbb{D}^2!(\mathcal{A}_1) \preceq \mathcal{E}_2!$ *and* $\mathcal{E}_2! \preceq \mathcal{E}^1_1!$

(ii) $\mathcal{A}_2 \preceq \mathcal{E}_2!$ *and* $\mathcal{A}^1_1 \preceq \mathcal{E}^1_1!$

(iii) $\mathbb{D}^2!(\mathcal{A}_1) \preceq \mathcal{A}_2$

(iv) $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{E}_2!$

(v) $\mathcal{E}_2!$ *is regular in Lusin's sense.*

Proof (i) Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\big[\forall i < 2[(\varphi|\alpha)^i = \alpha^i] \wedge \forall i \geq 2[(\varphi|\alpha)^i = \underline{1}]\big]$ and note that φ reduces $\mathbb{D}^2!(\mathcal{A}_1)$ to $\mathcal{E}_2!$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\psi|\alpha)(s) = 0 \leftrightarrow \exists n[s \sqsubset \underline{n} \wedge \overline{\alpha^n}s \sqsubset \underline{0}]]$ and note that ψ reduces $\mathcal{E}_2!$ to $\mathcal{E}^1_1!$.

(ii) Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[(\varphi|\alpha)^0 = \underline{0} \wedge \forall i[(\varphi|\alpha)^{i+1} = \alpha^i]$ and note that φ reduces $\mathcal{A}_2$ to $\mathcal{E}_2!$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that, for all α, $\forall n[(\psi|\alpha)(\overline{\underline{0}}n) = 0]$ and $\forall m\forall n\forall t[(\psi|\alpha)(\overline{\underline{0}}n * \langle m+1\rangle * t) = \alpha(t)]]$ and note that ψ reduces $\mathcal{A}^1_1$ to $\mathcal{E}^1_1!$.

(iii) Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that, for all α, for all n, $(\varphi|\alpha)^0(n) = \max\big(\alpha^0(n), \alpha^1(n)\big)$, and, for all i, $(\varphi|\alpha)^{i+1}(n) \neq 0$ if and only if either $\overline{\alpha^0}i \sqsubset \underline{0}$ or $\overline{\alpha^1}i \sqsubset \underline{0}$. Note that φ reduces $\mathbb{D}^2!(\mathcal{A}_1)$ to $\mathcal{A}_2$.

(iv) Assume that $\psi\colon \omega^\omega \to \omega^\omega$ maps $\mathbb{D}^2(\mathcal{A}_1)$ into $\mathcal{E}^1_1!$. We shall prove that ψ also maps the closure $\overline{\mathbb{D}^2(\mathcal{A}_1)}$ of $\mathbb{D}^2(\mathcal{A}_1)$ into $\mathcal{E}^1_1!$ and thus does not reduce $\mathbb{D}^2(\mathcal{A}_1)$ to $\mathcal{E}^1_1!$.

First, as in the proof of Theorem 3.6, define, for both $i < 2$, $\mathcal{P}_i := \{\beta \mid \beta^i = \underline{0}\}$. Note that $\mathcal{P}_0, \mathcal{P}_1$ are spreads and $\mathbb{D}^2(\mathcal{A}_1) = \mathcal{P}_0 \cup \mathcal{P}_1$. Assume that $\alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}$. We are going to prove: $\psi|\alpha \in \mathcal{E}^1_1!$. The following notion is useful. We define, for all s, s is *fine for* α if and only if $\exists m\forall\beta \in \mathbb{D}^2(\mathcal{A}_1)[\overline{\alpha}m \sqsubset \beta \to \exists\gamma \in \mathcal{F}_{\psi|\beta}[s \sqsubset \gamma]]$. We will prove that for each p there exists exactly one s such that $\text{length}(s) = p$ and s is fine for α. Define α_0, α_1 such that, for both $i < 2$, $(\alpha_i)^i = \underline{0}$ and $\forall j[\neg\exists n[j = \langle i, n\rangle] \to \alpha_i(j) = \alpha(j)]$. Define α_{01} such that $(\alpha_{01})^0 = (\alpha_{01})^1 = \underline{0}$ and $\forall j[\neg\exists i < 2\exists n[j = \langle i, n\rangle] \to \alpha_{01}(j) = \alpha(j)]$. Note that if $\alpha \# \alpha_0$, then $\alpha = \alpha_1 \in \mathcal{P}_1$, and, if $\alpha \# \alpha_1$, then $\alpha = \alpha_0 \in \mathcal{P}_0$, and, if $\alpha \# \alpha_{01}$, then either $\alpha \# \alpha_0$ or $\alpha \# \alpha_1$, and, therefore $\alpha \in \mathcal{P}_0 \cup \mathcal{P}_1 = \mathbb{D}^2(\mathcal{A}_1)$. Note that $\alpha_{01} \in \mathcal{P}_0 \cap \mathcal{P}_1$.

Let p be given. Note that $\forall i < 2\forall\beta \in \mathcal{P}_i\exists s[\mathit{length}(s) = p \wedge \exists\gamma \in \mathcal{F}_{\psi|\beta}[s \sqsubset \gamma]]$. Using Brouwer's Continuity Principle **BCP**, see Section 1.1.6, find s_0, s_1, m_0, m_1 such

[17] See Definition 13.

that $\text{length}(s_0) = \text{length}(s_1) = p$ and $\forall i < 2\forall\beta \in \mathcal{P}_i \cap \overline{\alpha_{01}}m_i\exists\gamma \in \mathcal{F}_{\psi|\beta}[s_i \sqsubset \gamma]$. Assume $s_0 \perp s_1$. Then $\exists\gamma \in \mathcal{F}_{\psi|\alpha_{01}}\exists\delta \in \mathcal{F}_{\psi|\alpha_{01}}[s_0 \sqsubset \gamma \wedge s_1 \sqsubset \delta]$ and this contradicts $\psi|\alpha_{01} \in \mathcal{E}_1^1!$. Conclude that $s_0 = s_1$. Define $s := s_0$ and $m := \max(m_0, m_1)$, and note that if $\overline{\alpha}m = \overline{\alpha_{01}}m$, then s is fine for α. Now assume that $\overline{\alpha}m \neq \overline{\alpha_{01}}m$. Find $k < 2$ such that $\alpha = \alpha_k$ and note $(\overline{\alpha}m)^{1-k} \perp \underline{0}$. Find s_2, m_2 such that $\text{length}(s_2) = p$ and $m < m_2$ and $\forall\beta \in \mathcal{P}_k \cap \overline{\alpha}m_2\exists\gamma \in \mathcal{F}_{\psi|\beta}[s_2 \sqsubset \gamma]$. As $(\overline{\alpha}m_2)^{1-k} \perp \underline{0}$, conclude that $\forall i < 2\forall\beta \in \mathcal{P}_i \cap \overline{\alpha}m_2\exists\gamma \in \mathcal{F}_{\psi|\beta}[s_2 \sqsubset \gamma]$, and s_2 is fine for α. Clearly then, for each p, one may find s such that $\text{length}(s) = p$ and s is fine for α.

Suppose s, t are given such that both s, t are fine for α. Find m such that $\forall\beta \in \mathbb{D}^2(\mathcal{A}_1)[\overline{\alpha}m \sqsubset \alpha \to (\exists\gamma \in \mathcal{F}_{\varphi|\beta}[s \sqsubset \gamma] \wedge \exists\gamma \in \mathcal{F}_{\varphi|\beta}[t \sqsubset \gamma])]$. Find $k < 2$ such that $\overline{\alpha}m \sqsubset \alpha_k$. Note that $\alpha_k \in \mathbb{D}^2(\mathcal{A}_1)$ and $\varphi|\alpha_k \in \mathcal{E}_1^1!$ and conclude that $s \sqsubseteq t \vee t \sqsubseteq s$. We thus see that if both s, t are fine for α, then $s \sqsubseteq t \vee t \sqsubseteq s$. We thus may define δ such that, for each p, $\overline{\delta}p$ is fine for α. Conclude that $\delta \in \mathcal{F}_{\psi|\alpha}$, and $\psi|\alpha \in \mathcal{E}_1^1$, ie $\psi|\alpha$ admits a path. We still have to prove that $\psi|\alpha$ admits *exactly one* path. Let η be given such that $\delta \# \eta$. Note that $\psi|\alpha_0 \in \mathcal{E}_1^1!$ and find λ in $\mathcal{F}_{\psi|\alpha_0}$. Using the co-transitivity of the relation #, distinguish two cases.

Case (a): $\eta \# \lambda$. Find n such that $(\psi|\alpha_0)(\overline{\eta}n) \neq 0$. Either $(\psi|\alpha)(\overline{\eta}n) = (\psi|\alpha_0)(\overline{\eta}n) \neq 0$, or $\alpha \# \alpha_0$ and $\alpha = \alpha_1$ and $\exists m[(\psi|\alpha)(\overline{\eta}m) \neq 0]$.

Case (b): $\delta \# \lambda$. Then $\alpha \# \alpha_0$ and $\alpha = \alpha_1$ and $\exists m[(\psi|\alpha)(\overline{\eta}m) \neq 0]$. We thus see that $\forall\eta[\eta \# \delta \to \exists p[(\psi|\alpha)(\overline{\eta}p) \neq 0]]$, and $\psi|\alpha \in \mathcal{E}_1^1!$.

Conclude that $\forall\alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\psi|\alpha \in \mathcal{E}_1^1!]$. Now assume that ψ reduces $\mathbb{D}^2(\mathcal{A}_1)$ to $\mathcal{E}_1^1!$. Conclude that $\forall\alpha \in \overline{\mathbb{D}^2(\mathcal{A}_1)}[\alpha \in \mathbb{D}^2(\mathcal{A}_1)]$. According to Theorem 1.3 (see Section 1.2.5) we have a contradiction.

(v) Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that, for all α, $(\varphi|\alpha)^{\alpha(0)} = \underline{0}$ and $\forall n < \alpha(0)[(\varphi|\alpha)^n = \overline{\underline{0}}\alpha^0(2n) * \langle\alpha^0(2n+1)+1\rangle * \alpha^{n+1}]$ and $\forall n > \alpha(0)[(\varphi|\alpha)^n = \overline{\underline{0}}\alpha^0(2n-2) * \langle\alpha^0(2n-1)+1\rangle * \alpha^n]$. Then $\varphi\colon \omega^\omega \rightarrowtail \omega^\omega$ and $\varphi|\omega^\omega = \mathcal{E}_2!$. Conclude that $\mathcal{E}_2!$ is regular in Lusin's sense. □

According to Theorem 4.6(iv), $\mathbb{D}^2(\mathcal{A}_1) \not\preceq \mathcal{E}_2!$, and, therefore, also $\mathcal{E}_2 \not\preceq \mathcal{E}_2!$. This is an *intuitionistic* phenomenon, as, in classical descriptive theory, $\mathcal{E}_2 \preceq \mathcal{E}_2!$. One may understand this classical fact by replacing $\mathcal{E}_2, \mathcal{E}_2!$ by sets that, from a constructive point of view, are extensions of them, although, classically, they would be judged to be the same. Theorem 4.7 will make this clear.

Definition 21 $\mathcal{ALMOST}$–$\mathcal{E}_2 := \{\alpha \mid \alpha \# \mathcal{A}_2\} = \{\alpha \mid \forall\gamma\exists n[\alpha^n(\gamma(n)) = 0]\}$, *and* $\mathcal{ALMOST}$–$\mathcal{E}_2! := \mathcal{ALMOST}$–$\mathcal{E}_2 \cap \{\alpha \mid \forall m\forall n[m \neq n \to \exists p[\alpha^m(p) \neq 0 \vee \alpha^n(p) \neq 0]\}$.

$\mathcal{ALMOST}\text{–}\mathcal{E}_2$ and $\mathcal{ALMOST}\text{–}\mathcal{E}_2!$ may be called $\mathbf{\Pi}^1_1$*–approximations* to $\mathcal{E}_2$ and $\mathcal{E}_2!$, respectively.

Theorem 4.7 $\mathcal{ALMOST}\text{–}\mathcal{E}_2 \preceq \mathcal{ALMOST}\text{–}\mathcal{E}_2!$.

Proof Define $\psi, \varphi\colon \omega^\omega \to \omega^\omega$ such that, for each α, $(\psi|\alpha)(0) = 0$ and $(\varphi|\alpha)^0 = \alpha^0 = \alpha^{(\psi|\alpha)(0)}$, and, for each n,

(1) *if* $\overline{\alpha^{(\psi|\alpha)(n)}}(n+1) \sqsubset \underline{0}$ then $(\psi|\alpha)(n+1) = (\psi|\alpha)(n)$ and $(\varphi|\alpha)^{n+1} = \underline{1}$, and

(2) *if* $\overline{\alpha^{(\psi|\alpha)(n)}}(n+1) \perp \underline{0}$ then $(\psi|\alpha)(n+1) = (\psi|\alpha)(n) + 1$ and $(\varphi|\alpha)^{n+1} = \alpha^{(\psi|\alpha)(n+1)}$.

The idea behind these definitions is the following. φ will be the function reducing $\mathcal{ALMOST}\text{–}\mathcal{E}_2$ to $\mathcal{ALMOST}\text{–}\mathcal{E}_2!$, and ψ will be an auxiliary function. Given α, we check its subsequences, $\alpha^0, \alpha^1, \ldots$ one by one. At stage 0, we start with studying α^0 and we define $(\psi|\alpha)(0) = 0$ and $(\varphi|\alpha)^0 = \alpha^0$. At every stage $n+1$, if $(\psi|\alpha)(n) = k$, we consider α^k, and we distinguish two cases. *Case 1.* We discover that $\alpha^k \# \underline{0}$, (as $\overline{\alpha^k}(n+1) \perp \underline{0}$). We now decide to study α^{k+1} at the next stage $n+1$, so we define $(\psi|\alpha)(n+1) = k+1$. We also define $(\varphi|\alpha)^{n+1} = \alpha^{k+1}$. *Case 2.* We do not yet see that $\alpha^k \# \underline{0}$ (as $\overline{\alpha^k}(n+1) \sqsubset \underline{0}$). We decide to continue our study of α^k at stage $n+1$, so we define $(\psi|\alpha)(n+1) = k$. We also define $(\varphi|\alpha)^{n+1} = \underline{1}$.

Note that for each α, for all k, if $\forall i < k[\alpha^i \# \underline{0}]$, then there exists j such that $(\psi|\alpha)(j) = k$. If j_0 is the least such j and $\alpha^k = \underline{0}$, then $(\varphi|\alpha)^{j_0} = \underline{0}$ and, for all $i \neq j_0$, one has $(\varphi|\alpha)^i \# \underline{0}$. Also note that for all n, m, if $n < m$, then either $(\psi|\alpha)(n) < (\psi|\alpha)(m)$ and $(\varphi|\alpha)^n \# \underline{0}$; or, $(\psi|\alpha)(n) = (\psi|\alpha)(m)$ and $(\varphi|\alpha)^m = \underline{1} \# \underline{0}$. Also note that for each n, $(\varphi|\alpha)^n = \underline{1}$ or $(\varphi|\alpha)^n = \alpha^{(\psi|\alpha)(n)}$.

We now prove that φ reduces $\mathcal{ALMOST}\text{–}\mathcal{E}_2$ to $\mathcal{ALMOST}\text{–}\mathcal{E}_2!$. Assume that $\alpha \in \mathcal{ALMOST}\text{–}\mathcal{E}_2$. Let γ be given. We want to find m such that $(\varphi|\alpha)^m\big(\gamma(m)\big) = 0$. Define δ such that $\delta(0) := 0$ and, for each n, if $\forall i \leq n[(\varphi|\alpha)^{\delta(i)} \circ \gamma \circ \delta(i) \neq 0]$ then $\delta(n+1) := \mu j[(\psi|\alpha)(j) = n+1]$; and, if not, then $\delta(n+1) := \delta(n)$. Note that for each n, if $\forall i < n[(\varphi|\alpha)^{\delta(i)} \circ \gamma \circ \delta(i) \neq 0]$, then $\forall i \leq n[(\varphi|\alpha)^{\delta(i)} = \alpha^i]]$. Define $n := \mu k[\alpha^k \circ \gamma \circ \delta(k) = 0]$. Conclude that $(\varphi|\alpha)^{\delta(n)} = \alpha^n$ and $(\varphi|\alpha)^{\delta(n)} \circ \gamma \circ \delta(n) = 0$ and $\exists m[(\varphi|\alpha)^m \circ \gamma(m) = 0]$. We thus see that $\forall \gamma \exists m[(\varphi|\alpha)^m \circ \gamma(m) = 0]$, ie $\varphi|\alpha \in \mathcal{ALMOST}\text{–}\mathcal{E}_2$. As we observed already: for all m, n, if $m \neq n$, then either $(\varphi|\alpha)^m \# \underline{0}$ or $(\varphi|\alpha)^n \# \underline{0}$. Conclude that $\varphi|\alpha \in \mathcal{ALMOST}\text{–}\mathcal{E}_2!$.

Now assume that $\varphi|\alpha \in \mathcal{ALMOST}\text{–}\mathcal{E}_2!$. Let γ be given. We want to find m such that $\alpha^m\big(\gamma(m)\big) = 0$. Define δ such that, for each n, $\delta(n) = \gamma\big((\psi|\alpha)(n)\big)$. Find n such that $(\varphi|\alpha)^n \circ \delta(n) = 0$. Note that $(\varphi|\alpha)^n \# \underline{1}$ and $(\varphi|\alpha)^n = \alpha^{(\psi|\alpha)(n)}$. Define

$m := (\psi|\alpha)(n)$ and note that $\alpha^m(\gamma(m)) = \alpha^{(\psi|\alpha)(n)}(\gamma\big((\psi|\alpha)(n)\big) = (\varphi|\alpha)^n\big(\delta(n)\big) = 0$. We thus see that $\forall\gamma\exists n[\alpha^n \circ \gamma(n) = 0]$, ie $\alpha \in \mathcal{ALMOST}{-}\mathcal{E}_2$. $\square$

The following definition has been given already in Section 1.2.7.

Definition 22 *For every* $\mathcal{X} \subseteq \omega^\omega$, *we define* $\mathsf{Perhaps}(\mathcal{X}) = \{\alpha \mid \exists\beta \in \mathcal{X}[\alpha \mathbin{\#} \beta \rightarrow \alpha \in \mathcal{X}]\}$. $\mathcal{X} \subseteq \omega^\omega$ *is called* perhapsive *if and only if* $\mathsf{Perhaps}(\mathcal{X}) = \mathcal{X}$.

Theorem 4.8

(i) $\mathcal{A}_1^1$ *is perhapsive.*
(ii) $\mathcal{E}_2!$ *is not perhapsive.*
(iii) $\mathcal{E}_2!$ *and* $\mathcal{E}_1^1!$ *are not* $\mathbf{\Pi}_1^1$.

Proof (i) See Theorem 1.5(iv).

(ii) Let $\mathcal{X}$ be the set of all α such that $\alpha(0) = 0$ and, for all n, if $n = \mu p[\alpha^0(p) \neq 0]$, then $\alpha^{n+1} = \underline{0}$ and, if $n \neq \mu p[\alpha^0(p) \neq 0]$, then $\alpha^{n+1} = \underline{1}$. We shall prove that $\mathcal{X}$ is a subset of $\mathsf{Perhaps}(\mathcal{E}_2!)$ but not of $\mathcal{E}_2!$ itself. It then follows that $\mathcal{E}_2!$ is not perhapsive. Define ζ such that $\zeta(0) = 0$ and $\zeta^0 = \underline{0}$ and $\forall n[\zeta^{n+1} = \underline{1}]$. Note that $\zeta \in \mathcal{X} \cap \mathcal{E}_2!$. Assume that $\alpha \in \mathcal{X}$ and $\alpha \mathbin{\#} \zeta$. Find i, n such that $\alpha^i(n) \neq \zeta^i(n)$. Either $i = 0$ and $\alpha^0(n) \neq 0$; or $i > 0$, $\alpha^i(n) \neq \zeta^i(n) = 1$ and $\alpha^0(i-1) \neq 0$. In both cases, $\alpha^0 \mathbin{\#} \underline{0}$ and $\alpha \in \mathcal{E}_2!$. We thus see that $\forall\alpha \in \mathcal{X}[\alpha \mathbin{\#} \zeta \rightarrow \alpha \in \mathcal{E}_2!]$ and conclude that $\mathcal{X} \subseteq \mathsf{Perhaps}(\mathcal{E}_2!)$.

Assume that $\mathcal{X} \subseteq \mathcal{E}_2!$. Note that $\mathcal{X}$ is a spread containing ζ. Using **BCP**, find m, n such that $\forall\alpha \in \mathcal{X}[\overline{\zeta}m \sqsubset \alpha \rightarrow \alpha^n = \underline{0}]$. In particular: $\zeta^n = \underline{0}$, and $n = 0$. But $\exists\alpha \in \mathcal{X}[\overline{\zeta}m \sqsubset \alpha \ \wedge \ \alpha^0 \mathbin{\#} \underline{0}]$, a contradiction. Conclude that $\mathcal{X} \nsubseteq \mathcal{E}_2!$ while $\mathcal{X} \subseteq \mathsf{Perhaps}(\mathcal{E}_2!)$, so $\mathsf{Perhaps}(\mathcal{E}_2!) \nsubseteq \mathcal{E}_2!$ and $\mathcal{E}_2!$ is not perhapsive.

(iii) Use (i), (ii), and Theorems 1.5(i), 4.1(ii) and 4.6(i). $\square$

5 $\mathcal{A}_1^1$ and $\mathcal{E}_1^1$

In this section, we study the sets $\mathcal{A}_1^1 = \mathcal{BAR} := \{\alpha \mid \forall\gamma\exists n[\alpha(\overline{\gamma}n) \neq 0]\}$ and $\mathcal{E}_1^1 = \mathcal{PATH} := \{\alpha \mid \exists\gamma\forall n[\alpha(\overline{\gamma}n) = 0]\}$. We have seen that $\mathcal{A}_1^1$ is $\mathbf{\Pi}_1^1$–complete and that $\mathcal{E}_1^1$ is $\mathbf{\Sigma}_1^1$–complete, see Theorems 4.1(ii) and 2.1(ii).

5.1 $\mathcal{A}^1_1$ positively fails to be strictly analytic

The following definitions have been given already in Section 1.1.2.

Definition 23 *For each* α, $T_\alpha := \{s \mid \forall t \sqsubset s[\alpha(t) = 0]\}$.

For all α, β, *for all* γ, *we define:* $\gamma : \alpha \leq^* \beta \leftrightarrow \big(\forall s[s \in T_\alpha \rightarrow \gamma(s) \in T_\beta] \ \wedge\ \forall s \forall t[s \sqsubset t \rightarrow \gamma(s) \sqsubset \gamma(t)]\big)$, *and* $\gamma : \alpha <^* \beta \leftrightarrow \big(\forall s[s \in T_\alpha \rightarrow \gamma(s) \in T_\beta] \ \wedge\ \forall s\forall t[s \sqsubset t \rightarrow \gamma(s) \sqsubset \gamma(t)] \ \wedge\ \gamma(\langle\ \rangle) \neq \langle\ \rangle\big)$.

For all α, β, *we define:* $\alpha <^* \beta \leftrightarrow \exists\gamma[\gamma : \alpha <^* \beta]$, *and* $\alpha \leq^* \beta \leftrightarrow \exists\gamma[\gamma : \alpha \leq^* \beta]$.

T_α is called the *tree determined by* α. Note that $\forall\alpha[0 = \langle\ \rangle \in T_\alpha]$.

$\alpha \leq^* \beta$ if and only if there exists a $\sqsubset$–preserving embedding of T_α into T_β.

$\alpha <^* \beta$ if and only if there exists n in ω and a $\sqsubset$–preserving embedding of T_α into $\{s \in T_\beta \mid \langle n\rangle \sqsubseteq s\}$.

Lemma 5.1

(i) *For all* α, β, γ, $\alpha \leq^* \alpha$ *and* $(\alpha \leq^* \beta \ \wedge\ \beta \leq^* \gamma) \rightarrow \alpha \leq^* \gamma$ *and* $\alpha <^* \beta \rightarrow \alpha \leq^* \beta$ *and* $(\alpha <^* \beta \ \wedge\ \beta \leq^* \gamma) \rightarrow \alpha <^* \gamma$ *and* $(\alpha \leq^* \beta \ \wedge\ \beta <^* \gamma) \rightarrow \alpha <^* \gamma$.

(ii) $\forall \alpha \in \mathcal{A}^1_1 \forall \beta \in \mathcal{A}^1_1[\alpha <^* \beta \rightarrow \alpha \mathrel{\#} \beta]$.

Proof (i) Note that for all $\alpha, \beta, \gamma, \delta, \varepsilon$, if $\delta : \alpha \leq^* \beta$ and $\varepsilon : \beta \leq^* \gamma$, then $\varepsilon \circ \delta : \alpha \leq^* \gamma$. Conclude that if $\alpha \leq^* \beta$ and $\beta \leq^* \gamma$, then $\alpha \leq^* \gamma$.

The proofs of the other statements are also straightforward.

(ii) Let α, β in $\mathcal{A}^1_1$ be given such that $\alpha <^* \beta$. Find γ such that $\forall s \in T_\alpha[\gamma(s) \in T_\beta]$ and $\forall s \forall t[s \sqsubset t \rightarrow \gamma(s) \sqsubset \gamma(t)]$ and $\gamma(\langle\ \rangle) \neq \langle\ \rangle$. Define ε such that $\varepsilon(0) = \langle\ \rangle$ and, for each n, $\varepsilon(n+1) = \gamma \circ \varepsilon(n)$. Note that for all n, $\varepsilon(n) \sqsubset \varepsilon(n+1)$, and, if $\varepsilon(n) \in T_\alpha$, then $\varepsilon(n+1) \in T_\beta$. Find δ such that $\forall n[\varepsilon(n) \sqsubset \delta]$ and note that $\exists n[\overline{\delta}n \notin T_\alpha]$. Conclude that $\exists m[\varepsilon(m) \notin T_\alpha]$ and define $p := \mu m[\varepsilon(m) \notin T_\alpha]$. Note that $p > 0$ and find q such that $p = q + 1$. Conclude that $\varepsilon(q) \in T_\alpha$ and $\varepsilon(p) \in T_\beta \setminus T_\alpha$ and $\alpha \mathrel{\#} \beta$. □

The next Theorem, Theorem 5.2, shows that $\mathcal{A}^1_1$ *positively fails to be strictly analytic or* $\mathbf{\Sigma}^{1*}_1$ in the following sense: given a (continuous) function from ω^ω into $\mathcal{A}^1_1$ one may construct an element of $\mathcal{A}^1_1$ that does not occur in the range of φ.

Theorem 5.2

(i) Cantor's diagonal argument: $\forall\varphi\colon\ \omega^\omega \to \mathcal{A}_1^1 \exists\alpha \in \mathcal{A}_1^1 \forall\beta[\alpha \# \varphi|\beta]$.
(ii) The Boundedness Theorem: $\forall\varphi\colon\ \omega^\omega \to \mathcal{A}_1^1 \exists\alpha \in \mathcal{A}_1^1 \forall\beta[\varphi|\beta \leq^* \alpha]$.

Proof (i) Assume that $\varphi\colon\ \omega^\omega \to \mathcal{A}_1^1$. We claim that there exists $\alpha : \omega^\omega \to \omega$ such that $\forall\beta[\alpha(\beta) = (\varphi|\beta)(\beta) + 1]$.

We first give an informal description of such an α. Given β, find $p := \mu n[(\varphi|\beta)(\overline{\beta}n) \neq 0]$. Now ensure that, for some q, $q = \mu n[\alpha(\overline{\beta}n) \neq 0]$ and $\alpha(\overline{\beta}q) = (\varphi|\beta)(\overline{\beta}p) + 1$. A precise definition of such an α is the following. Define α such that for each c, if there is no $b \sqsubseteq c$ such that $b < \text{length}(\varphi|c)$ and $(\varphi|c)(b) \neq 0$, then $\alpha(c) = 0$, and, if there is, then $\alpha(c) = (\varphi|c)(b_0) + 1$, where b_0 is the least such b. Note that $\alpha \in \mathcal{A}_1^1$ and, for each β, $\alpha \# \varphi|\beta$ as $\alpha(\beta) \neq (\varphi|\beta)(\beta)$.

(ii). Assume that $\varphi\colon\ \omega^\omega \to \mathcal{A}_1^1$. Note that $\forall\beta\forall\delta\exists n[(\varphi|\beta)(\overline{\delta}n) \neq 0]$, and $\forall\beta\forall\delta\exists n\exists m[\varphi^{\overline{\delta}n}(\overline{\beta}m) > 0\ \wedge\ \forall i < m[\varphi^{\overline{\delta}n}(\overline{\beta}i) = 0]]$. Define α such that $\forall s[\alpha(s) \neq 0 \leftrightarrow \exists t \sqsubseteq s_I \exists u \sqsubseteq s_{II}[\varphi^t(u) > 0\ \wedge\ \forall v \sqsubset u[\varphi^t(v) = 0]]]$. Note that $\alpha \in \mathcal{A}_1^1$. Let β be given. Define ε such that $\forall d\forall n[n = \text{length}(d) \to \varepsilon(d) = \ulcorner\overline{\beta}n, d\urcorner]$. Note that $\varepsilon : \varphi|\beta \leq^* \alpha$. We thus see that $\forall\beta[\varphi|\beta \leq^* \alpha]$. □

Using Lemma 5.1, one may obtain Theorem 5.2(i) from Theorem 5.2(ii), as follows. Assume $\alpha \in \mathcal{A}_1^1$ and $\forall\beta[\varphi|\beta \leq^* \alpha]$. Note[18]: $S^*(\alpha) \in \mathcal{A}_1^1$ and $\forall\beta[\varphi|\beta <^* S^*(\alpha)]$ and thus, according to Theorem 5.2(i), $\forall\beta[\varphi|\beta \# S^*(\alpha)]$.

5.2 $\mathcal{E}_1^1$ positively fails to be $\mathbf{\Pi}_1^1$

The next Theorem, Theorem 5.3, should prepare the reader for Theorem 5.4. The proof of Theorem 5.3 is elementary in the sense that no use is made of intuitionistic principles like Brouwer's Continuity Principle **BCP** or the Fan Theorem **FT**. The proof of Theorem 5.3(i) has been given in Veldman [35, Section 5.4]. Theorem 5.3(iii) is a rather weak statement if one compares it to the result of the Borel Hierarchy Theorem, Theorem 1.2 in Section 1.2.4. One should compare Theorem 5.3(iii) to Theorem 5.5(i).

Theorem 5.3

(i) $\mathcal{E}_2$ positively fails to be $\mathbf{\Pi}_2^0$: *if a continuous function maps $\mathcal{E}_2$ into $\mathcal{A}_2$, it also maps some element of $\mathcal{A}_2$ into $\mathcal{A}_2$:*

$$\forall\varphi\colon\ \omega^\omega \to \omega^\omega[\forall\alpha \in \mathcal{E}_2[\varphi|\alpha \in \mathcal{A}_2] \to \exists\alpha \in \mathcal{A}_2[\varphi|\alpha \in \mathcal{A}_2]]$$

[18] For each α, $S^*(\alpha)$ is the element β of ω^ω such that $\beta(0) = 0$ and $\forall n[\beta^n = \alpha]$; see Section 1.1.8. If $\alpha \in \mathcal{A}_1^1$, then also $S^*(\alpha) \in \mathcal{A}_1^1$. $S^*(\alpha)$ is called the *successor* of α.

(ii) *If $\mathcal{E}_2$ is contained in a set $\mathcal{X}$ that is a countable intersection of open sets, also some element of $\mathcal{A}_2$ is in $\mathcal{X}$:* $\forall\beta[\mathcal{E}_2 \subseteq \mathcal{F}^2_\beta \rightarrow \exists\alpha[\alpha \in \mathcal{A}_2 \cap \mathcal{F}^2_\beta]]$.

(iii) *The assumption that $\mathcal{A}_2$ is a countable union of* spreads *leads to a contradiction:* $\neg\exists\beta[\forall n[\mathrm{Spr}(\beta^n)] \wedge \mathcal{A}_2 = \bigcup_n \mathcal{F}_{\beta^n}]$.

Proof (i) Assume $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ and $\forall\alpha \in \mathcal{E}_2[\varphi|\alpha \in \mathcal{A}_2]$. Now define α such that, for all n, for all m, $\alpha^n(m) \neq 0$ if and only if $(\varphi|\overline{\alpha}m)^n \perp \underline{0}$. Note that for all n, $\alpha^n \# \underline{0}$ if and only if $(\varphi|\alpha)^n \# \underline{0}$.

We now prove that for all n, both α^n and $(\varphi|\alpha)^n$ are in $\mathcal{E}_1$. Let n be given. Define α_n such that $(\alpha_n)^n = \underline{0}$ and $\forall j[\neg\exists t[j = \langle n\rangle * t] \rightarrow \alpha_n(j) = \alpha(j)]$. Note that $\alpha_n \in \mathcal{E}_2$, $\varphi|\alpha_n \in \mathcal{A}_2$, and $(\varphi|\alpha_n)^n \perp \underline{0}$. Find $t \sqsubset \alpha_n$ such that $(\varphi|t)^n \perp \underline{0}$ and distinguish two cases. Either $t \sqsubset \alpha$ and $(\varphi|\alpha)^n \# \underline{0}$ and also $\alpha^n \# \underline{0}$; or, $t \perp \alpha$, $\alpha_n \perp \alpha$, $\alpha^n \# \underline{0}$, and also $(\varphi|\alpha)^n \# \underline{0}$. We thus see that for all n, $\alpha^n \# \underline{0}$ and $(\varphi|\alpha)^n \# \underline{0}$, ie $\alpha \in \mathcal{A}_2$ and $\varphi|\alpha \in \mathcal{A}_2$.

(ii) Let β given such that $\mathcal{E}_2 \subseteq \mathcal{F}^2_\beta$. Find $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ reducing $\mathcal{F}^2_\beta$ to $\mathcal{A}_2$. Note that $\forall\alpha \in \mathcal{E}_2[\varphi|\alpha \in \mathcal{A}_2]$. Applying (i), find α in $\mathcal{A}_2$ such that $\varphi|\alpha \in \mathcal{A}_2$, so $\alpha \in \mathcal{A}_2 \cap \mathcal{F}^2_\beta$.

(iii) Let β be given such that $\forall n[\mathrm{Spr}(\beta^n)]$ and $\mathcal{A}_2 = \mathcal{G}^2_\beta = \bigcup_n \mathcal{F}_{\beta_n}$. Find ρ such that, for each n, $\rho^n : \omega^\omega \rightarrow \mathcal{F}_{\beta^n}$ is the canonical retraction of ω^ω onto $\mathcal{F}_{\beta^n}$. Assume that $\alpha \in \mathcal{E}_2$. Note that $\forall\delta \in \mathcal{A}_2[\alpha \# \delta]$ and $\forall n\forall\delta \in \mathcal{F}_{\beta^n}[\alpha \# \delta]$ and $\forall n[\alpha \# \rho^n|\alpha]$ and $\forall n\exists m[\beta^n(\overline{\alpha}m) \neq 0]$ and $\forall n[\alpha \in \mathcal{G}_{\beta^n}]$ and $\alpha \in \mathcal{F}^2_\beta$. We thus see that $\forall\alpha \in \mathcal{E}_2[\alpha \in \mathcal{F}^2_\beta]$, ie $\mathcal{E}_2 \subseteq \mathcal{F}^2_\beta$. Applying (ii), we find $\alpha \in \mathcal{A}_2 \cap \mathcal{F}^2_\beta = \mathcal{G}^2_\beta \cap \mathcal{F}^2_\beta = \emptyset$, a contradiction. $\square$

The proof of the next theorem, Theorem 5.4, is also elementary.

Theorem 5.4

(i) $\mathcal{E}^1_1$ positively fails to be $\mathbf{\Pi}^1_1$: *If a continuous function from ω^ω to ω^ω maps $\mathcal{E}^1_1$ into $\mathcal{A}^1_1$, it also maps some element of $\mathcal{A}^1_1$ into $\mathcal{A}^1_1$:* $\forall\varphi\colon \omega^\omega \rightarrow \omega^\omega[\forall\alpha \in \mathcal{E}^1_1[\varphi|\alpha \in \mathcal{A}^1_1] \rightarrow \exists\alpha \in \mathcal{A}^1_1[\varphi|\alpha \in \mathcal{A}^1_1]]$.

(ii) *If $\mathcal{E}^1_1$ is contained in a $\mathbf{\Pi}^1_1$ set $\mathcal{X}$, also some element of $\mathcal{A}^1_1$ is in $\mathcal{X}$:* $\forall\beta[\mathcal{E}^1_1 \subseteq \mathcal{U}\mathcal{G}_\beta \rightarrow \exists\alpha[\alpha \in \mathcal{A}^1_1 \cap \mathcal{U}\mathcal{G}_\beta]]$.

Proof (i) Assume $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ and $\forall\alpha \in \mathcal{E}^1_1[\varphi|\alpha \in \mathcal{A}^1_1]$. Now define α such that, for all t, $\alpha(t) \neq 0$ if and only if $\exists s \sqsubseteq t[(\varphi|\overline{\alpha}t)(s) \neq 0]$. Note that for all γ, $\exists n[\alpha(\overline{\gamma}n) \neq 0]$ if and only if $\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0]$.

We now prove: for all γ, $\exists n[\alpha(\overline{\gamma}n) \neq 0]$ and $\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0]$.

Let γ be given. Define α_γ such that $\forall n[\alpha(\overline{\gamma}n) = 0]$ and $\forall t[t \perp \gamma \rightarrow \alpha_\gamma(t) = \alpha(t)]$. Note that $\alpha_\gamma \in \mathcal{E}^1_1$ and $\varphi|\alpha_\gamma \in \mathcal{A}^1_1$. Find m such that $(\varphi|\alpha_\gamma)(\overline{\gamma}m) \neq 0$. Find $t \sqsubset \alpha_\gamma$ such that $(\varphi|t)(\overline{\gamma}m) \neq 0$ and distinguish two cases. Either $t \sqsubset \alpha$, $(\varphi|\alpha)(\overline{\gamma}m) \neq 0$, and $\exists n \leq m[\alpha(\overline{\gamma}n) \neq 0]$; or, $t \perp \alpha$, $\alpha \perp \alpha_\gamma$, $\exists n[\alpha(\overline{\gamma}n) \neq 0]$, and $\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0]$.

We thus see that for all γ, $\exists n[\alpha(\overline{\gamma}n) \neq 0]$ and $\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0]$, ie $\alpha \in \mathcal{A}^1_1$ and $\varphi|\alpha \in \mathcal{A}^1_1$.

(ii) Let β given such that $\mathcal{E}^1_1 \subseteq \mathcal{UG}_\beta$. Find $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ reducing $\mathcal{UG}_\beta$ to $\mathcal{A}^1_1$. Note that $\forall \alpha \in \mathcal{E}^1_1[\varphi|\alpha \in \mathcal{A}^1_1]$. Applying (i), find α in $\mathcal{A}^1_1$ such that $\varphi|\alpha \in \mathcal{A}^1_1$, so $\alpha \in \mathcal{A}^1_1 \cap \mathcal{UG}_\beta$. □

5.3 May one prove: '$\mathcal{A}^1_1$ is not analytic'?

The following theorem should be compared to Veldman [35, Theorem 5.2(iv)].

Theorem 5.5

(i) *If* $\mathcal{A}_2$ *is a countable union of* closed *sets, there exists* α *not in either* $\mathcal{A}_2$ *or* $\mathcal{E}_2$*:* $\mathcal{A}_2 \preceq \mathcal{E}_2 \rightarrow \exists \alpha[\alpha \notin \mathcal{E}_2 \wedge \alpha \notin \mathcal{A}_2]$.

(ii) *If* $\mathcal{A}^1_1$ *is analytic, there exists* α *not in either* $\mathcal{A}^1_1$ *or* $\mathcal{E}^1_1$*:* $\mathcal{A}^1_1 \preceq \mathcal{E}^1_1 \rightarrow \exists \alpha[\alpha \notin \mathcal{E}^1_1 \wedge \alpha \notin \mathcal{A}^1_1]$.

Proof (i) Let $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ be given. Define α such that, for all n, for all m, $\alpha^n(m) \neq 0$ if and only if $(\varphi|\overline{\alpha}m)^n \perp \underline{0}$. Note that for all n, $\exists m[\alpha^n(m) \neq 0]$ if and only if $\exists m[(\varphi|\alpha)^n(m) \neq 0]$, so $\alpha^n \in \mathcal{E}_1$ if and only if $(\varphi|\alpha)^n \in \mathcal{E}_1$ and $\alpha^n \in \mathcal{A}_1$ if and only if $(\varphi|\alpha)^n \in \mathcal{A}_1$. Conclude that $\alpha \in \mathcal{E}_2$ if and only if $\varphi|\alpha \in \mathcal{E}_2$ and $\alpha \in \mathcal{A}_2$ if and only if $\varphi|\alpha \in \mathcal{A}_2$.

Now assume, in addition, that φ reduces $\mathcal{A}_2$ to $\mathcal{E}_2$. If $\alpha \in \mathcal{A}_2$, then both $\varphi|\alpha \in \mathcal{E}_2$ and $\varphi|\alpha \in \mathcal{A}_2$, a contradiction. If $\alpha \in \mathcal{E}_2$, then both $\varphi|\alpha \in \mathcal{E}_2$ and $\alpha \in \mathcal{A}_2$, a contradiction. We thus see that $\alpha \notin \mathcal{A}_2$ and $\alpha \notin \mathcal{E}_2$.

(ii) Let $\varphi\colon \omega^\omega \rightarrow \omega^\omega$ be given. Define α such that, for all t, $\alpha(t) \neq 0$ if and only if $\exists s \sqsubseteq t[(\varphi|\overline{\alpha}t)(s) \neq 0]$. Note that for each γ, $\exists n[\alpha(\overline{\gamma}n) \neq 0]$ if and only if $\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0]$. Conclude that $\alpha \in \mathcal{E}^1_1$ if and only if $\varphi|\alpha \in \mathcal{E}^1_1$ and $\alpha \in \mathcal{A}^1_1$ if and only if $\varphi|\alpha \in \mathcal{A}^1_1$.

Now assume, in addition, that φ reduces $\mathcal{A}^1_1$ to $\mathcal{E}^1_1$. If $\alpha \in \mathcal{A}^1_1$, then both $\varphi|\alpha \in \mathcal{E}^1_1$ and $\varphi|\alpha \in \mathcal{A}^1_1$, a contradiction. If $\alpha \in \mathcal{E}^1_1$, then both $\varphi|\alpha \in \mathcal{E}^1_1$ and $\alpha \in \mathcal{A}^1_1$, a contradiction. We thus see that $\alpha \notin \mathcal{A}^1_1$ and $\alpha \notin \mathcal{E}^1_1$. □

Markov's Principle **MP**, in our view a dubious assumption (see Section 1.1.11), proves, for all α,

$$\alpha \notin \mathcal{E}_2 \to \neg\exists n\forall m[\alpha^n(m) = 0] \to \forall n\neg\neg\exists m[\alpha^n(m) \neq 0]$$
$$\to \forall n\exists m[\alpha^n(m) \neq 0] \to \alpha \in \mathcal{A}_2$$

and thus, together with Theorem 5.5(i), $\mathcal{A}_2 \not\leq \mathcal{E}_2$.

MP also proves, for all α,

$$\alpha \notin \mathcal{E}^1_1 \to \neg\exists\gamma\forall n[\alpha(\overline{\gamma}n) = 0] \to \forall\gamma\neg\neg\exists n[\alpha(\overline{\gamma}n) \neq 0]$$
$$\to \forall\gamma\exists n[\alpha(\overline{\gamma}n) \neq 0] \to \alpha \in \mathcal{A}^1_1$$

and thus, together with Theorem 5.5(ii), $\mathcal{A}^1_1 \not\leq \mathcal{E}^1_1$.

Intuitionistically, one obtains the conclusion $\mathcal{A}_2 \not\leq \mathcal{E}_2$ as a corollary of a stronger statement proven from Brouwer's Continuity Principle **BCP**; see Theorem 1.2 in Section 1.2.4. No such argument seems to be available for the conclusion $\mathcal{A}^1_1 \not\leq \mathcal{E}^1_1$.

One may prove $\mathcal{A}^1_1 \not\leq \mathcal{E}^1_1$, avoiding **MP** but using **KS**; see Section 1.1.10. One may argue that $\mathcal{A}^1_1$ is *definite*, and therefore, if analytic, also strictly analytic; see Theorem 2.11 in Section 2.5. We have seen that $\mathcal{A}^1_1$ is not strictly analytic; see Theorem 5.2.

5.4 $\mathcal{E}^1_1$ and $\mathcal{A}^1_1$ positively fail to be (positively) Borel

In classical descriptive set theory, the following statement holds:

> *A continuous function $\varphi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{X} \subseteq \omega^\omega$ to $\mathcal{E}^1_1$ reduces $\omega^\omega \setminus \mathcal{X}$ to $\mathcal{A}^1_1$.*

So, if one has seen that every Borel $\mathcal{X} \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_1$ and reduces to $\mathcal{E}^1_1$, one may conclude that every Borel $\mathcal{X} \subseteq \omega^\omega$ reduces to $\mathcal{A}^1_1$ and is $\mathbf{\Pi}^1_1$. In our constructive context, this conclusion is wrong, see Theorems 2.1(iv) and Theorem 4.1(iv).

The following subtle Lemma 5.6 replaces the just mentioned statement.

Lemma 5.6 *For every complementary pair $(\mathcal{X}, \mathcal{Y})$ of positively Borel sets there exists $\varphi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{E}^1_1$ and mapping $\mathcal{Y}$ into $\mathcal{A}^1_1$.*

Proof We use induction on the class of complementary pairs of Borel sets and distinguish three cases.

Case 1. Let β be given such that $\mathcal{X} = \mathcal{G}_\beta = \{\alpha \mid \exists n[\beta(\overline{\alpha}n) \neq 0]\}$ and $\mathcal{Y} = \mathcal{F}_\beta = \{\alpha \mid \forall n[\beta(\overline{\alpha}n) = 0]\}$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[(\varphi|\alpha)(0) = 0 \ \wedge \ \forall s >$

$0[(\varphi|\alpha)(s) = 0 \leftrightarrow \beta\big(\overline{\alpha}(s(0))\big) \neq 0]]$. Note that φ simultaneously reduces $\mathcal{X}$ to $\mathcal{E}^1_1$ and $\mathcal{Y}$ to $\mathcal{A}^1_1$, because for each α:

$$\alpha \in \mathcal{G}_\beta \leftrightarrow \exists n[\beta(\overline{\alpha}n) \neq 0] \leftrightarrow \exists\gamma[\beta(\overline{\alpha}\gamma(0)) \neq 0] \leftrightarrow \exists\gamma[(\varphi|\alpha)(\langle\gamma(0)\rangle) = 0]$$
$$\leftrightarrow \exists\gamma\forall n[(\varphi|\alpha)(\overline{\gamma}n) = 0] \leftrightarrow \varphi|\alpha \in \mathcal{E}^1_1$$

and
$$\alpha \in \mathcal{F}_\beta \leftrightarrow \forall n[\beta(\overline{\alpha}n) = 0] \leftrightarrow \forall\gamma[\beta(\overline{\alpha}\gamma(0)) = 0] \leftrightarrow \forall\gamma[(\varphi|\alpha)(\langle\gamma(0)\rangle) \neq 0]$$
$$\leftrightarrow \forall\gamma\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0] \leftrightarrow \varphi|\alpha \in \mathcal{A}^1_1$$

Case 2. Let β be given such that $\mathcal{X} = \mathcal{F}_\beta$ and $\mathcal{Y} = \mathcal{G}_\beta$. Define $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\varphi|\alpha)(s) = 0 \leftrightarrow \forall j \leq s[\beta(\overline{\alpha}j) = 0]]$. Note that φ simultaneously reduces $\mathcal{X}$ to $\mathcal{E}^1_1$ and $\mathcal{Y}$ to $\mathcal{A}^1_1$, because, for each α:

$$\alpha \in \mathcal{F}_\beta \leftrightarrow \forall n[\beta(\overline{\alpha}n) = 0] \leftrightarrow \forall s[(\varphi|\alpha)(s) = 0] \leftrightarrow \forall\gamma\forall n[(\varphi|\alpha)(\overline{\gamma}n) = 0]$$
$$\leftrightarrow \exists\gamma\forall n[(\varphi|\alpha)(\overline{\gamma}n) = 0] \leftrightarrow \varphi|\alpha \in \mathcal{E}^1_1$$

and
$$\alpha \in \mathcal{G}_\beta \leftrightarrow \exists n[\beta(\overline{\alpha}n) \neq 0] \leftrightarrow \exists s\forall t \geq s[(\varphi|\alpha)(t) \neq 0]$$
$$\leftrightarrow \forall\gamma\exists n[(\varphi|\alpha)(\overline{\gamma}n) \neq 0] \leftrightarrow \varphi|\alpha \in \mathcal{A}^1_1$$

Case 3. Let $(\mathcal{X}_0, \mathcal{Y}_0), (\mathcal{X}_1, \mathcal{Y}_1), \ldots$ be an infinite sequence of complementary pairs of (positively) Borel sets and let φ be given such that, for each n, $\varphi^n : \omega^\omega \to \omega^\omega$ reduces $\mathcal{X}_n$ to $\mathcal{E}^1_1$ and maps $\mathcal{Y}_n$ into $\mathcal{A}^1_1$.

Case 3a. Define $\mathcal{X} = \bigcup_n \mathcal{X}_n$ and $\mathcal{Y} := \bigcap_n \mathcal{Y}_n$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha[(\psi|\alpha)(0) = 0 \ \wedge\ \forall n\forall s[(\psi|\alpha)(\langle n\rangle * s) = (\varphi^n|\alpha)(s)]]$. Note that ψ reduces $\mathcal{X}$ to $\mathcal{E}^1_1$ and maps $\mathcal{Y}$ into $\mathcal{A}^1_1$, because: for each α,

$$\alpha \in \mathcal{X} \leftrightarrow \exists n[\alpha \in \mathcal{X}_n] \leftrightarrow \exists n[\varphi^n|\alpha \in \mathcal{E}^1_1] \leftrightarrow \exists n\exists\gamma\forall m[(\varphi^n|\alpha)(\overline{\gamma}m) = 0]$$
$$\leftrightarrow \exists\gamma\forall m[(\psi|\alpha)(\overline{\gamma}m) = 0] \leftrightarrow \psi|\alpha \in \mathcal{E}^1_1$$

and
$$\alpha \in \mathcal{Y} \leftrightarrow \forall n[\alpha \in \mathcal{Y}_n] \to \forall n[\varphi^n|\alpha \in \mathcal{A}^1_1] \leftrightarrow \forall n\forall\gamma\exists m[(\varphi^n|\alpha)(\overline{\gamma}m) \neq 0]$$
$$\leftrightarrow \forall\gamma\exists m[(\psi|\alpha)(\overline{\gamma}m) \neq 0]$$

so $\alpha \in \mathcal{Y} \to \psi|\alpha \in \mathcal{A}^1_1$

Case 3b. Define $\mathcal{X} = \bigcap_n \mathcal{X}_n$ and $\mathcal{Y} := \bigcup_n \mathcal{Y}_n$. Define $\psi\colon \omega^\omega \to \omega^\omega$ such that $\forall\alpha\forall s[(\psi|\alpha)(s) = 0 \leftrightarrow \forall n \leq s\forall t \sqsubseteq s^n[(\varphi^n|\alpha)(t) = 0]]$. Note that ψ reduces $\mathcal{X}$ to $\mathcal{E}^1_1$ and maps $\mathcal{Y}$ into $\mathcal{A}^1_1$, because, for each α:

$$\alpha \in \mathcal{X} \leftrightarrow \forall n[\alpha \in \mathcal{X}_n] \leftrightarrow \forall n[\varphi^n|\alpha \in \mathcal{E}^1_1] \leftrightarrow \forall n \exists \gamma \forall m[(\varphi^n|\alpha)(\overline{\gamma}m) = 0]$$
$$\leftrightarrow {}^{19}\exists \gamma \forall n \forall m[(\varphi^n|\alpha)(\overline{\gamma^n}m) = 0] \leftrightarrow \exists \gamma \forall m[(\psi|\alpha)(\overline{\gamma}m) = 0] \leftrightarrow \psi|\alpha \in \mathcal{E}^1_1$$

and

$$\alpha \in \mathcal{Y} \leftrightarrow \exists n[\alpha \in \mathcal{Y}_n] \to \exists n[\varphi^n|\alpha \in \mathcal{A}^1_1] \leftrightarrow \exists n \forall \gamma \exists m[(\varphi^n|\alpha)(\overline{\gamma}m) \neq 0]$$
$$\to {}^{20}\forall \gamma \exists n \exists m[(\varphi^n|\alpha)(\overline{\gamma^n}m) \neq 0] \leftrightarrow \forall \gamma \exists m[(\psi|\alpha)(\overline{\gamma}m) \neq 0] \leftrightarrow \psi|\alpha \in \mathcal{A}^1_1$$

so $\alpha \in \mathcal{Y} \to \psi|\alpha \in \mathcal{A}^1_1$. □

Theorem 5.7 ($\mathcal{E}^1_1$ and $\mathcal{A}^1_1$ positively fail to be (positively) Borel)

(i) *For every σ in $\mathcal{HRS}$, for every $\varphi\colon \omega^\omega \to \omega^\omega$, if $\varphi|\mathcal{E}^1_1 \subseteq \mathcal{E}_\sigma$, then $\exists \alpha \in \mathcal{A}^1_1[\varphi|\alpha \in \mathcal{E}_\sigma]$.*

(ii) *For every $\mathcal{X}$ in $\mathfrak{Borel}$, if $\mathcal{E}^1_1 \subseteq \mathcal{X}$, then $\exists \alpha \in \mathcal{A}^1_1[\alpha \in \mathcal{X}]$.*

(iii) *For every σ in $\mathcal{HRS}$, for every $\varphi\colon \omega^\omega \to \omega^\omega$, if $\varphi|\mathcal{A}^1_1 \subseteq \mathcal{E}_\sigma$, then $\exists \alpha \in \mathcal{E}^1_1[\varphi|\alpha \in \mathcal{E}_\sigma]$.*

(iv) *For every $\mathcal{X}$ in $\mathfrak{Borel}$, if $\mathcal{A}^1_1 \subseteq \mathcal{X}$, then $\exists \alpha \in \mathcal{E}^1_1[\alpha \in \mathcal{X}]$.*

Proof (i) Let σ, φ be given such that $\sigma \in \mathcal{HRS}$ and $\varphi\colon \omega^\omega \to \omega^\omega$ and $\varphi|\mathcal{E}^1_1 \subseteq \mathcal{E}_\sigma$. Using Lemma 5.6, find $\psi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{A}_\sigma$ to $\mathcal{E}^1_1$ and mapping $\mathcal{E}_\sigma$ into $\mathcal{A}^1_1$. Note that $\varphi \star \psi$[21] maps $\mathcal{A}_\sigma$ into $\mathcal{E}_\sigma$. Applying the Borel Hierarchy Theorem, Theorem 1.2, find β in $\mathcal{E}_\sigma$ such that $(\varphi \star \psi)|\beta \in \mathcal{E}_\sigma$. Define $\alpha := \psi|\beta$ and note that $\alpha \in \mathcal{A}^1_1$ and $\varphi|\alpha \in \mathcal{E}_\sigma$.

(ii) Let $\mathcal{X}$ in $\mathfrak{Borel}$ be given such that $\mathcal{E}^1_1 \subseteq \mathcal{X}$. Find σ in $\mathcal{HRS}$ and $\varphi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{E}_\sigma$. Note $\varphi|\mathcal{E}^1_1 \subseteq \mathcal{E}_\sigma$. Applying (i), find α in $\mathcal{A}^1_1$ such that $\varphi|\alpha \in \mathcal{E}_\sigma$ and, therefore, $\alpha \in \mathcal{X}$.

(iii) Let σ, φ be given such that $\sigma \in \mathcal{HRS}$ and $\varphi\colon \omega^\omega \to \omega^\omega$ and $\varphi|\mathcal{A}^1_1 \subseteq \mathcal{E}_\sigma$. Using Lemma 5.6, find $\psi : \omega^\omega \to \omega^\omega$ reducing $\mathcal{E}_\sigma$ to $\mathcal{E}^1_1$ and mapping $\mathcal{A}_\sigma$ into $\mathcal{A}^1_1$. Note that $\varphi \star \psi$ maps $\mathcal{A}_\sigma$ into $\mathcal{E}_\sigma$. Applying the Borel Hierarchy Theorem, Theorem 1.2, find β in $\mathcal{E}_\sigma$ such that $(\varphi \star \psi)|\beta \in \mathcal{E}_\sigma$. Define $\alpha := \psi|\beta$ and note that $\alpha \in \mathcal{E}^1_1$ and $\varphi|\alpha \in \mathcal{E}_\sigma$.

(iv) Let $\mathcal{X}$ in $\mathfrak{Borel}$ be given such that $\mathcal{A}^1_1 \subseteq \mathcal{X}$. Find σ in $\mathcal{HRS}$ and $\varphi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{E}_\sigma$. Note $\varphi|\mathcal{A}^1_1 \subseteq \mathcal{E}_\sigma$. Applying (iii), find α in $\mathcal{E}^1_1$ such that $\varphi|\alpha \in \mathcal{E}_\sigma$ and, therefore, $\alpha \in \mathcal{X}$. □

[19] We are applying the Second Axiom of Countable Choice, $\mathbf{AC}_{0,1}$: $\forall m \exists \gamma[m\mathcal{R}\gamma] \to \exists \gamma \forall m[m\mathcal{R}\gamma^m]$, see Section 1.1.3.

[20] The contraposition of $\mathbf{AC}_{0,1}$: $\forall \gamma \exists m[m\mathcal{R}\gamma^m] \to \exists m \forall \gamma[m\mathcal{R}\gamma]$, is not constructively valid, and, therefore, we have here a single arrow only.

[21] For all $\varphi, \psi\colon \omega^\omega \to \omega^\omega$, also $\varphi * \psi\colon \omega^\omega \to \omega^\omega$ and, for all α, $\varphi * \psi|\alpha = \varphi|(\psi|\alpha)$, see Section 1.1.5.

5.5 Other results showing that $\mathcal{E}^1_1$ and $\mathcal{A}^1_1$ are not (positively) Borel

$\mathcal{MONPATH} := \{\alpha \mid \exists\gamma \in \mathcal{F}_\alpha \forall n[\gamma(n) \leq \gamma(n+1) \leq 1]\}$ is what might be called a *simple* $\mathbf{\Sigma}^1_1$ set, as, from a classical point of view, $\mathcal{MONPATH}$ is $\mathbf{\Pi}^0_1$. The assumption that $\mathcal{MONPATH}$ is (positively) Borel leads to a contradiction, see Veldman [33, Theorem 2.23(vi)]. It follows that $\mathcal{E}^1_1$ is not positively Borel, but the statement of Theorem 5.7(ii) is a stronger conclusion.

As we mentioned in Section 4.3, $\mathcal{ALMOST^*FIN} := \{\alpha \mid \forall\zeta \in [\omega]^\omega \exists n[\alpha \circ \zeta(n) = 0]\}$ is $\mathbf{\Pi}^1_1$ but *not* (positively) Borel. $\mathcal{ALMOST^*FIN}$ might be called a *simple* $\mathbf{\Pi}^1_1$ set, as, from a classical point of view, $\mathcal{ALMOST^*FIN}$ is $\mathbf{\Sigma}^0_2$. It follows that also $\mathcal{A}^1_1$ is *not* (positively) Borel, but the statement of Theorem 5.7(iv) is a stronger conclusion.

As one might expect, the results about $\mathcal{MONPATH}$ and $\mathcal{ALMOST^*FIN}$ strongly use Brouwer's Continuity Principle **BCP**.

5.6 One half of Souslin's Theorem

Theorem 5.8

(i) *For every σ in $\mathcal{STP}$, $\{\alpha \mid \alpha \leq^* \sigma\} \in \mathfrak{Borel}$.*

(ii) *Every $\mathcal{X} \subseteq \omega^\omega$ that is both* strictly *analytic and co-analytic is (positively) Borel: $\mathbf{\Sigma}^{1*}_1 \cap \mathbf{\Pi}^1_1 \subseteq \mathfrak{Borel}$.*

Proof (i) Note that $\forall\alpha[\alpha \leq^* 1^* \leftrightarrow \alpha(0) \neq 0]$. Also note that for all $\sigma \neq 1^*$ in $\mathcal{STP}$, $\forall\alpha[\alpha \leq^* \sigma \leftrightarrow \forall m \exists n[\alpha^m \leq^* \sigma^n]]$. Now use induction on $\mathcal{STP}$.

(ii) Assume that $\mathcal{X} \in \mathbf{\Sigma}^{1*}_1 \cap \mathbf{\Pi}^1_1$. If $\mathcal{X} = \emptyset$, clearly $\mathcal{X} \in \mathfrak{Borel}$. Assume $\mathcal{X}$ is inhabited. Find $\varphi\colon \omega^\omega \to \omega^\omega$ such that $\mathcal{X} = \varphi|\omega^\omega$. Find $\psi\colon \omega^\omega \to \omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{A}^1_1$. Using Theorem 5.2(ii), find β in $\mathcal{A}^1_1$ such that $\forall\alpha[(\psi \star \varphi)(\alpha) \leq^* \beta]$. Note that D_β is a bar in ω^ω. Using Brouwer's Thesis on bars in ω^ω **BT**, see Section 1.1.9, find a stump σ such that $D_\beta \cap T_\sigma$ is bar in ω^ω. Conclude that $\forall\alpha[\alpha \leq^* \beta \to \alpha \leq^* \sigma]$. Conclude, using (i): $\mathcal{X} = \{\gamma \mid \psi|\gamma \leq^* \sigma\} \in \mathfrak{Borel}$. □

Theorem 5.8(ii) is of limited application as every $\mathbf{\Pi}^1_1$ subset of ω^ω is perhapsive, see Theorems 4.8(i) and 1.5(i) , and "most" positively Borel sets are not. Therefore, there are not "many" positively Borel sets that are both co-analytic and strictly analytic. The converse of Theorem 5.8(ii), although classically a well-known fact, is far from true.

6 Countable and almost-countable spreads

6.1 Countable spreads

Countable closed subsets of the set of the real numbers were among the first objects studied by Cantor. One might say that this study led him to discover set theory.

In our constructive context we study *located* and closed subsets of ω^ω, ie *spreads*, and ask ourselves what could be a useful notion of countability.

Definition 24 *For each δ, we define $En_\delta = \{\delta^n \mid n \in \omega\}$. We also define:*

$$\mathcal{COUNT} := \{\beta \mid \mathrm{Spr}(\beta) \ \wedge \ \exists\delta[\mathcal{F}_\beta \subseteq En_\delta]\}$$

En_δ is called the subset of ω^ω *enumerated by* δ, see Section 1.1.2.

If $\beta \in \mathcal{COUNT}$, we call $\mathcal{F}_\beta$ an *(at most) countable spread.*

Definition 25 *$\mathcal{X} \subseteq \omega^\omega$ is called* discrete *if and only if* $\forall\alpha \in \mathcal{X}\forall\beta \in \mathcal{X}[\alpha \# \beta \vee \alpha = \beta]$.

Recall that $\mathcal{FIN}$ is the set of all α such that $\exists n\forall m \geq n[\alpha(m) = 0]$, ie $D_\alpha := \{m \mid \alpha(m) \neq 0\}$ is a *finite* subset of ω.

Like Theorem 2.8, the following Theorem 6.1 should be compared to a classical result due to W. Hurewicz, see Kechris [14, Theorem 27.5].

Theorem 6.1

(i) For every spread $\mathcal{F} \subseteq \omega^\omega$: $\mathcal{F}$ is (at most) countable if and only if $\mathcal{F}$ is discrete: $\forall\beta[\beta \in \mathcal{COUNT} \leftrightarrow (\mathrm{Spr}(\beta) \ \wedge \ \forall\gamma_0 \in \mathcal{F}_\beta\forall\gamma_1 \in \mathcal{F}_\beta[\gamma_0 \# \gamma_1 \ \vee \ \gamma_0 = \gamma_1])]$.

(ii) $\mathcal{FIN} \preceq \mathcal{COUNT}$.

(iii) $\mathcal{A}^1_1 \preceq \mathcal{COUNT}$.

(iv) $\mathcal{COUNT}$ is not the co-projection of a closed subset of ω^ω but it is the co-projection of a (positively) Borel subset of ω^ω: *$\mathcal{COUNT}$ is not $\mathbf{\Pi}^1_1$ but $\mathcal{COUNT}$ is $\mathbf{\Pi}^{1+}_1$.*

Proof (i) Assume $\beta \in \mathcal{COUNT}$, ie $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ is (at most) countable. Note that if $\beta(0) \neq 0$ then $\mathcal{F}_\beta = \emptyset$ is discrete. Assume that $\beta(0) = 0$ and find δ such that $\mathcal{F}_\beta \subseteq En_\delta$. Then $\forall\gamma \in \mathcal{F}_\beta\exists n[\gamma = \delta^n]$. Let γ_0, γ_1 in $\mathcal{F}_\beta$ be given. Using Brouwer's Continuity Principle **BCP** (see Section 1.1.6), find n_0, m_0, n_1, m_1 such that

$\forall i < 2 \forall \gamma \in \mathcal{F}_\beta[\overline{\gamma_i} m_i \sqsubset \gamma \to \gamma = \delta^{n_i}]$. Note that if $\overline{\gamma_0} m_0 \perp \overline{\gamma_1} m_1$, then $\gamma_0 \# \gamma_1$, and, if not, then $\gamma_0 = \delta^{n_0} = \gamma_1$. We thus see that $\forall \gamma_0 \in \mathcal{F}_\beta \forall \gamma_1 \in \mathcal{F}_\beta[\gamma_0 \# \gamma_1 \ \vee \ \gamma_0 = \gamma_1]$, ie $\mathcal{F}_\beta$ is discrete.

Now assume $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ is discrete. We may assume that $\beta(0) = 0$, ie $\mathcal{F}_\beta$ is inhabited. Define ε such that, for all s, $\varepsilon(s) = 0$ if and only if $\beta(s_I) = \beta(s_{II}) = 0$. Note that $\mathrm{Spr}(\varepsilon)$ and for all γ, $\gamma \in \mathcal{F}_\varepsilon$ if and only if both γ_I and γ_{II} are in $\mathcal{F}_\beta$. Conclude that $\forall \gamma \in \mathcal{F}_\varepsilon[\gamma_I \# \gamma_{II} \ \vee \ \gamma_I = \gamma_{II}]$. Using the First Axiom of Continuous Choice $\mathbf{AC}_{1,0}$ (see Section 1.1.6) find $\varphi \colon \mathcal{F}_\varepsilon \to \omega$ such that $\forall \gamma \in \mathcal{F}_\varepsilon[\big(\varphi(\gamma) = 0 \to \gamma_I \# \gamma_{II}\big) \ \wedge \ \big(\varphi(\gamma) > 0 \to \gamma_I = \gamma_{II}\big)]$. Note that $\forall \gamma \in \mathcal{F}_\beta[\varphi(\ulcorner \gamma, \gamma \urcorner) > 0]$ and, for all n, if $\beta(n) = 0$ and $\varphi | \ulcorner n, n \urcorner \perp \langle 0 \rangle$, then there exists exactly one $\gamma \in \mathcal{F}_\beta$ such that $n \sqsubset \gamma$. Find δ such that, for each n, if $\beta(n) = 0$ and $\varphi | \ulcorner n, n \urcorner \perp \langle 0 \rangle$, then $n \sqsubset \delta^n$ and $\delta^n \in \mathcal{F}_\beta$, and note that $\mathcal{F}_\beta \subseteq En_\delta$. We thus see that $\mathcal{F}_\beta$ is (at most) countable.

(ii) Define $\varphi \colon \omega^\omega \to \omega^\omega$ such that $\forall \alpha \forall s[(\varphi|\alpha)(s) = 0 \leftrightarrow \exists m \exists k[s = \overline{\alpha} m * \underline{\overline{0}} k]]$. We shall prove that φ reduces $\mathcal{FIN}$ to $\mathcal{COUNT}$. Note that for every α, $\mathrm{Spr}(\varphi|\alpha)$ and $\alpha \in \mathcal{F}_{\varphi|\alpha}$.

First, let α in $\mathcal{FIN}$ be given. Find m such that $\forall n \geq m[\alpha(n) = 0]$. Note that $\forall \gamma[\gamma \in \mathcal{F}_{\varphi|\alpha} \leftrightarrow \exists k \leq m[\gamma = \overline{\alpha} k * \underline{0}]]$. Define δ such that $\forall k \leq m[\delta^k = \overline{\alpha} k * \underline{0}]$. Note that $\mathcal{F}_{\varphi|\alpha} \subseteq En_\delta$ and $\varphi|\alpha \in \mathcal{COUNT}$. Clearly, for every α, if $\alpha \in \mathcal{FIN}$, then $\varphi|\alpha \in \mathcal{COUNT}$.

Now let α be given such that $\varphi|\alpha \in \mathcal{COUNT}$. According to (i): $\mathcal{F}_{\varphi|\alpha}$ is discrete. Note that $\alpha \in \mathcal{F}_{\varphi|\alpha}$. Using Brouwer's Continuity Principle **BCP**, see Section 1.1.6, find m such that $\forall \gamma \in \mathcal{F}_{\varphi|\alpha}[\overline{\alpha} m \sqsubset \gamma \to \alpha = \gamma]$. Conclude that $\forall n \geq m[\alpha(n) = 0]$ and $\alpha \in \mathcal{FIN}$. Clearly, for every α, if $\varphi|\alpha \in \mathcal{COUNT}$, then $\alpha \in \mathcal{FIN}$.

We thus see that φ reduces $\mathcal{FIN}$ to $\mathcal{COUNT}$.

(iii) Recall that we defined, for each α, $T_\alpha = \{t \mid \forall u \sqsubset t[\alpha(u) = 0]\}$. Define $\varphi \colon \omega^\omega \to \omega^\omega$ such that, for all α, for all s, $(\varphi|\alpha)(s) = 0$ if and only if $\exists t \in T_\alpha \exists k[s = t * \underline{\overline{0}} k]$. We shall prove that φ reduces $\mathcal{A}^1_1$ to $\mathcal{COUNT}$. Note that $\forall \alpha[\mathrm{Spr}(\varphi|\alpha)]$.

First, assume that $\alpha \in \mathcal{A}^1_1$. Let γ_0, γ_1 in $\mathcal{F}_{\varphi|\alpha}$ be given. Find $n_0 := \mu n[\alpha(\overline{\gamma_0} n) \neq 0]$ and $n_1 := \mu n[\alpha(\overline{\gamma_1} n) \neq 0]$. Note that $\overline{\gamma_0} n_0 \in T_\alpha$ and $\overline{\gamma_0}(n_0+1) \notin T_\alpha$ and $\gamma_0 = \overline{\gamma_0} n_0 * \underline{0}$. Similarly, $\gamma_1 = \overline{\gamma_1} n_1 * \underline{0}$. *If* $\overline{\gamma_0} n_0 \perp \overline{\gamma_1} n_1$, then $\gamma_0 \# \gamma_1$ and, *if not*, then $\gamma_0 = \gamma_1$. We thus see that $\forall \gamma_0 \in \mathcal{F}_{\varphi|\alpha} \forall \gamma_1 \in \mathcal{F}_{\varphi|\alpha}[\gamma_0 \# \gamma_1 \vee \gamma_0 = \gamma_1]$, ie $\mathcal{F}_{\varphi|\alpha}$ is discrete. Using (i), conclude that $\varphi|\alpha \in \mathcal{COUNT}$. Clearly, for each α, if $\alpha \in \mathcal{A}^1_1$, then $\varphi|\alpha \in \mathcal{COUNT}$.

Now let α be given such that $\varphi|\alpha \in \mathcal{COUNT}$. Let γ be given. Define γ^* such that, for each n, if $\overline{\gamma}(n+1) \in T_\alpha$, then $\gamma^*(n) = \gamma(n)$; and, if not, then $\gamma^*(n) = 0$. Note that $\gamma^* \in \mathcal{F}_{\varphi|\alpha}$. According to (i), $\mathcal{F}_{\varphi|\alpha}$ is discrete. Using Brouwer's Continuity

Principle **BCP**, find n such that $\forall\delta \in \mathcal{F}_{\varphi|\alpha}[\overline{\gamma^*}n \sqsubset \delta \rightarrow \gamma^* = \delta]$. Suppose that $\forall m \leq n[\alpha(\overline{\gamma^*}m) = 0]$. Then $\forall p[\overline{\gamma^*}n * \langle p\rangle \in T_\alpha$ and $(\varphi|\alpha)(\overline{\gamma^*}n * \langle p\rangle) = 0]$. Conclude that $\exists m \leq n[\alpha(\overline{\gamma^*}m) \neq 0]$, and $\exists m \leq n[\alpha(\overline{\gamma}m) \neq 0]$. We thus see that $\forall\gamma\exists m[\alpha(\overline{\gamma}m) \neq 0]$, ie $\alpha \in \mathcal{A}^1_1$. Clearly, for each α, if $\varphi|\alpha \in \mathcal{COUNT}$, then $\alpha \in \mathcal{A}^1_1$.

We thus see that φ reduces $\mathcal{A}^1_1$ to $\mathcal{COUNT}$.

(iv) As $\mathcal{FIN}$ reduces to $\mathcal{COUNT}$, see (ii), and $\mathcal{FIN}$ is not $\mathbf{\Pi}^1_1$, see Theorem 4.3(iii), also $\mathcal{COUNT}$ is not $\mathbf{\Pi}^1_1$.

Note, considering the proof of (i): for all β, $\beta \in \mathcal{COUNT}$ if and only if $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ is discrete, ie $\forall\gamma \in \mathcal{F}_\beta\exists n\forall s\forall t[(\beta(s) = \beta(t) = 0 \wedge \overline{\gamma}n \sqsubseteq s \wedge \overline{\gamma}n \sqsubseteq t) \rightarrow (s \sqsubseteq t \vee t \sqsubseteq s)]$. Conclude, using the last observation of Section 1.1.5, that for all β, $\beta \in \mathcal{COUNT}$ if and only if $\mathrm{Spr}(\beta)$ and $\forall\gamma\exists n\forall s\forall t[(\beta(s) = \beta(t) = 0 \wedge \overline{\gamma}n \sqsubseteq s \wedge \overline{\gamma}n \sqsubseteq t) \rightarrow (s \sqsubseteq t \vee t \sqsubseteq s)]$. Let $\mathcal{X}$ be the set of all β such that $\mathrm{Spr}(\beta_I)$ and either $\exists n[\beta_I(\overline{\beta_{II}}n) \neq 0]$ or $\exists n\forall s\forall t[(\beta_I(s) = \beta_I(t) = 0 \wedge \overline{\beta_{II}}n \sqsubseteq s \wedge \overline{\beta_{II}}n \sqsubseteq t) \rightarrow (s \sqsubseteq t \vee t \sqsubseteq s)]$ and note that $\mathcal{X} \in \mathbf{\Pi}^0_3$ and $\mathcal{COUNT} = \mathrm{Un}(\mathcal{X})$ and $\mathcal{COUNT}$ is $\mathbf{\Pi}^{1+}_1$. □

6.2 Almost-countable spreads

One might feel that the notion of a *countable spread* as introduced in Section 6.1 is perhaps too strong. We therefore introduce a weaker notion.

Note that for each δ, for each γ, if $\forall n[\gamma \# \delta^n]$, one may define α such that, for each n, $\alpha(n) = \mu(p)[\overline{\gamma}p \perp \delta^n]$. Conclude that $\forall n[\gamma \# \delta^n]$ if and only if $\exists\alpha\forall n[\overline{\gamma}\alpha(n) \perp \delta^n]$. One may consider α such that $\forall n[\overline{\gamma}\alpha(n) \perp \delta^n]$ as *evidence* for the fact that $\forall n[\gamma \# \delta^n]$.

Definition 26 *For all γ, δ, we define:* γ almost belongs to $En_\delta = \{\delta^n \mid n \in \omega\}$ *if and only if $\forall\alpha\exists n[\overline{\gamma}\alpha(n) \sqsubset \delta^n]$.*

So γ almost belongs to En_δ if every attempt to give evidence that γ is apart from every element of En_δ fails in finitely many steps.

Lemma 6.2

(i) *For all $\gamma, \delta, \varepsilon$, if $En_\delta \subseteq En_\varepsilon$ and γ almost belongs to En_δ, then γ almost belongs to En_ε.*

(ii) *For all $\gamma, \delta, \varepsilon$, if $En_\delta = En_\varepsilon$, then γ almost belongs to En_δ if and only if γ almost belongs to En_ε.*

Proof (i) Let δ, ε be given such that $En_\delta \subseteq En_\varepsilon$. Let γ be given such that $\forall \alpha \exists n[\overline{\gamma}\alpha(n) \sqsubset \delta^n]$. Using the First Axiom of Countable Choice $\mathbf{AC}_{0,0}$, see Section 1.1.3, find ζ such that $\forall n[\delta^n = \varepsilon^{\zeta(n)}]$. Let α be given. Find n such that $\overline{\gamma}\alpha \circ \zeta(n) \sqsubset \delta^n = \varepsilon^{\zeta(n)}$ and conclude that $\exists m[\overline{\gamma}\alpha(m) \sqsubset \varepsilon^m]$. Conclude that $\forall \alpha \exists n[\overline{\gamma}\alpha(n) \sqsubset \varepsilon^n]$.

(ii) immediately follows from (i). $\square$

Define δ such that $\forall n[\delta^n = n * \underline{0}]$ and note that $\mathcal{FIN} = \{\delta^n \mid n \in \omega\} = En_\delta$. Recall that $\mathcal{ALMOST}^*\mathcal{FIN} = \{\gamma \mid \forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) = 0]\}$; see Definition 17.

Lemma 6.3 *For each γ, $\gamma \in \mathcal{ALMOST}^*\mathcal{FIN}$ if and only if γ almost belongs to $\mathcal{FIN}$.*

Proof Let γ in $\mathcal{ALMOST}^*\mathcal{FIN}$ be given. We want to prove that γ almost belongs to $\mathcal{FIN} = \{n * \underline{0} \mid n \in \omega\}$. Let α be given. We want to prove: $\exists n[\overline{\gamma}\alpha(n) \sqsubset n * \underline{0}]$. To this end, we define ζ in $[\omega]^\omega$, step by step. If $\overline{\gamma}\alpha(0) \perp \underline{0}$, define $\zeta(0) = \mu i < \alpha(m)[\gamma(i) \neq 0]$; and, if not, define $\zeta(0) = 0$. Now assume $p > 0$ and we defined $\zeta(0), \zeta(1), \dots, \zeta(p-1)$. Define $m := \overline{\gamma}\big(\zeta(p-1)+1\big)$. *If* $\overline{\gamma}\alpha(m) \perp m * \underline{0}$, ie $\overline{\gamma}\alpha(\overline{\gamma}\big(\zeta(p-1)+1\big)) \perp \overline{\gamma}\big(\zeta(p-1)+1\big) * \underline{0}$, define $\zeta(p) = \mu i < \alpha(m)[i > \zeta(p-1) \ \wedge \gamma(i) \neq 0]$, and, *if not*, define $\zeta(p) = \zeta(p-1)+1$. Now find n such that $\gamma \circ \zeta(n) = 0$ and conclude that for some $p \leq n$ we must have seen $\overline{\gamma}\alpha(m) \sqsubset m * \underline{0}$, where $m = \overline{\gamma}\big(\zeta(p-1)+1\big)$. We thus see that γ almost belongs to $\mathcal{FIN}$.

Conversely, let γ be given such that γ almost belongs to $\mathcal{FIN}$, ie $\forall \alpha \exists n[\overline{\gamma}\alpha(n) \sqsubset n * \underline{0}]$. Assume that $\zeta \in [\omega]^\omega$. Find η in $[\omega]^\omega$ such that $\forall n[\zeta \circ \eta(n) > \mathit{length}(n)]$. Define α such that, for each n, $\alpha(n) = \zeta \circ \eta(n) + 1$. Find n such that $\overline{\gamma}\alpha(n) \sqsubset n * \underline{0}$ and conclude that $\gamma \circ \zeta \circ \eta(n) = 0$. We thus see that $\forall \zeta \in [\omega]^\omega \exists n[\gamma \circ \zeta(n) = 0]$, ie $\gamma \in \mathcal{ALMOST}^*\mathcal{FIN}$. $\square$

Definition 27 *For each δ, we let $\mathcal{ALMOST}^*(En_\delta)$ be the set of all γ that almost belong to En_δ, ie such that $\forall \alpha \exists n[\overline{\gamma}\alpha(n) \sqsubset \delta^n]$. We also define:*

$$\mathcal{ALMOST}^*\mathcal{COUNT} := \{\beta \mid \mathrm{Spr}(\beta) \ \wedge \ \exists \delta[\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\delta)]\}$$

If $\beta \in \mathcal{ALMOST}^*\mathcal{COUNT}$, we call $\mathcal{F}_\beta$ an *almost-countable spread.*

Lemma 6.4 *For each β, if $\mathcal{F}_\beta$ is an inhabited almost-countable spread, then there exists ε in $(\mathcal{F}_\beta)^\omega$ such that $\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\varepsilon)$.*

Proof Let β, δ be given such that $\mathrm{Spr}(\beta)$ and $\beta(0) = 0$ and $\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\delta)$. Let ρ be the retraction of ω^ω onto $\mathcal{F}_\beta$. Define ε such that $\forall n[\varepsilon^n = \rho|\delta^n]$ and note that $\forall n[\varepsilon^n \in \mathcal{F}_\beta]$. We now prove: $\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\varepsilon)$. Assume $\gamma \in \mathcal{F}_\beta$ and let α be given. Find n such that $\overline{\gamma}\alpha(n) \sqsubset \delta^n$. Conclude that $\beta\big(\overline{\gamma}\alpha(n)\big) = 0$ and $\overline{\gamma}\alpha(n) \sqsubset \varepsilon^n$. We thus see that $\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\varepsilon)$. $\square$

Lemma 6.5 *If $\mathcal{F}, \mathcal{H}$ are inhabited spreads and $\mathcal{F}$ maps onto $\mathcal{H}$ and $\mathcal{F}$ is almost-countable, also $\mathcal{H}$ is almost-countable.*

Proof Let β_0 and β_1 be spread-laws such that $\beta_0(0) = \beta_1(0) = 0$ and$\mathcal{F}_{\beta_0}$ is almost-countable. Assume $\varphi\colon \mathcal{F}_{\beta_0} \to \mathcal{F}_{\beta_1}$ is surjective. Using Lemma 6.4, find δ in $(\mathcal{F}_{\beta_0})^\omega$ such that $\mathcal{F}_{\beta_0} \subseteq \mathcal{ALMOST}^*(En_\delta)$. Define ε such that $\forall n[\varepsilon^n = \varphi|\delta^n]$. Assume that $\zeta \in \mathcal{F}_{\beta_1}$ and find γ in $\mathcal{F}_{\beta_0}$ such that $\varphi|\gamma = \zeta$. Let α be given. Find η such that $\forall n[\overline{\varepsilon^n}\alpha(n) \sqsubseteq \varphi|\overline{\delta^n}\eta(n)]$. Find n such that $\overline{\gamma}\eta(n) \sqsubset \delta^n$ and conclude that $\overline{\zeta}\alpha(n) = \overline{\varphi|\gamma}\alpha(n) \sqsubset \varphi|\delta^n = \varepsilon^n$. We thus see that $\forall \zeta \in \mathcal{F}_{\beta_1} \forall \alpha \exists n[\overline{\zeta}\alpha(n) \sqsubset \varepsilon^n]$, ie $\mathcal{F}_{\beta^1} \subseteq \mathcal{ALMOST}^*(En_\varepsilon)$ and $\mathcal{F}_{\beta_1}$ is almost-countable. $\square$

Theorem 6.6

(i) *For each β such that* $\mathrm{Spr}(\beta)$, *$\mathcal{F}_\beta$ is a countable spread if and only if $\mathcal{F}_\beta$ embeds into $\mathcal{FIN}$.*

(ii) *For each β such that* $\mathrm{Spr}(\beta)$, *if $\mathcal{F}_\beta$ is an almost-countable spread, then $\mathcal{F}_\beta$ embeds into $\mathcal{ALMOST}^*\mathcal{FIN}$.*

Proof (i) Assume that $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ is an inhabited countable spread. Find δ in $(\mathcal{F}_\beta)^\omega$ such that $\mathcal{F}_\beta = En_\delta$, ie $\forall \gamma \in \mathcal{F}_\beta \exists n[\gamma = \delta^n]$. Using the First Axiom of Continuous Choice $\mathbf{AC}_{1,0}$, see Section 1.1.6, find $\varphi\colon \mathcal{F}_\beta \to \omega$ such that $\forall \gamma \in \mathcal{F}_\beta[\gamma = \delta^{\varphi(\gamma)}]$. Define $\psi\colon \mathcal{F}_\beta \to \omega^\omega$ such that $\forall \gamma \in \mathcal{F}_\beta[\psi|\gamma = \overline{\underline{1}}\varphi(\gamma) * \underline{0}]$ and note that $\psi\colon \mathcal{F}_\beta \rightarrowtail \mathcal{FIN}$.

Conversely, assume that $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ embeds into $\mathcal{FIN}$. Find φ such that $\varphi\colon \mathcal{F}_\beta \rightarrowtail \mathcal{FIN}$. Note that $\mathcal{FIN}$ is discrete, ie for all δ_0, δ_1 in $\mathcal{FIN}$, either $\delta_0 = \delta_1$ or $\delta_0 \# \delta_1$. Conclude that for all γ_0, γ_1 in $\mathcal{F}_\beta$, either $\varphi|\gamma_0 = \varphi|\gamma_1$ or $\varphi|\gamma_0 \# \varphi|\gamma_1$, and, therefore, either $\gamma_0 = \gamma_1$ or $\gamma_0 \# \gamma_1$, ie $\mathcal{F}_\beta$ is discrete. Using Theorem 6.1(i), conclude that $\mathcal{F}_\beta$ is a countable spread.

(ii) Assume that $\mathrm{Spr}(\beta)$ and $\mathcal{F}_\beta$ is an inhabited almost-countable spread. Using Lemma 6.4, find δ in $(\mathcal{F}_\beta)^\omega$ such that $\mathcal{F}_\beta = \mathcal{ALMOST}^*(En_\delta)$.

We first prove the following observation: for all s such that $\beta(s) = 0$ there exists n such that $s \sqsubset \delta^n$. Let s be given such that $\beta(s) = 0$. Find γ in $\mathcal{F}_\beta$ such that $s \sqsubset \gamma$. Then find n such that $\overline{\gamma}length(s) \sqsubset \delta^n$ and conclude that $s \sqsubset \delta^n$.

Now define $\varphi\colon \mathcal{F}_\beta \to \omega^\omega$ such that, for all γ in $\mathcal{F}_\beta$, for all n, if $\mu p[\overline{\gamma}n \sqsubset \delta^p] < \mu p[\overline{\gamma}(n+1) \sqsubset \delta^p]]$, then $(\varphi|\gamma)(n) = \mu p[\overline{\gamma}(n+1) \sqsubset \delta^p]$; and, if $\mu p[\overline{\gamma}n \sqsubset \delta^p] = \mu p[\overline{\gamma}(n+1) \sqsubset \delta^p]]$, then $(\varphi|\gamma)(n) = 0$. We prove that φ is a strongly injective function from $\mathcal{F}_\beta$ into ω^ω. Let γ_0, γ_1 in $\mathcal{F}_\beta$ be given such that $\gamma_0 \mathbin{\#} \gamma_1$. Find n such that $\overline{\gamma_0}n \neq \overline{\gamma_1}n$. Note that $\mu p[\overline{\gamma_0}n \sqsubset \delta^p] \neq \mu p[\overline{\gamma_1}n \sqsubset \delta^p]$. Conclude that $\exists i \le n[(\varphi|\gamma_0)(i) \neq (\varphi|\gamma_1)(i)]$ and $\varphi|\gamma_0 \mathbin{\#} \varphi|\gamma_1$.

We prove that φ maps $\mathcal{F}_\beta$ into $\mathcal{ALMOST}^*\mathcal{FIN}$. Let γ in $\mathcal{F}_\beta$ be given and consider $\varphi|\gamma$. Let ζ in $[\omega]^\omega$ be given. Find n such that $\overline{\gamma}\big(\zeta(n)+1\big) \sqsubset \delta^n$. Assume that $\forall i \le n[(\varphi|\gamma)\big(\zeta(i)\big) \neq 0]$. Conclude that $\forall i < n[0 < (\varphi|\gamma)\big(\zeta(i)\big) < (\varphi|\gamma)\big(\zeta(i+1)\big)]$, and $(\varphi|\gamma)(\zeta(n)\big) \ge n+1$. Conclude that $\mu p[\overline{\gamma}\big(\zeta(n)+1\big) \sqsubset \delta^p] \ge n+1$ and also that $\overline{\gamma}\big(\zeta(n)+1\big) \sqsubset \delta^n$. This is a contradiction. Conclude that $\exists i \le n[(\varphi|\gamma)\big(\zeta(i)\big) = 0]$. Clearly, $\forall \zeta \in [\omega]^\omega \exists i[(\varphi|\gamma)\big(\zeta(i)\big) = 0]$, ie $\varphi|\gamma \in \mathcal{ALMOST}^*\mathcal{FIN}$. $\square$

For the converse of Theorem 6.6(ii), see Corollary 6.14.

6.3 Cantor–Bendixson sets

Definition 28 *Let ε, β be given. We define $\nu = \mathsf{CB}(\varepsilon, \beta)$ in 2^ω as follows. For each s, $\nu(s) = 0$ if and only if either $s \sqsubset \varepsilon$ or there exist m, n, t such that $s = \overline{\varepsilon}m * \langle n\rangle * t$ and $\varepsilon(m) \neq n$ and $\beta^{J(m,n)}(t) = 0$.*

Lemma 6.7 *Let ε, β be given.*

(i) *If, for all n, $\mathrm{Spr}(\beta^n)$, then $Spr\big(\mathsf{CB}(\varepsilon,\beta)\big)$.*

(ii) *If, for all n, $\beta^n \in \mathcal{ALMOST}^*\mathcal{COUNT}$, then $\mathsf{CB}(\varepsilon,\beta) \in \mathcal{ALMOST}^*\mathcal{COUNT}$.*

Proof (i) The proof is straightforward and left to the reader. If, for all n, $\mathrm{Spr}(\beta^n)$, and $\nu = \mathsf{CB}(\varepsilon, \beta)$, we call ε the *spine* of the spread $\mathcal{F}_\nu$.

(ii) Assume that for all n, $\beta^n \in \mathcal{ALMOST}^*\mathcal{COUNT}$. Using the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, see Section 1.1.3, find δ such that, for all n, $\mathcal{F}_{\beta^n} \subseteq \mathcal{ALMOST}^*(En_{\delta^n})$. Define η such that $\eta^0 = \varepsilon$ and, for all m, n, p, if $\varepsilon(m) \neq n$, then $\eta^{J(J(m,n),p)+1} = \overline{\varepsilon}m * \langle n\rangle * \delta^{J(m,n),p}$.

Define $\nu := \mathsf{CB}(\varepsilon, \beta)$. We prove that $\mathcal{F}_\nu$ is a subset of $\mathcal{ALMOST}^*(En_\eta)$. Assume that $\gamma \in \mathcal{F}_\nu$. Let α be given. We want to prove: $\exists n[\overline{\gamma}\alpha(n) \sqsubset \eta^n]$. If $\overline{\gamma}\alpha(0) \sqsubset \eta^o = \varepsilon$, we are done. Now assume that $\overline{\gamma}\alpha(0) \perp \eta^0 = \varepsilon$. Define $m := \mu p[\gamma(p) \neq \varepsilon(p)]$ and $n := \gamma(m)$. Define $k := J(m,n)$ and $s := \overline{\varepsilon}m * \langle n\rangle$. Note that $s \sqsubset \gamma$ and find μ such that $\gamma = s * \mu$. Note that $\mu \in \mathcal{F}_{\beta^k}$. Find p such that $\overline{\mu}\big(\alpha(J(k,p)+1)\big) \sqsubset$

$\delta^{k,p}$. Conclude that $\overline{\gamma}\alpha\big(J(k,p)+1\big) \sqsubset s * \overline{\mu}\alpha(\big(J(k,p)+1\big) \sqsubset s * \delta^{k,p} = \eta^{J(k,p)+1}$. Conclude that $\forall\alpha\exists n[\overline{\gamma}\alpha(n) \sqsubset \eta^n]$, ie $\gamma \in \mathcal{ALMOST}^*(En_\eta)$. We thus see that $\mathcal{F}_\nu \subseteq \mathcal{ALMOST}^*(En_\eta)$ and $\nu = \mathsf{CB}(\varepsilon,\beta) \in \mathcal{ALMOST}^*\mathcal{COUNT}$. $\square$

Definition 29 *We introduce a subset $\mathcal{CB}$ of ω^ω by means of the following inductive definition.*

(i) *For all β, if $\mathrm{Spr}(\beta)$ and $\beta(0) \neq 0$ (so $\mathcal{F}_\beta = \emptyset$) then $\beta \in \mathcal{CB}$.*
(ii) *For all ε, for all β, if, for all n, $\beta^n \in \mathcal{CB}$, then $\mathsf{CB}(\varepsilon,\beta) \in \mathcal{CB}$.*
(iii) *All members of $\mathcal{CB}$ are given by (i), (ii).*

The following theorem may be compared to Cantor's result [8, Theorem C] in [9, page 220], and to a related intuitionistic result: Veldman [38, Theorems 9.1 and 9.2].

Theorem 6.8 $\mathcal{ALMOST}^*\mathcal{COUNT} = \mathcal{CB}$.

Proof Using Lemma 6.7 and induction, we conclude that $\mathcal{CB} \subseteq \mathcal{ALMOST}^*\mathcal{COUNT}$.

We now prove that $\mathcal{ALMOST}^*\mathcal{COUNT}$ is a subset of $\mathcal{CB}$.

Let β in $\mathcal{ALMOST}^*\mathcal{COUNT}$ be given. One may assume that $\beta(0) = 0$. Using Lemma 6.4, find δ in $(\mathcal{F}_\beta)^\omega$ such that $\mathcal{F}_\beta \subseteq \mathcal{ALMOST}^*(En_\delta)$. Now define β^+ in 2^ω such that, for all c, $\beta^+(c) = 0$ if and only if $\forall i < \mathrm{length}(c)[\beta\big(c(i)\big) = 0] \wedge \big(i+1 < \mathrm{length}(c) \to c(i) \sqsubset c(i+1)\big)]$. Note that $\mathrm{Spr}(\beta^+)$. Define $B := \{c \mid \exists i < \mathrm{length}(c)[c(i) \sqsubset \delta^i]\}$. We now prove: B is a bar in $\mathcal{F}_{\beta^+}$. Let γ in $\mathcal{F}_{\beta^+}$ be given. Find ζ in $\mathcal{F}_\beta$ such that $\forall n[\gamma(n) \sqsubset \zeta]$. Find α such that $\forall n[\gamma(n) = \overline{\zeta}\alpha(n)]$. Find n such that $\overline{\zeta}\alpha(n) \sqsubset \delta^n$ and, therefore: $\gamma(n) \sqsubset \delta^n$ and $\overline{\gamma}(n+1) \in B$. We thus see that $\mathrm{Bar}_{\mathcal{F}_{\beta^+}}(B)$. We define: $\widetilde{\langle\,\rangle} = \langle\,\rangle$, and, for each $n > 0$, for each c in ω^n, $\tilde{c} := c(n-1)$. Define $C := \bigcup_n \{c \in \omega^n \mid \beta^+(c) = 0 \wedge \big(\forall i < n[c(i) \perp \delta^i] \to {}^{\tilde{c}}\beta \in \mathcal{CB}]\big)\}$. Note that $B \subseteq C$ and C is monotone in $\{s \mid \beta^+(s) = 0\}$. Let c, n be given such that $c \in \omega^n$ and $\beta^+(c) = 0$ and $\forall t[\beta^+(c * \langle t\rangle) = 0 \to c * \langle t\rangle \in C]$. Assume that $\forall i < n[c(i) \perp \delta^i]$. Note that for all t, if $\tilde{c} \sqsubset t$ and $\beta(t) = 0$ and $t \perp \delta^n$, then $c * \langle t\rangle \in C$ and ${}^t\beta \in \mathcal{CB}$. Find ε such that $\tilde{c} * \varepsilon \in \mathcal{F}_\beta$, and, if $\tilde{c} \sqsubset \delta^n$, then $\delta^n = \tilde{c} * \varepsilon$. Define $\nu := {}^{\tilde{c}}\beta$ and note that $\varepsilon \in \mathcal{F}_\nu$ and, for all s, if $\nu(s) = 0$ and $\varepsilon \perp s$, then ${}^s\nu \in \mathcal{CB}$. In particular, for all m, n, s, if $s = \overline{\varepsilon}m * \langle n\rangle$ and $\varepsilon(m) \neq n$, then ${}^s\nu \in \mathcal{CB}$. Conclude that $\nu = {}^{\tilde{c}}\beta \in \mathcal{CB}$, and $c \in C$. We thus see that C is inductive in $\{s \mid \beta^+(s) = 0\}$. Using the Principle of Bar Induction **BI**, see Section 1.1.9, we conclude that $\langle\,\rangle \in C$, ie $\beta \in \mathcal{CB}$.

We thus see that $\mathcal{ALMOST}^*\mathcal{COUNT} \subseteq \mathcal{CB}$. $\square$

6.4 Reducible spreads

Definition 30 *For each σ in $\mathcal{STP}$, we define the collection $\mathcal{RED}_\sigma$ of codes of σ–reducible spreads, as follows, by induction.*

(i) $\mathcal{RED}_{1^*} = \mathcal{RED}_{\underline{1}} := \{\underline{1}\}$.

(ii) *For every $\sigma \neq 1^*$ in $\mathcal{STP}$, $\mathcal{RED}_\sigma$ is the set of all β in 2^ω such that* $\mathrm{Spr}(\beta)$ *and, for some ε in $\mathcal{F}_\beta$:*

$$\forall m \forall n \Big[\big(\varepsilon(n) \neq m \ \wedge \ \beta(\overline{\varepsilon}n * \langle m\rangle) = 0\big) \to \exists p[{}^{\overline{\varepsilon}n*\langle m\rangle}\beta \in \mathcal{RED}_{\sigma^p}]\Big]$$

We also define $\mathcal{RED} := \bigcup_{\sigma\in\mathcal{STP}} \mathcal{RED}_\sigma$.

If $\beta \in \mathcal{RED}_\sigma$, then $\mathcal{F}_\beta$ is called a *σ–reducible spread*. If $\beta \in \mathcal{RED}$, then $\mathcal{F}_\beta$ is called a *reducible spread*.

The notion of a reducible spread goes back to Cantor. We here introduce this notion without bringing up the operation of taking the derivative of a given $\mathcal{X} \subseteq \omega^\omega$. Cantor defined a closed set to be *reducible* if one, by repeating the operation of taking the derivative, if needed transfinitely many times, ends up with the empty set.

Note that, for all σ in $\mathcal{STP}$, for all β such that $\mathrm{Spr}(\beta)$, $\mathcal{F}_\beta$ is σ–reducible if and only if $s * \mathcal{F}_\beta$ is σ–reducible. Also note that, for all β_0, β_1 such that $\forall i < 2[\mathrm{Spr}(\beta_i)]$ and $\mathcal{F}_{\beta_0} \subseteq \mathcal{F}_{\beta_1}$, for all σ in $\mathcal{STP}$, if $\mathcal{F}_{\beta_1}$ is σ–reducible, then $\mathcal{F}_{\beta_0}$ is σ–reducible.

Theorem 6.9 *$\mathcal{CB} = \mathcal{RED}$.*

Proof We first prove that $\mathcal{CB} \subseteq \mathcal{RED}$, using induction on $\mathcal{CB}$.

(1) For all β, if $\mathrm{Spr}(\beta)$ and $\beta(0) \neq 0$, then $\mathcal{F}_\beta = \emptyset$ and $\beta \in \mathcal{RED}_{1^*}$.

(2) Let β, ε be given such that $\mathrm{Spr}(\beta)$ and $\varepsilon \in \mathcal{F}_\beta$ and $\forall n \exists \sigma \in \mathcal{STP}[\beta^n \in \mathcal{RED}_\sigma]$. Using $\mathbf{AC}_{0,1}$, find τ in $\mathcal{STP}$ such that $\tau(0) = 0$ and $\forall n[\beta^n \in \mathcal{RED}_{\tau^n}]$. Conclude that $\mathsf{CB}(\varepsilon, \beta) \in \mathcal{RED}_\tau$.

(3) Using induction on $\mathcal{CB}$, conclude that $\mathcal{CB} \subseteq \bigcup_{\sigma\in\mathcal{STP}} \mathcal{RED}_\sigma$.

We now prove that $\mathcal{RED} \subseteq \mathcal{CB}$, using induction on $\mathcal{STP}$.

(1) For all σ in $\mathcal{STP}$, for all β, if $\sigma(0) \neq 0$ and $\beta \in \mathcal{RED}_\sigma$, then $\mathcal{F}_\beta = \emptyset$ and $\beta \in \mathcal{CB}$.

(2) Let σ in $\mathcal{STP}$ be given such that $\sigma(0) = 0$ and $\forall n[\mathcal{RED}_{\sigma^n} \subseteq \mathcal{CB}]$.

Let β in $\mathcal{RED}_\sigma$ be given. Find ε in $\mathcal{F}_\beta$ such that $\forall s[(\beta(s) = 0 \ \wedge \ s \perp \varepsilon) \to \exists n[{}^s\beta \in \mathcal{RED}_{\sigma^n}]]$. Conclude $\forall s[(\beta(s) = 0 \ \wedge \ s \perp \varepsilon) \to {}^s\beta \in \mathcal{CB}]$, and $\beta \in \mathcal{CB}$. Define γ

such that, for all m,n, if $\varepsilon(m) \neq n$, then $\gamma^{J(m,n)} = {}^{\overline{\varepsilon}m*\langle n\rangle}\beta$ and, if $\varepsilon(m) = n$, then $\gamma^{J(m,n)} = \underline{1}$. Note that for all n, $\gamma^n \in \mathcal{CB}$ and $\beta = \mathsf{CB}(\varepsilon,\gamma) \in \mathcal{CB}$.

We thus see that $\mathcal{RED}_\sigma \subseteq \mathcal{CB}$.

(3) Using induction on $\mathcal{STP}$, we conclude that $\forall\sigma \in \mathcal{STP}[\mathcal{RED}_\sigma \subseteq \mathcal{CB}]$. □

6.5 Perhaps$_\sigma$–countable spread

In this section we will see that there are many notions of countability for spreads in between the notion of a countable spread, see Section 6.1, and the notion of an almost-countable spread, see Section 6.2.

Definition 31 *For each inhabited $\mathcal{X} \subseteq \omega^\omega$, for each σ in $\mathcal{STP}$, we define $\mathbb{P}(\sigma,\mathcal{X}) \subseteq \omega^\omega$, the σ–th perhapsive extension of $\mathcal{X}$, as follows, by induction. For every σ in $\mathcal{STP}$,*

(i) *if $\sigma(0) \neq 0$ then $\mathbb{P}(\sigma,\mathcal{X}) = \mathcal{X}$; and*
(ii) *if $\sigma(0) = 0$, then $\mathbb{P}(\sigma,\mathcal{X}) = \{\alpha \mid \exists\beta \in \mathcal{X}[\alpha \# \beta \rightarrow \exists n[\alpha \in \mathbb{P}(\sigma^n,\mathcal{X})]]\}$.*

In Veldman [36, Theorem 3.19], one may find the straightforward proof that, for all inhabited $\mathcal{X},\mathcal{Y} \subseteq \omega^\omega$, for all σ,τ in $\mathcal{STP}$, if $\mathcal{X} \subseteq \mathcal{Y}$ and $\sigma \leq \tau$, then $\mathbb{P}(\sigma,\mathcal{X}) \subseteq \mathbb{P}(\tau,\mathcal{Y})$.

Definition 32 *Let β,σ be given such that* $\mathrm{Spr}(\beta)$ *and $\sigma \in \mathcal{STP}$. The spread $\mathcal{F}_\beta$ is called* perhaps$_\sigma$–countable *if and only if $\exists\delta[\mathcal{F}_\beta \subseteq \mathbb{P}(\sigma, En_\delta)]$.*

The proof of the third item of the next theorem, Theorem 6.10, resembles the proof of '$\mathcal{ALMOST}^*\mathcal{COUNT} \subseteq \mathcal{CB}$'; see Theorem 6.8.

Theorem 6.10

(i) $\forall\delta[\mathcal{ALMOST}^*(En_\delta) = \bigcup_{\sigma\in\mathcal{STP}} \mathbb{P}(\sigma, En_\delta)]]$
(ii) $\mathcal{ALMOST}^*\mathcal{FIN} = \bigcup_{\sigma\in\mathcal{STP}} \mathbb{P}(\sigma,\mathcal{FIN})$
(iii) *For all β,δ,φ, if* $\mathrm{Spr}(\beta)$ *and $\varphi\colon \mathcal{F}_\beta \to \mathcal{ALMOST}^*(En_\delta)$, then $\exists\sigma \in \mathcal{STP}[\varphi\colon \mathcal{F}_\beta \to \mathbb{P}(\sigma, En_\delta)]$*
(iv) *For all β,φ, if* $\mathrm{Spr}(\beta)$ *and $\varphi\colon \mathcal{F}_\beta \to \mathcal{ALMOST}^*(\mathcal{FIN})$, then $\exists\sigma \in \mathcal{STP}[\varphi\colon \mathcal{F}_\beta \to \mathbb{P}(\sigma,\mathcal{FIN})]$*
(v) $\forall\beta \in \mathcal{CB}\exists\sigma \in \mathcal{STP}\exists\varphi[\varphi\colon \mathcal{F}_\beta \rightarrowtail \mathbb{P}(\sigma,\mathcal{FIN})]$

Proof (i) Let δ be given. We first prove $\bigcup_{\sigma\in\mathcal{STP}}\mathbb{P}(\sigma,En_\delta)\subseteq\mathcal{ALMOST}^*(En_\delta)$, using induction on $\mathcal{STP}$. First note that $\mathbb{P}(1^*,En_\delta)=En_\delta\subseteq\mathcal{ALMOST}^*(En_\delta)$. Now let σ in $\mathcal{STP}$ be given such that $\sigma\neq 1^*$ and $\forall n[\mathbb{P}(\sigma^n,En_\delta)\subseteq\mathcal{ALMOST}^*(En_\delta)]$. Assume that $\gamma\in\mathbb{P}(\sigma,En_\delta)$. Find n such that $\gamma\ \#\ \delta^n\to\exists m[\gamma\in\mathbb{P}(\sigma^m,En_\delta)]$. Let α be given and distinguish two cases. *Case (a)*: $\overline{\gamma}\alpha(n)\sqsubset\delta^n$. *Case (b)*: $\overline{\gamma}\alpha(n)\perp\delta^n$. Find m such that $\gamma\in\mathbb{P}(\sigma^m,En_\delta)$. Conclude that $\gamma\in\mathcal{ALMOST}^*(En_\delta)$ and $\exists p[\overline{\gamma}\alpha(p)\sqsubset\delta^p]$. We thus see, in both cases, that $\exists p[\overline{\gamma}\alpha(p)\sqsubset\delta^p]$. Conclude that $\forall\gamma\in\mathbb{P}(\sigma,En_\delta)\forall\alpha\exists p[\overline{\gamma}\alpha p\sqsubset\delta^p]$, ie $\mathbb{P}(\sigma,En_\delta)\subseteq\mathcal{ALMOST}^*(En_\delta)$. Using induction on $\mathcal{STP}$, conclude that $\bigcup_{\sigma\in\mathcal{STP}}\mathbb{P}(\sigma,En_\delta)\subseteq\mathcal{ALMOST}^*(En_\delta)$.

We now prove: $\mathcal{ALMOST}^*(En_\delta)\subseteq\bigcup_{\sigma\in\mathcal{STP}}\mathbb{P}(\sigma,En_\delta)$. Let γ in $\mathcal{ALMOST}^*(En_\delta)$ be given. Define $B:=\bigcup_p\{a\in\omega^p\mid\exists i<p[\overline{\gamma}a(i)\sqsubset\delta^i]\}$ and note that B is a bar in ω^ω. Define $C:=\bigcup_p\{a\in\omega^p\mid\forall i<p[\overline{\gamma}a(i)\perp\delta^i]\to\exists\sigma\in\mathcal{STP}[\gamma\in\mathbb{P}(\sigma,En_\delta)]\}$. Note that $B\subseteq C$ and C is monotone. We now prove that C is inductive. Let a be given such that $\forall n[a*\langle n\rangle\in C]$. Define $p:=\mathrm{length}(a)$. Assume $\forall i<p[\overline{\gamma}a(i)\perp\delta^i]$. Using the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, see Section 1.1.3, find τ in $\mathcal{STP}$ such that $\forall b[\overline{\gamma}b\perp\delta^p\to\gamma\in\mathbb{P}(\tau^b,En_\delta)]$. Conclude that if $\gamma\perp\delta^p$, then $\exists b[\gamma\in\mathbb{P}(\tau^b,En_\delta)]$, ie $\gamma\in\mathbb{P}(\tau,En_\delta)$. We thus see that if $\forall i<\mathrm{length}(a)[\overline{\gamma}a(i)\perp\delta^i]$, then $\exists\tau[\gamma\in\mathbb{P}(\tau,En_\delta)]$, ie $a\in C$. Conclude that C is inductive. Using the Principle of Bar Induction **BI** (see Section 1.1.9) we find that $\langle\ \rangle\in C$, ie $\exists\tau[\gamma\in\mathbb{P}(\tau,En_\delta)]$.

We thus see that $\mathcal{ALMOST}^*(En_\delta)\subseteq\bigcup_{\sigma\in\mathcal{STP}}\mathbb{P}(\sigma,En_\delta)$.

(ii) This follows from (i) and Lemma 6.3.

(iii) Let β,δ,φ be given such that $\mathrm{Spr}(\beta)$ and $\varphi\colon\mathcal{F}_\beta\to\mathcal{ALMOST}(En_\delta)$. Note that $\forall\gamma\in\mathcal{F}_\beta\forall\alpha\exists n[\overline{\varphi|\gamma}\alpha(n)\sqsubset\delta^n]$. Define β^+ such that, for each c, $\beta^+(c)=0$ if and only if $\forall i[i+1<\mathrm{length}(c)\to c(i)\sqsubset c(i+1)]$ and $\forall i<\mathrm{length}(c)[\beta(c_I(i))=0\ \wedge\ \mathit{length}\big(\varphi|c_I(i)\big)\geq c_{II}(i)]]$. Note that $\mathrm{Spr}(\beta^+)$. Define $B:=\bigcup_p\{c\in\omega^p\mid\exists i<p[\overline{\varphi|c_I(i)}c_{II}(i)\sqsubset\delta^i]\}$. We now prove that B is a bar in $\mathcal{F}_{\beta^+}$. Let γ in $\mathcal{F}_{\beta^+}$ be given. Find ζ in $\mathcal{F}_\beta$ such that $\forall n[\gamma_I(n)\sqsubset\zeta]$. Find n such that $\overline{\zeta}\gamma_{II}(n)\sqsubset\delta^n$ and, therefore: $\overline{\gamma}(n+1)\in B$. Conclude that $\mathrm{Bar}_{\mathcal{F}_{\beta^+}}(B)$.

For each c such that $\beta^+(c)=0$ we define $\tilde{c}$ as follows. $\tilde{0}=0$ and, for each c, for all n, if $n=\mathrm{length}(c)>0$, then $\tilde{c}:=c_I(n-1)$. Let C be the set of all c such that $\beta^+(c)=0$ and, if $\forall i<\mathit{length}(c)[\overline{\varphi|c_I(i)}c_{II}(i)\perp\delta^i]$, then $\exists\sigma\in\mathcal{STP}[\varphi\colon\mathcal{F}_\beta\cap\tilde{c}\to\mathbb{P}(\sigma,En_\delta)]$. Note that $B\subseteq C$ and C is monotone in $\{s\mid\beta^+(s)=0\}$. We now prove that C is inductive in $\{s\mid\beta^+(s)=0\}$. Let c be given such that $\beta^+(c)=0$ and $\forall t[\beta^+(c*\langle t\rangle)=0\to c*\langle t\rangle\in C]$. Find $n:=\mathrm{length}(c)$. Assume that $\forall i<n[\overline{\varphi|c_I(i)}c_{II}(i)\perp\delta^i]$. Note that $\forall t[(\beta^+(c*\langle t\rangle)=0\ \wedge\ \overline{\varphi|t_I}t_{II}(n)\perp\delta^n)\to\exists\sigma\in\mathcal{STP}[\varphi\colon\mathcal{F}_\beta\cap t_I(n)\to\mathbb{P}(\sigma,En_\delta)]]$. Using the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, see Section 1.1.3,

find τ in $\mathcal{STP}$ such that, for all t, if $\beta^+(c * \langle t \rangle) = 0$ and $\overline{\varphi|t_I} t_{II}(n) \perp \delta^n$, then $\varphi\colon\ \mathcal{F}_\beta \cap t_I(n) \to \mathbb{P}(\tau^t, En_\delta)$. Clearly, $\forall \gamma \in \mathcal{F}_\beta \cap \tilde{c}[\varphi|\gamma \mathbin{\#} \delta^n \to \exists t[\varphi|\gamma \in \mathbb{P}(\tau^t, En_\delta)]]$ and $\varphi\colon\ \mathcal{F}_\beta \cap \tilde{c} \to \mathbb{P}(\tau, En_\delta)$. We thus see that C is inductive in $\{s \mid \beta^+(s) = 0\}$.

Using the Principle of Bar Induction **BI** (see Section 1.1.9) we conclude that $\langle\ \rangle \in C$, ie $\exists \sigma \in \mathcal{STP}[\varphi\colon\ \mathcal{F}_\beta \to \mathbb{P}(\sigma, En_\delta)]$.

(iv) This is an immediate consequence of (iii), as $\exists \delta[\mathcal{FIN} = En_\delta]$.

(v) This follows from (iii) and Theorem 6.6(ii). □

6.6 Special and very special Cantor–Bendixson sets

Definition 33 *We define a function $\sigma \mapsto cb_\sigma$ from $\mathcal{STP}$ to ω^ω, as follows.*

(i) $cb_{1^*} = \underline{1}$

(ii) *For all $\sigma \neq 1^*$ in $\mathcal{STP}$, cb_σ satisfies $\forall m[cb_\sigma(\overline{0}m) = 0]$ and $\forall m \forall n \forall s[cb_\sigma(\overline{0}m * \langle n+1 \rangle * s) = cb_{\sigma^n}(s)]$.*

Note that if $\sigma \neq 1^*$, then $cb_\sigma = \mathsf{CB}(\underline{0}, \beta)$, where, for all m, n, $\beta^{J(m,n+1)} = cb_{\sigma^n}$.

We also define a function $\sigma \mapsto cb^\lozenge_\sigma$ from $\mathcal{STP}$ to ω^ω, as follows.

(i) $cb^\lozenge_{1^*} = \underline{1}$

(ii) *For all $\sigma \neq 1^*$ in $\mathcal{STP}$, $cb^\lozenge_\sigma$ satisfies $\forall m[cb^\lozenge_\sigma(\overline{0}m) = 0]$, $\forall m \forall s[cb^\lozenge_\sigma(\overline{0}m * \langle 1 \rangle * s) = cb^\lozenge_{\sigma^{L(m)}}(s)]$ and $\forall m \forall n \forall s[cb^\lozenge_\sigma(\overline{0}m * \langle n+2 \rangle * s) = 1]$.*

Note that if $\sigma \neq 1^*$, then $cb^\lozenge_\sigma = \mathsf{CB}(\underline{0}, \beta)$, where, for all m, $\beta^{J(m,1)} = cb^\lozenge_{\sigma^{L(m)}}$ and, for all m, n, $\beta^{J(m,n+2)} = \underline{1}$.

Note that for each σ in $\mathcal{STP}$, cb_σ is a spread-law and $cb^\lozenge_\sigma$ is a fan-law and $\mathcal{F}^\lozenge_{cb_\sigma} \subseteq 2^\omega$. Note that for each σ in $\mathcal{STP}$, for each n, $\mathcal{F}_{cb_\sigma}$ embeds into $\mathcal{F}_{cb_\sigma} \cap \underline{\overline{0}}n$, and $\mathcal{F}_{cb^\lozenge_\sigma}$ embeds into $\mathcal{F}^\lozenge_{cb_\sigma} \cap \underline{\overline{0}}n$.

The sets $\mathcal{F}_{cb_\sigma}$, where $\sigma \in \mathcal{STP}$, are called *special Cantor–Bendixson sets*. The sets $\mathcal{F}^\lozenge_{cb_\sigma}$, where $\sigma \in \mathcal{STP}$, are called *very special Cantor–Bendixson sets*. The latter sets occur in Veldman [33] and [36].

Lemma 6.11

(i) *For all σ in $\mathcal{STP}$, $\mathcal{F}_{cb_\sigma}$ and $\mathcal{F}_{cb^\lozenge_\sigma}$ are subsets of $\mathcal{ALMOST}^*\mathcal{FIN}$.*

(ii) *For all σ in $\mathcal{STP}$, $\mathcal{F}_{cb_\sigma}$ embeds into $\mathcal{F}_{cb^\lozenge_\sigma}$.*

Proof We use induction on $\mathcal{STP}$. Note that $\mathcal{F}_{cb_{1^*}} = \mathcal{F}_{cb^{\Diamond}_{1^*}} = \emptyset \subseteq \mathcal{ALMOST}^*\mathcal{FIN}$. Let $\sigma \neq 1^*$ be given such that, for each n, $\mathcal{F}_{cb_{\sigma^n}}$ and $\mathcal{F}_{cb^{\Diamond}_{\sigma^n}}$ are subsets of $\mathcal{ALMOST}^*\mathcal{FIN}$. Note that, for each α in $\mathcal{F}_{cb_\sigma}$, if $\alpha \# \underline{0}$, then there exist m, n, β such that $\alpha = \overline{\underline{0}}m * \langle n+1\rangle * \beta$, $\beta \in \mathcal{ALMOST}^*\mathcal{FIN}$, and $\alpha \in \mathcal{ALMOST}^*\mathcal{FIN}$. Conclude that $\mathcal{F}_{cb_\sigma} \subseteq \mathcal{ALMOST}^*\mathcal{FIN}$. For similar reasons, $\mathcal{F}_{cb^{\Diamond}_\sigma} \subseteq \mathcal{ALMOST}^*\mathcal{FIN}$.

(ii) We use induction on $\mathcal{STP}$. First note that $\mathcal{F}_{cb_{1^*}} = \mathcal{F}_{cb^{\Diamond}_{1^*}} = \emptyset$; so, for $\sigma = 1^*$ the statement is trivial. Let $\sigma \neq 1^*$ in $\mathcal{STP}$ be given such that, for all n, $\mathcal{F}_{cb_{\sigma^n}}$ embeds into $\mathcal{F}_{cb^{\Diamond}_{\sigma^n}}$. Using $\mathbf{AC}_{0,1}$, find φ such that, for all n, φ^n embeds $\mathcal{F}_{cb_{\sigma^n}}$ into $\mathcal{F}_{cb^{\Diamond}_{\sigma^n}}$. Define $\psi\colon \mathcal{F}_{cb_\sigma} \to \omega^\omega$ such that $\psi|\underline{0} = \underline{0}$ and for all m, n, for all α in $\mathcal{F}_{cb_{\sigma^n}}$, $\psi|\overline{\underline{0}}m * \langle n+1\rangle * \alpha = \overline{\underline{0}}J(n,m) * \langle 1\rangle * \varphi^n|\alpha$. Then ψ embeds $\mathcal{F}_{cb_\sigma}$ into $\mathcal{F}_{cb^{\Diamond}_\sigma}$. □

The proof of the following lemma does not use the Fan Theorem.

Lemma 6.12 (The Fan Theorem for very special Cantor–Bendixson sets) *For every σ in $\mathcal{STP}$, for every $B \subseteq \omega$, every bar in $\mathcal{F}_{cb^{\Diamond}_\sigma}$ has a finite subbar.*

Proof We use induction on $\mathcal{STP}$. Assume $\sigma \in \mathcal{STP}$. If $\sigma = 1^*$, there is nothing to prove. So assume $\sigma \neq 1^*$ and, for each n, every bar in $\mathcal{F}_{cb^{\Diamond}_{\sigma^n}}$ has a finite subbar. Now assume $B \subseteq \omega$ is a bar in $\mathcal{F}_{cb^{\Diamond}_\sigma}$. Find n such that $\overline{\underline{0}}n \in B$. Using the induction hypothesis, find finite subsets $B_0, B_1, \ldots, B_{n-1}$ of B such that, for each $i < n$, B_i is bar in $\mathcal{F}_{cb^{\Diamond}_\sigma} \cap \overline{\underline{0}}i * \langle 1\rangle$. Note that the finite set $\{\overline{\underline{0}}n\} \cup \bigcup_{i<n} B_i$ is bar in $\mathcal{F}_{cb_\sigma}$. □

The next theorem shows that every Cantor–Bendixson set is, in a certain sense, equinumerous to a special Cantor–Bendixson set.

Theorem 6.13 *For every Cantor–Bendixson set $\mathcal{F}$ there exists a special Cantor–Bendixson set $\mathcal{H}$ such that $\mathcal{H}$ maps onto $\mathcal{F}$ and $\mathcal{F}$ embeds into $\mathcal{H}$:*

$$\forall \beta \in \mathcal{CB} \exists \sigma \in \mathcal{STP}\left[\exists \varphi[\varphi\colon \mathcal{F}_{cb_\sigma} \twoheadrightarrow \mathcal{F}_\beta] \wedge \exists \psi[\psi\colon \mathcal{F}_\beta \rightarrowtail \mathcal{F}_{cb_\sigma}]\right]$$

Proof We use induction on $\mathcal{CB}$. If $\beta(0) \neq 0$, so $\mathcal{F}_\beta = \emptyset$, one may take $\sigma = 1^*$, as also $\mathcal{F}_{cb_\sigma} = \emptyset$. Now let β, ε be given such that $\mathrm{Spr}(\beta)$ and $\varepsilon \in \mathcal{F}_\beta$ and, for all m, n, s, if $\varepsilon(n) \neq m$ and $s = \overline{\varepsilon}n * \langle m\rangle$, then there exist σ in $\mathcal{STP}$ such that $\mathcal{F}_{cb_\sigma}$ maps onto $\mathcal{F}_{^s\beta}$ and $\mathcal{F}_{^s\beta}$ embeds into $\mathcal{F}_{cb_\sigma}$. Using the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, see Section 1.1.3, find τ, φ, ψ such that $1^* \neq \tau \in \mathcal{STP}$ and, for all m, n, s, if $\varepsilon(m) \neq n$ and $s = \overline{\varepsilon}m * \langle n\rangle$, then $\varphi^s\colon \mathcal{F}_{cb_{\tau^s}} \twoheadrightarrow \mathcal{F}_{^s\beta}$ and $\psi^s\colon \mathcal{F}_{^s\beta} \rightarrowtail \mathcal{F}_{cb_{\tau^s}}$. Define

$C := \{s \mid \beta(s) = 0 \ \wedge\ \exists m \exists n[s = \overline{\varepsilon}m * \langle n\rangle \ \wedge\ \varepsilon(m) \neq n]\}$. Define $\rho\colon \mathcal{F}_{cb_\tau} \to \omega^\omega$ such that $\rho|\underline{0} = \varepsilon$ and, for all s, if $s \in C$, then, for all γ in $\mathcal{F}_{cb_{\tau^s}}$, $\rho|(\overline{\underline{0}}s * \langle s+1\rangle * \gamma) = s * \varphi^s|\gamma$ and, for each δ in $\mathcal{F}_{cb_\tau}$, if there is no s in C such that $\overline{\underline{0}}s * \langle s+1\rangle \sqsubset \delta$, then $\rho|\delta = \varepsilon$. Clearly, ρ maps $\mathcal{F}_{cb_\tau}$ onto $\mathcal{F}_\beta$. Define $\chi\colon \mathcal{F}_\beta \to \omega^\omega$ such that $\chi|\varepsilon = \underline{0}$ and, for all s in C, for all $\gamma \in \mathcal{F}_{^s\beta}$, $\chi|(s * \gamma) = \overline{\underline{0}}s * \langle s+1\rangle * \psi^s|\gamma$. Clearly, χ embeds $\mathcal{F}_\beta$ into $\mathcal{F}_{cb_\tau}$. □

The following corollary proves the converse of Theorem 6.6(ii).

Corollary 6.14 *Every almost-countable spread embeds into* $\mathcal{ALMOST}^*\mathcal{FIN}$.

Proof Use Theorem 6.13 and Lemma 6.11(i). □

The next result, Theorem 6.15, gives a refinement of Theorem 6.13: every *finitary* Cantor–Bendixson set is, what one might call, equinumerous to a *very* special Cantor–Bendixson set.

Theorem 6.15 *For every Cantor–Bendixson set* $\mathcal{F}$ *that is a fan there exists a very special Cantor–Bendixson set* $\mathcal{H}$ *such that* $\mathcal{H}$ *maps onto* $\mathcal{F}$ *and* $\mathcal{F}$ *embeds into* $\mathcal{H}$*:*

$$\forall \beta \in \mathcal{CB}\left[\mathrm{Fan}(\beta) \to \exists \sigma \in \mathcal{STP}[\exists \varphi[\varphi\colon \mathcal{F}_{cb^\lozenge_\sigma} \twoheadrightarrow \mathcal{F}_\beta] \ \wedge\ \exists \psi[\psi\colon \mathcal{F}_\beta \rightarrowtail \mathcal{F}_{cb^\lozenge_\sigma}]\right]$$

Proof We use induction on $\mathcal{CB}$. If $\beta(0) \neq 0$, take $\sigma = 1^*$, and note that $\mathcal{F}_\beta = \mathcal{F}_{cb_{1^*}} = \emptyset$. Now let β, ε be given such that $\mathrm{Fan}(\beta)$ and $\varepsilon \in \mathcal{F}_\beta$ and for all m, n, s, if $\varepsilon(m) \neq n$ and $s = \overline{\varepsilon}m * \langle n\rangle$, then there exist σ in $\mathcal{STP}$ such that $\mathcal{F}^\lozenge_{cb_\sigma}$ maps onto $\mathcal{F}_{^s\beta}$ and $\mathcal{F}_{^s\beta}$ embeds into $\mathcal{F}^\lozenge_{cb_\sigma}$.

Using the Second Axiom of Countable Choice $\mathbf{AC}_{0,1}$, see Section 1.1.3, find τ, φ, ψ such that $1^* \neq \tau \in \mathcal{STP}$ and, for all m, n, s, if $\varepsilon(m) \neq n$ and $s = \overline{\varepsilon}m * \langle n\rangle$, then $\varphi^s\colon \mathcal{F}_{cb_{\tau^s}} \twoheadrightarrow \mathcal{F}^\lozenge_{^s\beta}$ and $\psi^s\colon \mathcal{F}_{^s\beta} \rightarrowtail \mathcal{F}^\lozenge_{cb_{\tau^s}}$. Define $C := \{s \mid \beta(s) = 0 \ \wedge\ \exists m \exists n[s = \overline{\varepsilon}m * \langle n\rangle \ \wedge\ \varepsilon(m) \neq n]\}$. Note that $\mathrm{Fan}(\beta)$, and thus: $\forall m \exists p \forall s \geq p[s \in C \to \mathrm{length}(s) \geq m]$. Using the First Axiom of Countable Choice $\mathbf{AC}_{0,0}$, see Section 1.1.3, find ζ such that $\forall m \forall s \geq \zeta(m)[s \in C \to \mathrm{length}(s) \geq m]$. Define $\rho : \mathcal{F}^\lozenge_{cb_\tau} \to \omega^\omega$ such that $\rho|\underline{0} = \varepsilon$ and, for all s in C, for all $\gamma \in \mathcal{F}^\lozenge_{cb_{\tau^n}}$, $\rho|(\overline{\underline{0}}J(s,0) * \langle 1\rangle * \gamma) = s * \varphi^s|\gamma$ and, for all δ in $\mathcal{F}_{cb_\tau}$, if there is no s in C such that $\overline{\underline{0}}J(s,0) * \langle 1\rangle \sqsubset \delta$, then $\rho|\delta = \varepsilon$. Note that ρ is well-defined and $\forall m \forall \gamma \in \mathcal{F}_{cb_\tau}[\overline{\underline{0}}J\big(\zeta(m), 0\big) \sqsubset \gamma \to \overline{\varepsilon}m \sqsubset \rho|\gamma]$. Clearly, $\rho : \mathcal{F}_{cb_\tau} \twoheadrightarrow \mathcal{F}_\beta$.

Define $\chi : \mathcal{F}_\beta \to \omega^\omega$ such that $\chi|\varepsilon = \underline{0}$ and, for all s in C, for all γ in $\mathcal{F}_{^s\beta}$, $\chi|(s * \gamma) = \overline{\underline{0}}J(s,0) * \langle 1\rangle * \psi^s|\gamma$. Clearly, $\chi : \mathcal{F}_\beta \rightarrowtail \mathcal{F}_{cb_\tau}$. □

Corollary 6.16 *Let β be given such that* $\mathrm{Spr}(\beta)$.

$\mathcal{F}_\beta$ *is almost-countable if and only if* $\exists\sigma \in \mathcal{STP}\exists\varphi[\varphi\colon\ \mathcal{F}_{cb_\sigma} \twoheadrightarrow \mathcal{F}_\beta]$.

Proof Use Theorems 6.8 and 6.13 and Lemma 6.5. □

The second item of the following Theorem seems to be of some interest in itself. It is an extension of Theorem 2.7(iii).

Theorem 6.17

(i) *For all β, if* $\forall i < 2[\mathrm{Spr}(\beta^i)]$ *and* $\exists\varphi[\varphi\colon\ \mathcal{F}_{\beta^0} \twoheadrightarrow \mathcal{F}_{\beta^1}]$, *then* $\exists\psi[\psi\colon\ \mathcal{F}_{\beta^1} \rightarrowtail \mathcal{F}_{\beta^0}]$.

(ii) *For all β, if* $\mathrm{Spr}(\beta^0)$ *and* $\mathrm{Fan}(\beta^1)$ *and* $\exists\psi[\psi\colon\ \mathcal{F}_{\beta^1} \rightarrowtail \mathcal{F}_{\beta^0}]$, *then* $\exists\varphi[\varphi\colon\ \mathcal{F}_{\beta^0} \twoheadrightarrow \mathcal{F}_{\beta^1}]$.

Proof (i) Let β, φ be given such that $\varphi\colon\ \mathcal{F}_{\beta^0} \twoheadrightarrow \mathcal{F}_{\beta^1}$, and, therefore $\forall\gamma \in \mathcal{F}_{\beta^1}\exists\alpha \in \mathcal{F}_{\beta^0}[\varphi|\alpha = \gamma]$. Using the Second Axiom of Continuous Choice $\mathbf{AC}_{1,1}$, see Section 1.1.6, find $\psi\colon\ \mathcal{F}_{\beta^1} \to \mathcal{F}_{\beta^0}$ such that $\forall\gamma \in \mathcal{F}_{\beta^1}[\varphi|(\psi|\gamma) = \gamma]$. We prove that ψ is strongly injective. Let γ, δ in $\mathcal{F}_{\beta^1}$ be given such that $\gamma \mathrel{\#} \delta$. Find n such that $\overline{\gamma}n \perp \delta$. Find m such that $\forall\alpha \in \mathcal{F}_{\beta^0}[\overline{\psi|\gamma}m = \overline{\alpha}m \to \overline{\varphi|(\psi|\gamma)}n = \overline{\varphi|\alpha}n]$. Consider $\alpha := \psi|\delta$ and conclude that $\overline{\psi|\gamma}m \neq \overline{\psi|\delta}m$. We thus see that $\forall\gamma \in \mathcal{F}_{\beta^1}\forall\delta \in \mathcal{F}_{\beta^1}[\gamma \mathrel{\#} \delta \to \psi|\gamma \mathrel{\#} \psi|\delta]$, ie $\psi : \mathcal{F}_{\beta^1} \rightarrowtail \mathcal{F}_{\beta^0}$.

(ii) Let β, ψ be given such that $\mathrm{Spr}(\beta^0)$ and $\mathrm{Fan}(\beta^1)$ and $\psi\colon\ \mathcal{F}_{\beta^1} \rightarrowtail \mathcal{F}_{\beta^0}$.

We first define δ such that $\forall s[\delta(s) = 0 \leftrightarrow \exists\alpha \in \mathcal{F}_{\beta^1}[s \sqsubset \psi|\alpha]]$. Let s be given. Note $\forall\alpha \in \mathcal{F}_{\beta^1}\exists m[s \sqsubset \psi|\overline{\alpha}m \ \vee\ s \perp \psi|\overline{\alpha}m]$. Using the Fan Theorem **FT**, see Section 1.1.7, find m such that $\forall\alpha \in \mathcal{F}_{\beta^1}[s \sqsubset \psi|\overline{\alpha}m \ \vee\ s \perp \psi|\overline{\alpha}m]$, ie $\forall t \in \omega^m[\beta^1(t) = 0 \to s \sqsubset \psi|t \ \vee\ s \perp \psi|t]$. Define $\delta(s) := 0$ if $\exists t \in \omega^m[\beta^1(t) = 0 \ \wedge\ s \sqsubset \psi|t]$ and $\delta(s) := 1$ if $\forall t \in \omega^m[\beta^1(t) = 0 \to s \perp \psi|t]$. Conclude that $\forall s[\delta(s) = 0 \leftrightarrow \exists\alpha \in \mathcal{F}_{\beta^1}[s \sqsubset \psi|\alpha]]$. Note that $\mathrm{Spr}(\delta)$. Also note, using **FT** again: for each m, the set $\{\overline{\psi|\alpha m} \mid \alpha \in \mathcal{F}_{\beta^1}\}$ is finite. Conclude that $\mathrm{Fan}(\delta)$.

We now construct $\tau\colon\ \mathcal{F}_\delta \to \mathcal{F}_{\beta^1}$ such that $\forall\varepsilon \in \mathcal{F}_\delta[\psi|(\tau|\varepsilon) = \varepsilon]$. Let ε in F_δ be given. We claim that for all s, t if $\beta^1(s) = \beta^1(t) = 0$ and $s \perp t$, then there exists n such that either $\forall\alpha \in \mathcal{F}_{\beta^1} \cap s[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$ or $\forall\alpha \in \mathcal{F}_{\beta^1} \cap t[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$. We prove this claim as follows. Let s, t be given such that $\beta^1(s) = \beta^1(t) = 0$ and $s \perp t$. Note that $\forall\alpha \in \mathcal{F}_{\beta^1} \cap s\forall\gamma \in \mathcal{F}_{\beta^1} \cap t[\psi|\alpha \perp \psi|\gamma]$. Conclude that $\forall\alpha \in \mathcal{F}_{\beta^1} \cap s\forall\gamma \in \mathcal{F}_{\beta^1} \cap t\exists n[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n \ \vee\ \psi|\overline{\gamma}n \perp \overline{\varepsilon}n]$. Using the Fan Theorem **FT**, find n such that $n \geq \mathit{length}(s)$ and $n \geq \mathit{length}(t)$ and $\forall\alpha \in \mathcal{F}_{\beta^1} \cap s\forall\gamma \in$

$\mathcal{F}_{\beta^1} \cap t[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n \ \vee \ \psi|\overline{\gamma}n \perp \overline{\varepsilon}n]$. Define $A := \{u \in \omega^n \mid \beta^1(u) = 0 \ \wedge \ s \sqsubseteq u\}$ and $B := \{u \in \omega^n \mid \beta^1(u) = 0 \ \wedge \ t \sqsubseteq u\}$. Note that $\forall u \in A \forall v \in B[\psi|u \perp \overline{\varepsilon}n \ \vee \ \psi|v \perp \overline{\varepsilon}n]$. Note that A, B are finite sets. Conclude, using Lemma 2.6, either $\forall u \in A[\psi|u \perp \overline{\varepsilon}n]$ or $\forall v \in B[\psi|v \perp \overline{\varepsilon}n]$, ie either $\forall \alpha \in \mathcal{F}_{\beta^1} \cap s[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$ or $\forall \alpha \in \mathcal{F}_{\beta^1} \cap t[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$.

Using the above fact repeatedly and keeping in mind that $\{k \mid \beta^1(\langle k \rangle) = 0\}$ is a finite set, conclude that $\exists k \exists n[\beta^1(\langle k \rangle) = 0 \ \wedge \ \forall \alpha \in \mathcal{F}_{\beta^1}[\alpha(0) \neq k \rightarrow \psi|\overline{\alpha}n \perp \overline{\varepsilon}n]]$.

We now define the promised τ, inductively, first specifying τ^0, then τ^1, and so on. We start with τ^0. Let s be given and define $n := \text{length}(s)$. Find out if there exists k such that $\beta^1(\langle k \rangle) = 0$ and $\forall j[\big(j \neq k \ \wedge \ \beta^1(a * \langle j \rangle) = 0\big) \rightarrow \forall \alpha \in \mathcal{F}_{\beta^1} \cap a * \langle j \rangle[\psi|\overline{\alpha}n \perp s]]$. If so, find such k and define $\tau^{m+1}(s) = k + 1$, and, if not, define $\tau^{m+1}(s) = 0$. Assume that $m > 0$ is given and $\tau^0, \tau^1, \ldots \tau^{m-1}$ have been defined. We define τ^m as follows. Let s be given. If $\delta(s) \neq 0$ or $\exists i < m \neg \exists j < \text{length}(s)[\tau^i(\overline{s}j) > 0]$, define $\tau^{m+1}(s) = 0$. Assume $\delta(s) = 0$ and $\forall i < m \exists j < \text{length}(s)[\tau^i(\overline{s}j) > 0]$. Find a such that $\text{length}(a) = m$ and $\forall i < m \exists j < \text{length}(s)[\tau^i(\overline{s}j) = a(i) + 1]$. (One might say that $a := \overline{\tau|s}m$, although this is a little previous, as τ is still under construction.) Note that $\{k \mid \beta^1(a * \langle k \rangle) = 0\}$ is a finite set. Define $n := \text{length}(s)$. Again using the claim we proved a moment ago, find out if there exists k such that $\beta^1(a * \langle k \rangle) = 0$ and $\forall j[\big(j \neq k \ \wedge \ \beta^1(a * \langle j \rangle) = 0\big) \rightarrow \forall \alpha \in \mathcal{F}_{\beta^1} \cap a * \langle j \rangle[\psi|\overline{\alpha}n \perp s]]$. If so, find such k and define $\tau^{m+1}(s) = k + 1$; if not, define $\tau^{m+1}(s) = 0$. Note that $\tau \colon \mathcal{F}_\delta \rightarrow \omega^\omega$ and $\forall \varepsilon \in \mathcal{F}_\delta[\tau|\varepsilon \in \mathcal{F}_{\beta^1} \ \wedge \ \forall \alpha \in \mathcal{F}_{\beta_1}[\alpha \perp (\tau|\varepsilon) \rightarrow \psi|\alpha \perp \varepsilon]]$. In particular: $\forall \alpha \in \mathcal{F}_{\beta^1}[\alpha \perp \big(\tau|(\psi|\alpha)\big) \rightarrow \psi|\alpha \perp \psi|\alpha]$. Conclude that $\forall \alpha \in \mathcal{F}_{\beta^1}[\tau|(\psi|\alpha) = \alpha]$ and $\tau : \mathcal{F}_\delta \twoheadrightarrow \mathcal{F}_{\beta^1}$.

Assume that $\varepsilon \in \mathcal{F}_\delta$ and $\psi|(\tau|\varepsilon) \perp \varepsilon$. Find m such that $\psi|(\tau|\overline{\varepsilon}m) \perp \varepsilon$. Note that $\forall \alpha \in \mathcal{F}_{\beta^1}[(\tau|\overline{\varepsilon}m) \sqsubset \alpha \rightarrow \psi|\alpha \perp \varepsilon]$. Conclude $\forall \alpha \in \mathcal{F}_{\beta^1}[\psi|\alpha \perp \varepsilon]$ and $\forall \alpha \in \mathcal{F}_{\beta^1} \exists n[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$. Using **FT** again, find n such that $\forall \alpha \in \mathcal{F}_{\beta^1}[\psi|\overline{\alpha}n \perp \overline{\varepsilon}n]$, and we have to conclude that $\delta(\overline{\varepsilon}n) \neq 0$ and $\varepsilon \notin \mathcal{F}_\delta$, a contradiction. Conclude that $\forall \varepsilon \in \mathcal{F}_\delta[\psi|(\tau|\varepsilon) = \varepsilon]$.

Let $\rho \colon \omega^\omega \rightarrow \mathcal{F}_\delta$ be the canonical retraction of ω^ω onto $\mathcal{F}_\delta$. Define $\varphi \colon \mathcal{F}_{\beta^0} \rightarrow \mathcal{F}_{\beta^1}$ such that $\forall \gamma \in \mathcal{F}_{\beta^0}[\varphi|\gamma = \tau|(\rho|\gamma)]$. Note that $\forall \alpha \in \mathcal{F}_{\beta^1}[\varphi|(\psi|\alpha) = \alpha]$ and $\varphi \colon \mathcal{F}_{\beta^0} \twoheadrightarrow \mathcal{F}_{\beta^1}$. □

Corollary 6.18 *Let β be given such that* $\text{Fan}(\beta)$*. $\mathcal{F}_\beta$ is almost-countable if and only if $\exists \sigma \in \mathcal{STP} \exists \varphi[\varphi \colon \mathcal{F}_\beta \rightarrowtail \mathcal{F}_{cb_\sigma}]$.*

Proof Every almost-countable spread $\mathcal{F}_\beta$ embeds into some $\mathcal{F}_{cb_\sigma}$, see Theorem 6.13. Conversely, if $\text{Fan}(\beta)$ and $\mathcal{F}_\beta$ embeds into some $\mathcal{F}_{cb_\sigma}$, then, according to Theorem 6.17, $\exists \psi[\psi \colon \mathcal{F}_{cb_\sigma} \twoheadrightarrow \mathcal{F}_\beta]$, and, according to Lemma 6.5, $\mathcal{F}_\beta$ is almost-countable. □

6.6.1 A comment

G. Ronzitti, on page 63 of her Ph.D. dissertation [24] and in the last definition of her paper [25], suggested[22] to call a spread $\mathcal{F}_\beta$ *countable* if and only if $\exists\sigma \in \mathcal{STP}\exists\varphi[\varphi\colon \mathcal{F}_{cb^\lozenge_\sigma} \twoheadrightarrow \mathcal{F}_\beta]$. Unfortunately, following this suggestion, one would have to call the set $\{\underline{n} \mid n \in \omega\}$ a not-countable set. Corollary 6.16 shows the suggestion makes sense if one uses the non-compact Cantor–Bendixson sets given by the function $\sigma \mapsto cb_\sigma$. The suggestion is also a good suggestion if one restricts oneself to fans, rather than spreads, see Theorem 6.15 and Lemma 6.5.

6.7 The Cantor–Bendixson Hierarchy

Lemma 6.19 *For all σ in $\mathcal{STP}$, for all δ, if $\mathcal{F}_{cb_\sigma}$ embeds into En_δ, then $\sigma \le S^*(1^*)$.*

Proof Let σ, δ be given such that $\sigma \in \mathcal{STP}$ and $\mathcal{F}_{cb_\sigma}$ embeds into En_δ. Then, according to Theorem 6.1(i), $\forall\gamma_0 \in \mathcal{F}_{cb_\sigma}\forall\gamma_1 \in \mathcal{F}_{cb_\sigma}[\gamma_0 = \gamma_1 \ \vee\ \gamma_0 \# \gamma_1]$. Using **BCP**, find m such that $\forall\gamma \in \mathcal{F}_{cb_\sigma}[\overline{0}m \sqsubset \gamma \to \underline{0} = \gamma]$. Conclude that $\forall n[\mathcal{F}_{cb_{\sigma^n}} = \emptyset]$ and $\forall n[\sigma^n \le 1^*]$ and $\sigma \le S^*(1^*)$. □

Theorem 6.20 (The Cantor–Bendixson Hierarchy Theorem)

(i) *For all σ, τ in $\mathcal{STP}$, if $\mathcal{F}_{cb_\sigma}$ is τ–reducible, ie $cb_\sigma \in \mathcal{RED}_\tau$,*[23] *then $\sigma \le \tau$.*

(ii) *For all σ, τ in $\mathcal{STP}$, for all δ, if $\mathcal{F}_{cb_\sigma}$ embeds into $\mathbb{P}(\tau, En_\delta)$, then $\sigma \le S^*(\tau)$.*

(iii) *For all σ, τ in $\mathcal{STP}$, if $\mathcal{F}_{cb_\sigma}$ embeds into $\mathbb{P}(\tau, \mathcal{FIN})$, then $\sigma \le S^*(\tau)$.*

(iv) *For all σ, τ in $\mathcal{STP}$, for all δ in $(\mathcal{F}_{cb_\sigma})^\omega$, if $\mathcal{F}_{cb_\sigma} \subseteq \mathbb{P}(\tau, En_\delta)$, then $\sigma \le S^*(\tau)$.*

Proof (i) We use induction on $\mathcal{STP}$. First, note that, for each σ in $\mathcal{STP}$, $\mathcal{F}_{cb_\sigma}$ is 1^*–reducible if and only if $\mathcal{F}_{cb_\sigma} = \emptyset$ if and only if $\sigma = 1^*$ if and only if $\sigma \le 1^*$. Next, assume that we are given $\tau \ne 1^*$ in $\mathcal{STP}$ such that, for each n, for each σ in $\mathcal{STP}$, if $\mathcal{F}_{cb_\sigma}$ is τ^n–reducible, then $\sigma \le \tau^n$.

Assume that we are given σ such that $\mathcal{F}_{cb_\sigma}$ is τ–reducible. Find ε in $\mathcal{F}_{cb_\sigma}$, such that for all m, n, if $\varepsilon(m) \ne n$ and $\beta(\overline{\varepsilon}m * \langle n\rangle) = 0$, then, for some p, $\mathcal{F}_{cb_\Sigma} \cap \overline{\varepsilon}m * \langle n\rangle$ is τ^p–reducible. Let p be given. Consider $s := \langle p+1\rangle$ and $t := \langle 0, p+1\rangle$ and note that either $s \perp \varepsilon$ or $t \perp \varepsilon$. Find m such that either $\mathcal{F}_{cb_\sigma} \cap \langle p+1\rangle = \langle p+1\rangle * \mathcal{F}_{cb_{\sigma^p}}$ is τ^m–reducible, or $\mathcal{F}_{cb_\sigma} \cap \langle 0, p+1\rangle = \langle 0, p+1\rangle * \mathcal{F}_{cb_{\sigma^p}}$ is τ^m–reducible. Conclude that $\mathcal{F}_{cb_{\sigma^p}}$ is τ^m–reducible and $\sigma^p \le \tau^m$. Conclude that $\forall p\exists m[\sigma^p \le \tau^m]$ and $\sigma \le \tau$.

[22] We describe her suggestion in the language of this paper.
[23] See Definition 30.

(ii) We use induction on $\mathcal{STP}$. By Lemma 6.19, for each σ in $\mathcal{STP}$, for each δ, if $\mathcal{F}_{cb_\sigma}$ embeds into $\mathbb{P}(1^*, En_\delta) = En_\delta$, then $\sigma \le S^*(1^*)$. Next, assume that we are given $\tau \neq 1^*$ in $\mathcal{STP}$ such that, for each n, for each σ in $\mathcal{STP}$, for each δ, if $\mathcal{F}_{cb_\sigma}$ embeds into $\mathbb{P}(\tau^n, En_\delta)$, then $\sigma \le S^*(\tau^n)$. Further assume that we are given σ, δ such that $\sigma \in \mathcal{STP}$ and $\mathcal{F}_{cb_\sigma}$ embeds into $\mathbb{P}(\tau, En_\delta)$. Find φ embedding $\mathcal{F}_{cb_\sigma}$ into $\mathbb{P}(\tau, En_\delta)$. Note that $\forall\gamma \in \mathcal{F}_{cb_\sigma}\exists p[\varphi|\gamma \mathrel{\#} \delta^p \to \exists n[\varphi|\gamma \in \mathbb{P}(\tau^n, En_\delta)]]$. Using Brouwer's Continuity Principle **BCP**, see Section 1.1.6, find m, p such that $\forall\gamma \in \mathcal{F}_{cb_\sigma}[(\overline{\underline{0}}m \sqsubset \gamma \ \wedge\ \varphi|\gamma \mathrel{\#} \delta^p) \to \exists n[\varphi|\gamma \in \mathbb{P}(\tau^n, En_\delta)]]$. Consider $\gamma_0 := \overline{\underline{0}}m * \langle p+1\rangle * \underline{0}$ and $\gamma_1 := \overline{\underline{0}}(m+1) * \langle p+1\rangle * \underline{0}$. Note $\varphi|\gamma_0 \mathrel{\#} \varphi|\gamma_1$ and find $i < 2$ such that $\varphi|\gamma^i \mathrel{\#} \delta^p$. Find j, n such that $\varphi|\overline{\gamma_i}j \perp \overline{\delta^p}n$. Note that $\forall\gamma \in \mathcal{F}_{cb_\sigma} \cap \overline{\gamma^i}j\exists i[\varphi|\gamma \in \mathbb{P}(\tau^i, En_\delta)]$. Using **BCP** again, find k, l such that $k > j$ and $\forall\gamma \in \mathcal{F}_{cb_\sigma}[\overline{\gamma_i}k \sqsubset \gamma \to \varphi|\gamma \in \mathbb{P}(\tau^l, En_\delta)]$. Note that $\mathcal{F}_{cb_{\sigma^p}}$ embeds into $\mathcal{F}_{cb_\sigma} \cap \overline{\gamma_i}k$ and φ embeds $\mathcal{F}_{cb_\sigma} \cap \overline{\gamma_i}k$ into $\mathbb{P}(\tau^l, En_\delta)$. Conclude that $\mathcal{F}_{cb_{\sigma^p}}$ embeds into $\mathbb{P}(\tau^l, En_\delta)$, and $\sigma^p \le S^*(\tau^l)$. Conclude that $\forall p\exists l[\sigma^p \le S^*(\tau^l) \le \tau = (S^*(\tau)^l]$ and $\sigma \le S^*(\tau)$.

(iii) Note that $\exists\delta[\mathcal{FIN} = En_\delta]$ and apply (ii).

(iv) This is an immediate consequence of (ii). $\square$

7 The second level and the collapse of the projective hierarchy

7.1 The classes $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$

Some relevant definitions may be found in Section 1.2.6.

Definition 34 *$\mathcal{X} \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$ if and only if there exists β such that:*

$$\mathcal{X} = \mathcal{EUG}_\beta := Ex\big(\mathrm{Un}(\mathcal{G}_\beta)\big) = \{\alpha \mid \exists\delta\forall\gamma[\ulcorner\ulcorner\alpha, \gamma\urcorner, \delta\urcorner \in \mathcal{G}_\beta]\}$$

$\mathcal{X} \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_2$ if and only if there exists β such that:

$$\mathcal{X} = \mathcal{UEF}_\beta := Un\big(Ex(\mathcal{F}_\beta)\big) = \{\alpha \mid \forall\delta\exists\gamma[\ulcorner\ulcorner\alpha, \gamma\urcorner, \delta\urcorner \in \mathcal{F}_\beta]\}$$

Let $\beta, \varepsilon, \zeta$ be given such that $\varepsilon \in \mathcal{EUG}_\beta$ and $\zeta \in \mathcal{UEF}_\beta$. Find δ such that $\forall\gamma\exists n[\beta(\overline{\ulcorner\ulcorner\varepsilon, \gamma\urcorner, \delta\urcorner}n) \neq 0]$. Find γ such that $\forall n[\beta(\overline{\ulcorner\ulcorner\zeta, \gamma\urcorner, \delta\urcorner}n) = 0]$. Find n such that $\beta(\overline{\ulcorner\ulcorner\varepsilon, \gamma\urcorner, \delta\urcorner}n) \neq 0]$ and conclude that $\overline{\varepsilon}n \neq \overline{\zeta}n$ and $\varepsilon \mathrel{\#} \zeta$.

We thus see that, for each β, $\mathcal{EUG}_\beta \mathrel{\#} \mathcal{UEF}_\beta$.

The next theorem shows some properties of the classes $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$. Note that we do not prove that the class $\mathbf{\Pi}^1_2$ is closed under the operation of countable union or even under the operation of finite union.

Theorem 7.1

(i) $\mathcal{US}^1_2 := \{\alpha \mid \alpha_{II} \in \mathcal{EUG}_{\alpha_I}\}$ *is* $\mathbf{\Sigma}^1_2$*–universal and* $\mathcal{UP}^1_2 := \{\alpha \mid \alpha_{II} \in \mathcal{UEG}_{\alpha_I}\}$ *is* $\mathbf{\Pi}^1_2$*–universal*

(ii) $\mathcal{E}^1_2 := \{\alpha \mid \exists\delta\forall\gamma\exists n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) \neq 0]\}$ *is* $\mathbf{\Sigma}^1_2$*–complete and* $\mathcal{A}^1_2 := \{\alpha \mid \forall\delta\exists\gamma\forall n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) = 0]\}$ *is* $\mathbf{\Pi}^1_2$*–complete*

(iii) $\mathbf{\Sigma}^1_2$ *is closed under the operations of countable union and countable intersection and* $\mathbf{\Pi}^1_2$ *is closed under the operation of countable intersection:*

$$\forall\beta\exists\varepsilon\exists\zeta\big[\bigcup_m \mathcal{EUG}_{\beta^m} = \mathcal{EUG}_\varepsilon \ \wedge\ \bigcap_m \mathcal{UEF}_{\beta^m} = \mathcal{UEF}_\varepsilon \ \wedge\ \bigcap_m \mathcal{EUG}_{\beta^m} = \mathcal{EUG}_\zeta\big]$$

(iv) *For all* $\mathcal{X} \subseteq \omega^\omega$*, if* $\mathcal{X} \in \mathbf{\Sigma}^1_2$*, then* $Ex(\mathcal{X}) \in \mathbf{\Sigma}^1_2$*, and, if* $\mathcal{X} \in \mathbf{\Pi}^1_2$*, then* $\mathrm{Un}(\mathcal{X}) \in \mathbf{\Pi}^1_2$*:*

$$\forall\beta\exists\eta[Ex(\mathcal{EUG}_\beta) = \mathcal{EUG}_\eta \ \wedge\ \mathrm{Un}(\mathcal{UEF}_\beta) = \mathcal{UEF}_\eta]$$

(v) *For all* $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$ *such that* $\mathcal{X} \preceq \mathcal{Y}$*, if* $\mathcal{Y} \in \mathbf{\Sigma}^1_2$*, then* $\mathcal{X} \in \mathbf{\Sigma}^1_2$*, and, if* $\mathcal{Y} \in \mathbf{\Pi}^1_2$*, then* $\mathcal{X} \in \mathbf{\Pi}^1_2$*:*

$$\forall\beta\forall\varphi\colon \omega^\omega \to \omega^\omega\exists\gamma \\ \big[\{\alpha \mid \varphi|\alpha \in \mathcal{EUG}_\beta\} = \mathcal{EUG}_\gamma \ \wedge\ \{\alpha \mid \varphi|\alpha \in \mathcal{UEF}_\beta\} = \mathcal{UEF}_\gamma\big]$$

(vi) $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1 \subseteq \mathbf{\Sigma}^1_2 \cap \mathbf{\Pi}^1_2$

Proof (i) Note that for each α, $\alpha \in \mathcal{US}^1_2 \leftrightarrow \alpha_{II} \in \mathcal{EUG}_{\alpha_I} \leftrightarrow \exists\delta[\ulcorner\alpha_{II},\delta\urcorner \in \mathcal{UG}_{\alpha_I}] \leftrightarrow \exists\delta\forall\gamma[\ulcorner\ulcorner\alpha_{II},\delta\urcorner,\gamma\urcorner \in \mathcal{G}_{\alpha_I}] \leftrightarrow \exists\delta\forall\gamma\exists n[\alpha_I(\overline{\ulcorner\ulcorner\alpha_{II},\delta\urcorner,\gamma\urcorner}n) \neq 0]$. Define β such that, for all a,c,d, if $\mathrm{length}(a) = \mathrm{length}(d) = \mathrm{length}(c)$ then $\beta(\ulcorner\ulcorner a,d\urcorner,c\urcorner) = a_I(\ulcorner\ulcorner a_{II},d\urcorner,c\urcorner)$. Note that for all α, $\alpha \in \mathcal{US}^1_2 \leftrightarrow \exists\delta\forall\gamma\exists n[\beta(\overline{\ulcorner\ulcorner\alpha,\delta\urcorner,\gamma\urcorner}n) \neq 0] \leftrightarrow \alpha \in \mathcal{EUG}_\beta$. Conclude that $\mathcal{US}^1_2 \in \mathbf{\Sigma}^1_2$. Also note that for each ε, $\mathcal{EUG}_\varepsilon = \mathcal{US}^1_2 \upharpoonright \varepsilon$ We thus see that $\mathcal{US}^1_2$ is $\mathbf{\Sigma}^1_2$–universal.

Similarly, for each α, $\alpha \in \mathcal{UP}^1_2 \leftrightarrow \forall\delta\exists\gamma\forall n[\alpha_I(\overline{\ulcorner\ulcorner\alpha_{II},\delta\urcorner,\gamma\urcorner}n) = 0]]$. Define β as above and conclude that $\mathcal{UP}^1_2 = \mathcal{UEF}_\beta \in \mathbf{\Pi}^1_2$. Note that for each ε, $\mathcal{UEF}_\varepsilon = \mathcal{UP}^1_2 \upharpoonright \varepsilon$. We thus see that $\mathcal{UP}^1_2$ is $\mathbf{\Pi}^1_2$–universal.

(ii) Define β such that, for all a,c,d, $\beta(a,c,d) \neq 0$ if and only if $\mathrm{length}(a) = \mathrm{length}(c) = \mathrm{length}(d) > 0$ and $\exists i < \mathrm{length}(a)[a(\ulcorner\overline{c}i,\overline{d}i\urcorner) \neq 0]$. Note that for each α, $\exists\delta\forall\gamma\exists n[\alpha(\ulcorner\overline{\gamma}n,\overline{\delta}n\urcorner) \neq 0]$ if and only if $\exists\delta\forall\gamma\exists n[\beta(\overline{\ulcorner\gamma,\delta\urcorner}n) \neq 0]$, and $\forall\delta\exists\gamma\forall n[\alpha(\ulcorner\overline{\gamma}n,\overline{\delta}n\urcorner) = 0]$ if and only if $\forall\delta\exists\gamma\forall n[\beta(\overline{\ulcorner\gamma,\delta\urcorner}n) = 0]$. Conclude that $\mathcal{E}^1_2 = \mathcal{EUG}_\beta \in \mathbf{\Sigma}^1_2$ and $\mathcal{A}^1_2 = \mathcal{UEF}_\beta \in \mathbf{\Pi}^1_2$.

Let ε be given. Define $\varphi\colon\ \omega^\omega \to \omega^\omega$ such that, for all α, for all c, d, if length(c) $=$ length(d) then $(\varphi|\alpha)(\ulcorner c, d\urcorner) = \varepsilon(\ulcorner\overline{\alpha}n, c\urcorner, d\urcorner)]$. Note that for all α,

$$\exists\gamma\forall\delta\exists n[\varepsilon(\overline{\ulcorner\alpha,\gamma\urcorner,\delta\urcorner}n) \neq 0]$$

if and only if
$$\exists\gamma\forall\delta\exists n[(\varphi|\alpha)(\ulcorner\overline{\gamma}n, \overline{\delta}n\urcorner) \neq 0]$$

ie $\alpha \in \mathcal{EUG}_\varepsilon$ if and only if $\varphi|\alpha \in \mathcal{E}^1_2$, and $\forall\gamma\exists\delta\forall n[\varepsilon(\overline{\ulcorner\alpha,\gamma\urcorner,\delta\urcorner}n) = 0]$ if and only if $\forall\gamma\exists\delta\forall n[(\varphi|\alpha)(\ulcorner\overline{\gamma}n, \overline{\delta}n\urcorner) = 0]$; ie $\alpha \in \mathcal{UEF}_\varepsilon$ if and only if $\varphi|\alpha \in \mathcal{A}^1_2$. We thus see that φ reduces the pair $(\mathcal{EUG}_\varepsilon, \mathcal{UEF}_\varepsilon)$ to the pair $(\mathcal{E}^1_2, \mathcal{A}^1_2)$.

We may conclude that $\mathcal{E}^1_2$ is $\mathbf{\Sigma}^1_2$–complete and that $\mathcal{A}^1_2$ is $\mathbf{\Pi}^1_2$–complete.

(iii) Let β be given. For each α, $\alpha \in \bigcup_m \mathcal{UEG}_{\beta^m}$ if and only if $\exists m\exists\delta\forall\gamma\exists n$ $[\beta^m(\overline{\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner}n) \neq 0]$. Define ε such that, for all m, a, c, d, $\varepsilon(\ulcorner\ulcorner a, c\urcorner, \langle m\rangle * d\urcorner) = \beta^m(\ulcorner\ulcorner a, c\urcorner, d\urcorner)]$, and $\beta(\ulcorner\ulcorner 0, 0\urcorner, 0\urcorner) = 0$. Note that, for each m, for all α, γ, δ, $\ulcorner\ulcorner\alpha,\gamma\urcorner, \langle m\rangle * \delta\urcorner \in \mathcal{G}_\varepsilon$ if and only if $\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner \in \mathcal{G}_{\beta^m}$. Therefore, for each α, $\alpha \in \mathcal{EUG}_\varepsilon$ if and only if $\exists m[\alpha \in \mathcal{EUG}_{\beta^m}]$ and $\mathcal{EUG}_\varepsilon = \bigcup_m \mathcal{EUG}_{\beta^m}$. Also note that, for each m, for all α, γ, δ, $\ulcorner\ulcorner\alpha,\gamma\urcorner, \langle m\rangle * \delta\urcorner \in \mathcal{F}_\varepsilon$ if and only if $\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner \in \mathcal{F}_{\beta^m}$. Therefore, for each α, $\alpha \in \mathcal{UEF}_\varepsilon$ if and only if $\forall m[\alpha \in \mathcal{UEF}_{\beta^m}]$, ie $\mathcal{UEF}_\varepsilon = \bigcap_m \mathcal{UEF}_{\beta^m}$. Also, for each α, $\alpha \in \bigcap_m \mathcal{UEG}_{\beta^m}$ if and only if $\forall m\exists\delta\forall\gamma\exists n[\beta^m(\overline{\ulcorner\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner}n) \neq 0]$. Then, by $\mathbf{AC}_{0,1}$, $\alpha \in \bigcap_m \mathcal{UEG}_{\beta^m}$ if and only if $\exists\delta\forall m\forall\gamma\exists n[\beta^m(\overline{\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta^m\urcorner}n) \neq 0]$ if and only if $\exists\delta\forall\gamma\exists n[\beta^{\gamma(0)}(\overline{\ulcorner\ulcorner\alpha,\gamma\circ S\urcorner,\delta^{\gamma(0)}\urcorner}n) \neq 0]$. Define ζ such that, for all a, c, d, $\zeta(\ulcorner\ulcorner a, c\urcorner, d\urcorner) \neq 0$ if and only if $\mathit{length}(a) = \text{length}(c) = \text{length}(d) > 0$ and $\exists i \leq \mathit{length}(a)[\beta^{c(0)}(\ulcorner\ulcorner\overline{a}i, \overline{c\circ S}i\urcorner, \overline{d^{c(0)}}i\urcorner) \neq 0]$. Note that, for all α, δ, $\forall\gamma\exists n[\beta^{\gamma(0)}(\overline{\ulcorner\ulcorner\alpha,\gamma\circ S\urcorner,\delta^{\gamma(0)}\urcorner}n) \neq 0]$ if and only if $\forall\gamma\exists n[\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner \in \mathcal{G}_\zeta]$. Conclude that for all α, $\alpha \in \bigcap_m \mathcal{UEG}_{\beta^m}$ if and only if $\alpha \in \mathcal{UEG}_\zeta$, ie $\mathcal{EUG}_\zeta = \bigcap_m \mathcal{EUG}_{\beta^m}$.

(iv) Let β be given. Note that for all α,

$$\alpha \in \mathit{Ex}(\mathcal{EUG}_\beta) \text{ if and only if } \exists\varepsilon\exists\delta\forall\gamma\exists n[\beta(\overline{\ulcorner\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner,\varepsilon\urcorner}n) \neq 0$$

and
$$\alpha \in \text{Un}(\mathcal{UEF}_\beta) \text{ if and only if } \forall\varepsilon\forall\delta\exists\gamma\forall n[\beta(\overline{\ulcorner\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner,\varepsilon\urcorner}n) = 0]$$

Define η such that, for all a, c, d, if length(a) $=$ length(c) $=$ length(d), then $\eta(\ulcorner\ulcorner a, c\urcorner, d\urcorner) = \beta(\ulcorner\ulcorner\ulcorner a, c\urcorner, d_I\urcorner, d_{II}))]$. One easily verifies that $\mathit{Ex}(\mathcal{EUG}_\beta) = \mathcal{EUG}_\eta$ and $\text{Un}(\mathcal{UEF}_\beta) = \mathcal{UEF}_\eta$.

(v) Let β, φ be given such that $\varphi\colon\ \omega^\omega \to \omega^\omega$. Note that, for each α, $\varphi|\alpha \in \mathcal{EUG}_\beta$ if and only if $\exists\delta\forall\gamma\exists n[\overline{\ulcorner\ulcorner\varphi|\alpha,\gamma\urcorner,\delta\urcorner}n) \neq 0]$ and $\varphi|\alpha \in \mathcal{UEF}_\beta$ if and only if $\forall\delta\exists\gamma\forall n[\overline{\ulcorner\ulcorner\varphi|\alpha,\gamma\urcorner,\delta\urcorner}n) = 0]$. Define ε such that for all a, c, d if length(a) $=$ length(c) $=$ length(d), then $\varepsilon(\ulcorner\ulcorner a, c\urcorner, d\urcorner) \neq 0$ if and only if $\exists i[\mathit{length}(\varphi|a) \geq i \wedge \beta(\ulcorner\ulcorner\overline{\varphi|a}i, \overline{c}i\urcorner, \overline{d}i\urcorner) \neq 0]$.

Then $\{\alpha \mid \varphi|\alpha \in \mathcal{UEG}_\beta\} = \mathcal{UEG}_\varepsilon$ and $\{\alpha \mid \varphi|\alpha \in \mathcal{EUF}_\beta\} = \mathcal{EUF}_\varepsilon$. $\square$

7.2 The collapse of the projective hierarchy

Theorem 7.2

(i) *For all* $\mathcal{X} \subseteq \omega^\omega$*, if* $\mathcal{X} \in \mathbf{\Sigma}^1_2$*, then* $\mathrm{Un}(\mathcal{X}) \in \mathbf{\Sigma}^1_2$*:* $\forall\beta\exists\varepsilon[\mathrm{Un}(\mathcal{EUG}_\beta) = \mathcal{EUG}_\varepsilon]$.

(ii) $\mathbf{\Pi}^1_2 \subseteq \mathbf{\Sigma}^1_2$*, and for all* $\mathcal{X} \subseteq \omega^\omega$ *, if* $\mathcal{X}$ *is (positively) projective, then* $\mathcal{X} \in \mathbf{\Sigma}^1_2$.

Proof (i) Let β be given. Using $\mathbf{AC}_{1,1}$,[24] note that for all α, $\alpha \in \mathrm{Un}(\mathcal{EUG}_\beta)$ if and only if $\forall\varepsilon\exists\delta\forall\gamma\exists n[\beta(\overline{\ulcorner\ulcorner\ulcorner\alpha,\gamma\urcorner,\delta\urcorner,\varepsilon\urcorner}n) \neq 0]$ if and only if $\exists\varphi[\varphi \in \mathcal{A}^1_1 \;\wedge\; \varphi(0) = 0 \;\wedge\; \forall\varepsilon\forall\gamma\exists n[\beta(\overline{\ulcorner\ulcorner\ulcorner\alpha,\gamma\urcorner,\varphi|\varepsilon\urcorner,\varepsilon\urcorner}n) \neq 0]]$ if and only if $\exists\varphi[\varphi \in \mathcal{A}^1_1 \;\wedge\; \varphi(0) = 0 \wedge \forall\varepsilon\forall\gamma\exists n\exists m[\mathit{length}(\varphi|\overline{\varepsilon}m) \geq n \;\wedge\; \beta(\ulcorner\ulcorner\ulcorner\overline{\alpha}n, \overline{\gamma}n\urcorner, \overline{(\varphi|\overline{\varepsilon}m)}n\urcorner, \overline{\varepsilon}n\urcorner) \neq 0]]$.

Using Theorem 7.1, we conclude that $\mathrm{Un}(\mathcal{EUG}_\beta) \in \mathbf{\Sigma}^1_2$.

(ii) This follows from (i). $\square$

Theorem 7.2 shows that, in intuitionistic mathematics, $\mathbf{\Sigma}^1_2$ is the class of all positively projective sets. Many difficult questions remain, for instance, if $\mathbf{\Pi}^1_2$ is a proper subclass of $\mathbf{\Sigma}^1_2$ and if the class $\mathbf{\Pi}^1_2$ is closed under the operation of disjunction. We were unable to answer these questions.

Note that the projection of a positively Borel set is analytic. It is not true however, that the co-projection of a positively Borel set is always co-analytic, for the simple reason that some positively Borel sets, like $\mathbb{D}^2(\mathcal{A}_1)$,[25] are not co-analytic.

Lemma 7.3 $\forall\varphi\colon \omega^\omega \to \omega^\omega\exists\alpha[(\alpha \in \mathcal{E}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{E}^1_2) \;\wedge\; (\alpha \in \mathcal{A}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{A}^1_2)]$.

Proof Let $\varphi\colon \omega^\omega \to \omega^\omega$ be given. Define α such that for all p, c, d, if $\mathrm{length}(c) = \mathrm{length}(d)$ and $p = \ulcorner c, d\urcorner$, then $\alpha(p) \neq 0$ if and only if, for some $m \leq \mathit{length}(c)$, $\ulcorner\overline{c}m, \overline{d}m\urcorner < \mathit{length}(\varphi|\overline{\alpha}p)$ and $(\varphi|\overline{\alpha}p)(\ulcorner\overline{c}m, \overline{d}m\urcorner) \neq 0]$. Note that, for all γ, δ, $\exists m[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}m) \neq 0]$ if and only if $\exists m[(\varphi|\alpha)(\overline{\ulcorner\gamma,\delta\urcorner}m) \neq 0]$. Conclude that $\exists\gamma\forall\delta\exists n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) \neq 0] \leftrightarrow \exists\gamma\forall\delta\exists n[(\phi|\alpha)(\overline{\ulcorner\gamma,\delta\urcorner}n) \neq 0]$, ie $\alpha \in \mathcal{E}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{E}^1_2$, and also that $\forall\gamma\exists\delta\forall n[\alpha(\overline{\ulcorner\gamma,\delta\urcorner}n) = 0] \leftrightarrow \forall\gamma\exists\delta\forall n[(\varphi|\alpha)(\overline{\ulcorner\gamma,\delta\urcorner}n) = 0]$, ie $\alpha \in \mathcal{A}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{A}^1_2$. $\square$

Note that the classical mathematician would conclude, from Lemma 7.3, that $\mathcal{A}^1_2 \not\preceq \mathcal{E}^1_2$ and $\mathcal{E}^1_2 \not\preceq \mathcal{A}^1_2$.

[24] For $\mathbf{AC}_{1,1}$ see Section 1.1.6. Note that $\varphi\colon \omega^\omega \to \omega^\omega$ if and only if $\varphi(0) = 0$ and, for each n, $\varphi^n : \omega^\omega \to \omega$. Note that $\varphi\colon \omega^\omega \to \omega^\omega$ if and only if only if $\varphi \in \mathcal{A}^1_1$ and $\varphi(0) = 0$, see also Section 1.1.5.

[25] See Theorem 4.1(iv).

Theorem 7.4

(i) $\exists\alpha[\alpha \notin \mathcal{E}^1_2 \ \wedge \ \alpha \notin \mathcal{A}^1_2]$
(ii) $\exists\gamma[\gamma \notin \mathcal{US}^1_2 \ \wedge \ \gamma \notin \mathcal{UP}^1_2]$

Proof (i) Using Theorems 7.2(i) and 7.1(ii), find $\varphi\colon\ \omega^\omega \to \omega^\omega$ reducing $\mathcal{A}^1_2$ to $\mathcal{E}^1_2$. Applying Lemma 7.3, find α such that $\alpha \in \mathcal{E}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{E}^1_2$ and $\alpha \in \mathcal{A}^1_2 \leftrightarrow \varphi|\alpha \in \mathcal{A}^1_2$. Assume that $\alpha \in \mathcal{E}^1_2$. Conclude that $\varphi|\alpha \in \mathcal{A}^1_2$ and $\alpha \in \mathcal{A}^1_2$. This is a contradiction, as $\mathcal{A}^1_2 \# \mathcal{E}^1_2$. Conclude that $\alpha \notin \mathcal{E}^1_2$ and $\varphi|\alpha \notin \mathcal{E}^1_2$ and $\alpha \notin \mathcal{A}^1_2$.

(ii) Define $\mathcal{DP}^1_2 := \{\alpha \mid \ulcorner\alpha,\alpha\urcorner \in \mathcal{UP}^1_2\}$. According to Theorem 7.2(i), $\mathcal{DP}^1_2 \in \mathbf{\Sigma}^1_2$. Using Theorem 7.1(iii), find β such that $\mathcal{DP}^1_2 = \mathcal{US}^1_2\upharpoonright\beta$. Note that for every α, $\ulcorner\alpha,\alpha\urcorner \in \mathcal{UP}^1_2 \leftrightarrow \alpha \in \mathcal{DP}^1_2 \leftrightarrow \ulcorner\beta,\alpha\urcorner \in \mathcal{US}^1_2$. Define $\gamma := \ulcorner\beta,\beta\urcorner$, and note that $\gamma \notin \mathcal{US}^1_2$ and $\gamma \notin \mathcal{UP}^1_2$, as $\mathcal{US}^1_2 \# \mathcal{UP}^1_2$. □

Theorem 7.4 has some noteworthy consequences. Assume that $\alpha \notin \mathcal{E}^1_2 \cup \mathcal{A}^1_2$. Then:

(i) $\neg\exists\delta\forall\gamma\exists n[\alpha(\ulcorner\overline{\gamma}n,\overline{\delta}n\urcorner) \neq 0$,
(ii) $\neg\forall\delta\exists\gamma\forall n[\alpha(\ulcorner\overline{\gamma}n,\overline{\delta}n\urcorner) = 0]$, and
(iii) $\forall\delta\forall\gamma\forall n[\ulcorner\alpha(\overline{\gamma}n,\overline{\delta}n\urcorner) = 0 \ \vee \ \alpha(\ulcorner\overline{\gamma}n,\overline{\delta}n\urcorner) \neq 0]$.

Theorem 7.4 thus shows that, in intuitionistic mathematics it is possible that statements

(i) $\neg\exists \mathsf{x}\forall \mathsf{y}\exists \mathsf{z}[\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$,
(ii) $\neg\forall \mathsf{x}\exists \mathsf{y}\forall \mathsf{z}[\neg\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$, and
(iii) $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}[\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z}) \ \vee \ \neg\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$

are simultaneously true. The example depends on $\mathbf{AC}_{1,1}$. Another example, depending only on **BCP**, has been given in Veldman [35, Section 5.5]:

(i) $\neg\exists\alpha\forall n\exists m[\alpha(n) = 0 \ \wedge \ \alpha(m) \neq 0]$,
(ii) $\neg\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \ \vee \ \alpha(m) = 0]$, and
(iii) $\forall\alpha\forall n\forall m[\big(\alpha(n) = 0 \ \wedge \ \alpha(m) \neq 0\big) \ \vee \ \big(\alpha(n) \neq 0 \ \vee \ \alpha(m) = 0\big)]$.

7.3 A parallel: the collapse of the (positive) arithmetical hierarchy

It has been observed by J.R. Moschovakis that, in the context of intuitionistic arithmetic, Church's Thesis **CT** causes the collapse of the (positively) arithmetical hierarchy, just as $\mathbf{AC}_{1,1}$ causes the collapse of the (positively) projective hierarchy; see J.R. Moschovakis [18] and [20]. It seems useful to explain this.

Let $T \subseteq \omega^3$ be Kleene's T–predicate. T is a *(Kalmár–)elementary* subset of ω^3 and, for all e,n,z, $T(e,n,z)$ stands for: *'z is the code of a successful computation according* *to the algorithm coded by e at the argument n*'. Let U be the elementary function from ω to ω extracting from each successful computation z its outcome $U(z)$. Every e determines a *partial* function φ_e from ω to ω by:

$$\forall n[\varphi_e(n) \simeq U(\mu z[T(e,n,z)])]$$

For each e, $W_e := \{n \mid \exists z[T(e,n,z)]\}$ is the domain of the partial function φ_e.

For every $X \subseteq \omega$, we define the *projection* $\mathrm{Ex}_0(X) := \{m \mid \exists n[\langle m,n\rangle \in X]\}$ and the *co-projection* $\mathrm{Un}_0(X) := \{m \mid \forall n[\langle m,n\rangle \in X]\}$.

One defines $\Sigma^0_1 := \{W_e \mid e \in \omega\}$ and $\Pi^0_1 := \{\omega \setminus W_e \mid e \in \omega\}$, and, for each $m > 0$, $\Sigma^0_{m+1} := \{\mathrm{Ex}_0(X) \mid X \in \Pi^0_m\}$ and $\Pi^0_{m+1} := \{\mathrm{Un}_0(X) \mid X \in \Sigma^0_m\}$.

One may prove: $\forall m > 0[\Sigma^0_m \cup \Pi^0_m \subseteq \Sigma^0_{m+1} \cap \Pi^0_{m+1}]$.

Using the following strong form of *Church's Thesis*

$$\mathbf{CT}: \text{ for every } R \subseteq \omega\times\omega, \quad \forall m\exists n[mRn] \to \exists e\forall m\exists z[T(e,m,z) \ \wedge \ mRU(z)]$$

one may prove that, for every X in Σ^0_3, also $\mathrm{Un}_0(X) \in \Sigma^0_3$, as follows.

Assume $X \in \Sigma^0_3$. Find e such that $X = \mathrm{Ex}_0\big(\mathrm{Un}_0(W_e)\big)$. Consider:

$$\begin{aligned} Y = \mathrm{Un}_0(X) &= \{m \mid \forall q[\langle m,q\rangle \in X]\} = \{m \mid \forall q\exists n\forall p\exists z[T(e,\langle m,p,n,q\rangle,z)]\} \\ &= \{m \mid \exists f\forall q\forall p\exists u\exists z[T(f,q,u) \ \wedge \ T(e,\langle m,p,U(u),q\rangle,z)]\} \in \Sigma^0_3 \end{aligned}$$

One may conclude that $\Pi^0_3 \subseteq \Sigma^0_3$ and $\bigcup_m \Sigma^0_m \subseteq \Sigma^0_3$.

Find f such that $\{e \mid \forall p\exists n\forall z[\neg T(e,\langle e,n,p\rangle,z)]\} = \{m \mid \exists p\forall n\exists z[T(f,\langle m,n,p\rangle,z)]\}$, and note that $\forall p\exists n\forall z[\neg T(f,\langle f,n,p\rangle,z)] \leftrightarrow \exists p\forall n\exists z[T(f,\langle f,n,p\rangle,z)]$; therefore:

$$\neg\forall p\exists n\forall z[\neg T(f,\langle f,p,n\rangle,z)] \text{ and } \neg\exists p\forall n\exists z[T(f,\langle f,p,n\rangle,z)]$$

Again, we see that three statements of the form

(i) $\neg\exists \mathsf{x}\forall \mathsf{y}\exists \mathsf{z}[\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$,
(ii) $\neg\forall \mathsf{x}\exists \mathsf{y}\forall \mathsf{z}[\neg\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$, and
(iii) $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}[\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z}) \vee \neg\mathsf{P}(\mathsf{x},\mathsf{y},\mathsf{z})]$

may be true simultaneously.

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