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Polish topologies on groups of non-singular transformations

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Abstract: In this paper, we prove several results concerning Polish group topologies on groups of non-singular transformations. We first prove that the group of measure-preserving transformations of the real line whose support has finite measure carries no Polish group topology. We then characterize the Borel σ–finite measures λ on a standard Borel space for which the group of λ–preserving transformations has the automatic continuity property. We finally show that the natural Polish topology on the group of all non-singular transformations is actually its only Polish group topology.

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1 Introduction

The study of measure-preserving (or more generally non-singular) transformations on a standard measured space (Y, λ) is broadened once one realises that such transformations form a Polish group. Indeed, the Baire Category Theorem is then available and so the question of generic properties of such transformations arises naturally.

As a somewhat degenerate case, one may first look at the case where the measure λ is completely atomic. Then $\mathrm{Aut}(Y, \lambda)$ only acts by permuting atoms of the same measure and thus splits as a direct product of permutation groups. In the case where all the atoms have the same measure and λ is infinite, we get the Polish group $\mathfrak{S}_\infty$ of permutations of the integers. In this group, the generic permutation has only finite orbits and infinitely many orbits of size n for every $n \in \mathbb{N}$. Such permutations thus form a comeager conjugacy class.

Actually a much stronger property called *ample generics* holds for the Polish group $\mathfrak{S}_\infty$. This has several nice consequences as was shown by Kechris and Rosendal [9], among which the *automatic continuity property*, which in turn implies that its Polish group topology is unique.

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Definition 1.1 A Polish group G has the **automatic continuity property** if whenever $\pi : G \to H$ is a group homomorphism taking values in a separable topological group H, the homomorphism π has to be continuous.

It is well-known that as soon as λ has a non-atomic part, the group $\mathrm{Aut}(Y,\lambda)$ fails to have ample generics. However, it was shown by Ben Yaacov, Berenstein and Melleray that when λ is a non-atomic *finite* measure, $\mathrm{Aut}(X,\lambda)$ still has the automatic continuity property [2] (see also [12, Section 2] for a more direct proof). Later on Sabok developed a framework to show automatic continuity for automorphism groups of metric structures [15]. In particular, he got another proof of automatic continuity for $\mathrm{Aut}(Y,\lambda)$, and then Malicki simplified his approach [13] . We first observe that this framework can also be applied when λ is infinite.

Theorem 1.2 *Let (Y,λ) be a standard Borel space equipped with a non-atomic σ–finite infinite measure λ. Then $\mathrm{Aut}(Y,\lambda)$ has the automatic continuity property.*

Note that as a concrete example for the above result, one can take Y to be the reals equipped with the Lebesgue measure. In general, we can actually characterize when $\mathrm{Aut}(Y,\eta)$ has the automatic continuity as follows, where the *η–atomic multiplicity* of a real $r>0$ is the number of atoms whose measure is equal to r.

Theorem 1.3 *Let (Y,η) be a standard Borel space equipped with a Borel σ–finite measure η. Then the following are equivalent:*

- *(i)* $\mathrm{Aut}(Y,\eta)$ *has the automatic continuity property.*
- *(ii) There are only finitely many positive reals whose η–atomic multiplicity belongs to $[2,+\infty[$.*

Let us now consider the group $\mathrm{Aut}^*(Y,\eta)$ of non-singular transformations of (Y,η), ie the group of Borel bijections which preserve η–null sets. If η_{at} denotes the atomic part of λ and η_{cont} denotes the atomless part, we see that $\mathrm{Aut}^*(Y,\eta)$ splits as a direct product:

$$\mathrm{Aut}^*(Y,\eta) = \mathrm{Aut}^*(Y,\eta_{at}) \times \mathrm{Aut}^*(Y,\eta_{cont})$$

The group $\mathrm{Aut}(Y,\eta_{at})$ is a permutation group, so it has the automatic continuity and thus we focus on $\mathrm{Aut}(Y,\eta_{cont})$, assuming that η_{cont} is non-trivial. Observe that η_{cont} is then equivalent to an atomless probability measure, so we may as well assume η_{cont} is a probability measure. We are thus led to ask:

Question *Let (X,μ) be a standard probability space. Does the group $\mathrm{Aut}^*(X,\mu)$ of all non-singular transformations of (X,μ) have the automatic continuity property ?*

The main difficulty with this question is that the framework of Sabok is not available for $\mathrm{Aut}^*(X,\mu)$ because it cannot be the automorphism group of a complete homogeneous metric structure as was recently shown by Ben Yaacov [1]. While we cannot answer this question, we still manage to obtain a basic consequence of automatic continuity, namely having a unique Polish group topology.

Theorem 1.4 *Let (X,μ) be a standard probability space. The group $\mathrm{Aut}^*(X,\mu)$ has a unique Polish group topology, namely the strong topology.*

The techniques we use to prove the above theorem are quite standard, except for the fact that we use the automatic continuity for $\mathrm{Aut}(X,\mu)$ so as to know that $\mathrm{Aut}(X,\mu)$ is a Borel subgroup of $\mathrm{Aut}^*(X,\mu)$ for any Polish group topology on $\mathrm{Aut}^*(X,\mu)$. This trick may be of use for other Polish groups.

Finally, we prove a result in the line of research started by Rosendal [14] by showing that the group of all measure-preserving transformations of the real line which have finite support cannot carry any Polish group topology.

The paper is organized in two independent sections. Section 2 deals with groups of measure-preserving transformations over a σ–finite space. After a preliminary section, we start with the above-mentioned absence of Polish group topology on the group of finite support transformations in Section 2.2. We then check that $\mathrm{Aut}(Y,\nu)$ has the automatic continuity property in Section 2.3, and we prove Theorem 1.3 in Section 2.4. Section 3 is finally devoted to the proof of the uniqueness of the Polish group topology of $\mathrm{Aut}^*(X,\mu)$ (Theorem 1.4).

Remark Throughout the paper, we will often neglect what happens on null sets without explicit mention.

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2 Groups of transformations preserving a σ–finite measure

2.1 Preliminaries

A **standard σ–finite space** is a standard Borel space equipped with a Borel *nonatomic* σ–finite infinite measure. All such spaces are isomorphic to $\mathbb{R}$ equipped with the Lebesgue measure, and we fix from now on such a standard σ–finite space (Y,λ).

The first group we are interested in is denoted by $\mathrm{Aut}(Y,\lambda)$ and consists of all Borel bijections $T\colon Y\to Y$ which preserve the measure λ: for all Borel $A\subseteq Y$, we have $\lambda(A)=\lambda(T^{-1}(A))$. As usual, two such bijections are identified if they coincide on a conull set.

Consider the space $\mathrm{MAlg}_f(Y,\lambda)$ of finite measure Borel subsets of Y where we identify A and B if $\lambda(A\bigtriangleup B)=0$. It is equipped with the metric $d_\lambda(A,B):=\lambda(A\bigtriangleup B)$. ($d_\lambda$ would only be a pseudo-metric if we had not identified sets up to measure zero.) We have the following well-known lemma.

Lemma 2.1 *The metric space $(\mathrm{MAlg}_f(Y,\lambda),d_\lambda)$ is complete and separable.*

Proof We first prove completeness. Let (A_n) be a Cauchy sequence, up to taking a subsequence we may assume that for all $n\in\mathbb{N}$, $\lambda(A_n\bigtriangleup A_{n+1})<2^{-n}$. It then follows from the Borel–Cantelli Lemma that for almost every $y\in Y$, we have some $N\in\mathbb{N}$ such that $y\notin A_n\bigtriangleup A_{n+1}$ for all $n\geqslant N$.

Now denote by A the set of all $y\in Y$ such that there is $N\in\mathbb{N}$ such that for all $n\geqslant N$, $y\in A_n$. It follows from the second-to-last sentence that if $y\notin A$ then there is $N\in\mathbb{N}$ such that for all $n\geqslant N$, $y\notin A_n$. We thus have that for all $N\in\mathbb{N}$, $A\bigtriangleup A_N\subseteq\bigcup_{n\geqslant N}(A_n\bigtriangleup A_{n+1})$. Since the latter has measure at most $\sum_{n\geqslant N}2^{-n}=2^{-N+1}$ which tends to zero, we conclude that $A_n\to A$ as wanted.

The separability is then obtained by noting that we may as well assume $Y=\mathbb{R}$ endowed with the Lebesgue measure, and then finite unions of rational endpoints intervals are dense in $\mathrm{MAlg}(Y,\lambda)$. □

Now, if (X,μ) is a standard probability space then every Borel subset has finite measure, and by definition the measure algebra $(\mathrm{MAlg}(X,\mu),d_\mu)$ is defined as its set of Borel subsets up to measure zero, equipped with the metric $d_\mu(A,B):=\mu(A\bigtriangleup B)$. If (Z,ν) is another standard probability space, any isometry between $(\mathrm{MAlg}(X,\mu),d_\mu)$ and $(\mathrm{MAlg}(Z,\nu),d_\nu)$ sending $\emptyset$ to $\emptyset$ comes from a measure-preserving bijection which is unique up to a null set (see Kechris [8, Section 1 (B)]). Using the σ–finiteness of (Y,ν) and the above fact, we easily get the following proposition.

Proposition 2.2 $\mathrm{Aut}(Y,\lambda)$ *is equal to the group of isometries of* $(\mathrm{MAlg}_f(Y,\lambda), d_\lambda)$ *which fix* $\emptyset$.

The above proposition implies that $\mathrm{Aut}(Y,\lambda)$ is a Polish group as it is a closed subgroup of the isometry group of a separable complete metric space. The corresponding topology is called the weak topology; it is thus defined by $T_n \to T$ if and only if for all $A \subseteq Y$ of finite measure, one has:

$$\lambda(T_n(A) \triangle T(A)) \to 0$$

Note that since $\lambda(T_n(A)) = \lambda(T(A))$, this condition is in turn equivalent to $\lambda(T_n(A) \setminus T(A)) \to 0$.

For $T \in \mathrm{Aut}(Y,\lambda)$, we define its **support** to be the following Borel set, which is only well-defined up to measure zero:

$$\mathrm{supp}\, T := \{y \in Y : T(y) \neq y\}$$

Note that we have the following relation: for all $S, T \in \mathrm{Aut}(Y,\lambda)$,

$$\mathrm{supp}(STS^{-1}) = S(\mathrm{supp}\, T).$$

Definition 2.3 The group $\mathrm{Aut}_f(Y,\lambda)$ is the normal subgroup of $\mathrm{Aut}(Y,\lambda)$ consisting of all $T \in \mathrm{Aut}(Y,\lambda)$ such that $\lambda(\mathrm{supp}(T)) < +\infty$.

2.2 Absence of Polish group topology on $\mathrm{Aut}_f(Y,\lambda)$

2.2.1 Non-Polishability

Our first lemma is well-known, we provide a proof for the reader's convenience.

Lemma 2.4 *For all* $R > 0$, *the set of* $T \in \mathrm{Aut}(Y,\lambda)$ *such that* $\lambda(\mathrm{supp}\, T) \leqslant R$ *is closed.*

Proof Take T such that $\lambda(\mathrm{supp}\, T) > R$, then there exists a partition of $\mathrm{supp}\, T$ in countably many sets of positive measure $(A_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$, we have $\lambda(T(A_i) \cap A_i) = 0$. By our hypothesis, we may then find $n \in \mathbb{N}$ such that $\lambda(A_1 \sqcup \cdots \sqcup A_n) > R$, and up to shrinking each A_i we may furthermore assume $\lambda(A_1 \sqcup \cdots \sqcup A_n) < +\infty$.

Let $\epsilon = (\lambda(A_1 \sqcup \cdots \sqcup A_n) - R)/n$. Now take $T' \in \mathrm{Aut}(Y,\lambda)$ such that $\lambda(T(A_i) \triangle T'(A_i)) < \epsilon$ for $i = 1, ..., n$, and let $B_i := A_i \setminus T'(A_i)$. By construction we have $\lambda(B_i) > \lambda(A_i) - \epsilon$. Moreover $T'(B_i)$ is disjoint from B_i so each B_i is contained in the support of T', and since they are disjoint we conclude that the support of T' has measure greater than $\lambda(A_1 \sqcup \cdots \sqcup A_n) - n\epsilon > R$. □

Definition 2.5 A subgroup H of a Polish group G is called **Polishable** if it admits a Polish group topology which refines the topology of G.

Remark By a direct application of the Lusin–Suslin theorem (see, eg, Kechris [7, Theorem 15.1]), every Polishable subgroup of a Polish group G is a Borel subset of G. Here it follows from the above lemma that $\mathrm{Aut}_f(Y,\lambda)$ is F_σ in $\mathrm{Aut}(Y,\lambda)$ (in particular Borel). Nevertheless, we have the following result.

Theorem 2.6 *The subgroup* $\mathrm{Aut}_f(Y,\lambda) \leqslant \mathrm{Aut}(Y,\lambda)$ *is not Polishable.*

Proof Suppose that $\mathrm{Aut}_f(Y,\lambda)$ is Polishable. Then by definition its Polish group topology τ refines the weak topology. For each $n\in\mathbb{N}$, let:

$$F_n := \{T\in \mathrm{Aut}_f(Y,\lambda) : \lambda(\operatorname{supp} T)\leqslant n\}$$

By the previous lemma, each F_n is closed in $\mathrm{Aut}_f(Y,\lambda)$. Since $\mathrm{Aut}_f(Y,\lambda)=\bigcup_{n\in\mathbb{N}} F_n$, the Baire Category Theorem yields that there is $n\in\mathbb{N}$ such that F_n has nonempty interior. Since τ is second-countable, we deduce that $\mathrm{Aut}_f(Y,\lambda)$ is covered by countably many F_n–translates. This means that $\mathrm{Aut}_f(Y,\lambda)$ contains a countable set which is n–dense[1] for the metric d_λ given by:

$$d_\lambda(T,T') := \lambda\left(\{x\in Y : T(x)\neq T'(x)\}\right)$$

Let us explain why this cannot happen.

Fix a Borel set $A\subseteq Y$ of measure $3n$, and identify A with the circle $\mathbb{S}^1$ equipped with the finite measure $3n\lambda$, where λ is the Haar measure on $\mathbb{S}^1$. Take $z\in\mathbb{S}^1$ and consider T_z the translation by z in $\mathbb{S}^1$, which we can see through our identification as a measure preserving transformation of (Y,λ) supported on A. Observe that for $z\neq z'$, we have $d_\lambda(T_z,T_{z'})=3n$. So in $\mathrm{Aut}_f(Y,\lambda)$ there is an uncountable subgroup all whose distinct elements are $3n$–apart for the metric d_λ, contradicting the fact that $\mathrm{Aut}_f(Y,\lambda)$ contains a countable set which is n–dense for the metric d_λ by the pigeonhole principle. □

2.2.2 Non-existence of a Polish group topology

We now upgrade the previous theorem to see that $G:=\mathrm{Aut}_f(Y,\lambda)$ cannot carry a Polish group topology. Fortunately, the arguments we need were carried out by Kallman in [6]

[1] By definition, a subset A of a metric space (X,d) is n–dense if every element of X is at distance at most n from some element of A.

to prove the uniqueness of the Polish topology of $\mathrm{Aut}(Y,\lambda)$. We reproduce them here for the convenience of the reader.

For a Borel subset $A \subseteq Y$ we let:

$$G_A := \{T \in \mathrm{Aut}_f(Y,\lambda) : \operatorname{supp} T \subseteq A\}$$

For a subset $F \subseteq \mathrm{Aut}_f(Y,\lambda)$ we let

$$\mathcal{C}(F) := \{U \in \mathrm{Aut}_f(Y,\lambda) : TU = UT \text{ for all } T \in F\}$$

denote its centraliser.

Lemma 2.7 *We have* $\mathcal{C}(G_A) = G_{Y\setminus A}$*.*

Proof We clearly have $G_{Y\setminus A} \leqslant \mathcal{C}(G_A)$.

Take $T \notin G_{Y\setminus A}$. By definition the support of T intersects A, so there is $B \subseteq A$ with $T(B)$ disjoint from B. But clearly T does not commute with non-trivial elements of $\mathrm{Aut}_f(Y,\lambda)$ supported in B, in particular $T \notin C(G_A)$. □

By the previous lemma and the fact that centralizers are always closed in topological groups, whenever τ is a Hausdorff group topology on $\mathrm{Aut}_f(Y,\lambda)$ we have that the set $G_{Y\setminus A}$ is τ–closed. Also note that for all $T \in \mathrm{Aut}_f(Y,\lambda)$ and all $A \subseteq Y$, we have:

$$G_{T(A)} = TG_AT^{-1} \tag{1}$$

Denote by $G(A,B)$ the set of $T \in \mathrm{Aut}_f(Y,\lambda)$ such that $T(A) \subseteq B$.

Lemma 2.8 *Let* τ *be a Hausdorff group topology on* $G = \mathrm{Aut}_f(Y,\lambda)$*. For all* $A,B \subseteq Y$*, the set* $G(A,B)$ *is* τ*–closed.*

Proof Observe first that $A \subseteq B$ if and only if $G_A \leqslant G_B$. The direct implication is clear; conversely, if A is not a subset of B then we find a transformation supported on $A \setminus B$, thus witnessing that $G_A \nleqslant G_B$. By equality (1) we then have $G(A,B) = \{T \in \mathrm{Aut}_f(Y,\lambda) : TG_AT^{-1} \subseteq G_B\}$. So, by the previous lemma $G(A,B)$ is the set of all $T \in \mathrm{Aut}_f(Y,\lambda)$ such that for all $U \in G_A$, TUT^{-1} commutes with every element of $G_{X\setminus B}$. This is clearly a τ–closed condition. □

Now take $A \subseteq Y$, let $\epsilon > 0$, and pick $B \subseteq Y$ containing A such that $\lambda(B \setminus A) = \epsilon$.

Lemma 2.9 $G_{Y\setminus A} \cdot G(A,B) = \{T \in \mathrm{Aut}_f(Y,\lambda) : \lambda(T(A) \setminus A) \leqslant \epsilon\}$.

Proof Note that $G_{Y\setminus A}$ is a group, and that the set $F := \{T \in \mathrm{Aut}_f(Y,\lambda) : \lambda(T(A)\setminus A) \leqslant \epsilon\}$ is left $G_{Y\setminus A}$–invariant. Moreover, since $\lambda(B\setminus A) = \epsilon$ we clearly have $G(A,B) \subseteq F$, so $G_{Y\setminus A_m} \cdot G(A_n,B) \subseteq F$.

For the reverse inclusion, take $T \in F$. Since $\lambda(T(A)\setminus A) \leqslant \epsilon$ and $\lambda(B\setminus A_m) = \epsilon$ we may find $U \in G_{Y\setminus A_m}$ such that $U(T(A)\setminus A) \subseteq B\setminus A$. We conclude that $UT \in G(A,B)$ so $T \in G_{Y\setminus A} \cdot G(A,B)$. □

The above lemma implies that if τ is a Polish group topology on $\mathrm{Aut}_f(Y,\lambda)$, then for all $A \subseteq Y$ and $\epsilon > 0$, the set $\{T \in \mathrm{Aut}_f(Y,\lambda) : \lambda(T(A)\setminus A) \leqslant \epsilon\}$ is analytic (it is the pointwise product of two closed sets) hence Baire-measurable.

Denote by w the weak topology on $\mathrm{Aut}(Y,\lambda)$, and denote by w' the topology it induces on $\mathrm{Aut}_f(Y,\lambda)$. Observe that for all $T \in \mathrm{Aut}_f(Y,\lambda)$, we have $\lambda(A\setminus T(A)) = \lambda(A \bigtriangleup T(A))/2$. It follows that sets of the form

$$\{T \in \mathrm{Aut}_f(Y,\lambda) : \lambda(T(A)\setminus A) \leqslant \epsilon\} \tag{2}$$

form a subbasis of neighborhoods of the identity in $\mathrm{Aut}_f(Y,\lambda)$ for the topology w'. In particular their left translates generate the Borel σ–algebra of $\mathrm{Aut}_f(Y,\lambda)$ associated to the topology w'. Since τ is a group topology and sets of the form (2) are τ–Baire measurable, we conclude that the identity map $(\mathrm{Aut}_f(Y,\lambda),\tau) \to (\mathrm{Aut}(Y,\lambda), w)$ is Baire-measurable. So the inclusion map is continuous by the Pettis Lemma (see, eg, [5, Theorem 2.3.2]). But this is impossible by Theorem 2.6. This proves the following result.

Theorem 2.10 *The group* $\mathrm{Aut}_f(Y,\lambda)$ *cannot carry a Polish group topology.*

2.3 Automatic continuity for $\mathrm{Aut}(Y,\lambda)$

Let us now briefly indicate why $\mathrm{Aut}(Y,\lambda)$ has the automatic continuity property when (Y,λ) is a standard σ–finite space (ie, λ is a non-atomic σ–finite infinite measure on the standard Borel space Y). To do so, we will simply check the criterions given by Sabok [15] and then simplified by Malicki [13]. We won't give full details since the proofs adapt verbatim and we refer the reader to Malicki's paper for definitions of the terms used thereafter.

As explained in Section 2.1, we may view the group $\mathrm{Aut}(Y,\lambda)$ as the group of automorphisms of the metric structure $(\mathrm{MAlg}_f(Y,\lambda), d_\lambda, \bigtriangleup, \cap)$ where

- $\mathrm{MAlg}_f(Y,\lambda)$ is the set of finite measure Borel subsets of Y, up to measure zero;

- $d_\lambda(A, B) = \lambda(A \bigtriangleup B)$; and
- $\bigtriangleup$ and $\cap$ are the usual set-theoretic operations, thus making $(\mathrm{MAlg}_f(Y, \lambda), \bigtriangleup, \cap)$ a Boolean ring (without unit).

It is well-known that $\mathrm{MAlg}_f(Y, \lambda)$ is homogeneous and complete as a metric structure.

Remark Note that Malicki and Sabok make a slightly different definition by naming the empty set and keeping only the operation $\bigtriangleup$, in line with Proposition 2.2. Here we prefer to provide a definition of the structure on $\mathrm{MAlg}_f(Y, \lambda)$ which makes finitely generated substructures easier to understand.

Lemma 2.11 *Finite tuples of disjoint finite measure subsets of (Y, λ) are ample and relevant.*

Proof The proof of [13, Lemma 6.2] adapts verbatim. □

Lemma 2.12 $\mathrm{MAlg}_f(Y, \lambda)$ *locally has finite automorphisms and has the extension property.*

Proof Every finitely generated substructure of $\mathrm{MAlg}_f(Y, \lambda)$ has a unit X so that we may see it as a substructure of the measure algebra $\mathrm{MAlg}(X, \lambda_X)$, which up to rescaling is the measure algebra over a standard probability space. The result then follows from [15, Lemma 8.1 and Lemma 8.2]. □

As a consequence of Malicki's theorem [13, Theorem 3.4], we thus have the following result.

Theorem 2.13 *The group $\mathrm{Aut}(Y, \lambda)$ of measure-preserving transformation of an infinite σ–finite standard measured space has the automatic continuity property.*

Corollary 2.14 (Kallman [6]) *The group $\mathrm{Aut}(Y, \lambda)$ has a unique Polish group topology.*

Remark Let $\mathrm{MAlg}_1(Y, \lambda)$ denote the closed set of all $A \in \mathrm{MAlg}_f(Y, \lambda)$ whose measure is at most 1. It is easy to check that the $\mathrm{Aut}(Y, \lambda)$–action on $\mathrm{MAlg}_1(Y, \lambda)$ is approximately oligomorphic and that $\mathrm{Aut}(Y, \lambda)$ is a closed subgroup of the isometry group of $\mathrm{MAlg}_1(Y, \lambda)$. By Ben Yaacov and Tsankov [3, Theorem 2.4], we conclude that $\mathrm{Aut}(Y, \lambda)$ is a Roelcke-precompact Polish group.

2.4 A characterization of automatic continuity

We finally use the previous results to characterize automatic continuity for $\mathrm{Aut}(Y,\eta)$, where (Y,η) be a standard Borel space equipped with a Borel σ–finite measure η, possibly with atoms. Recall that for such a measure there are only countably many atoms and they have finite measure (by σ–finiteness), and that each atom is a singleton (because Y is standard). Let us say that the η–**atomic multiplicity** of a positive real r is the (possibly infinite) number of atoms in Y whose measure is equal to r.

Theorem 2.15 *Let (Y,η) be a standard Borel space equipped with a Borel σ–finite measure η. Then the following are equivalent:*

(i) $\mathrm{Aut}(Y,\eta)$ *has the automatic continuity property.*

(ii) There are only finitely many positive reals whose η–atomic multiplicity belongs to $[2,+\infty[$.

Proof We first prove the contrapositive of (i)⇒(ii). Suppose there are infinitely many positive reals whose η–atomic multiplicity belongs to $[2,+\infty[$ and enumerate them as $(r_n)_{n\in\mathbb{N}}$. Then if A_n is the set of atoms of measure r_n, we see that each A_n is $\mathrm{Aut}(Y,\eta)$–invariant and we thus get natural surjection:

$$\mathrm{Aut}(Y,\eta)\twoheadrightarrow\prod_n\mathfrak{S}(A_n)$$

For each n, let σ_n be the signature map $\mathfrak{S}(A_n)\twoheadrightarrow\{\pm1\}$. By composing our previous homomorphism with $(\sigma_n)_{n\in\mathbb{N}}$ we get a continuous surjection $\mathrm{Aut}(Y,\eta)\twoheadrightarrow\{\pm1\}^{\mathbb{N}}$. Since the latter has $2^{2^{\aleph_0}}$ distinct homomorphisms onto $\{\pm1\}$ (indeed each ultrafilter on $\mathbb{N}$ provides such a homomorphism) and there are at most $2^{\aleph_0}$ continuous homomorphisms $\mathrm{Aut}(Y,\eta)\to\{\pm1\}$, we conclude that $\mathrm{Aut}(Y,\eta)$ does not have the automatic continuity property.

We now prove (ii)⇒(i). Let $(r_i)_{i=1}^n$ be the reals whose η–atomic multiplicity belongs to $[2,+\infty[$ and let A_i be the set of atoms of measure r_i. Let $(s_j)_{j\in J}$ denote the reals whose η–atomic multiplicity is infinite and let B_j be the set of atoms of measure s_j. Finally, let $\eta_{n.a.}$ be the non-atomic part of η. We then have a decomposition

$$\mathrm{Aut}(Y,\eta)=\mathrm{Aut}(Y,\eta_{n.a.})\times\prod_{i=1}^n\mathfrak{S}(A_i)\times\prod_{j\in J}\mathfrak{S}(B_j), \tag{3}$$

where $\mathfrak{S}(B_j)$ is equipped with the topology of pointwise convergence, viewing B_j as a discrete set.

Let us show that $\mathrm{Aut}(Y,\eta_{n.a.})$, $\prod_{i=1}^n\mathfrak{S}(A_i)$ and $\prod_{j\in J}\mathfrak{S}(B_j)$ have automatic continuity. Since η is σ–finite, $\eta_{n.a.}$ also is. We then have three cases to check.

- If $\eta_{n.a.}$ is trivial, $\mathrm{Aut}(Y, \eta_{n.a.})$ also is and hence has automatic continuity.
- If $\eta_{n.a.}$ is finite, $\mathrm{Aut}(Y, \eta_{n.a.})$ has automatic continuity by Ben Yaacov, Berenstein and Melleray [2, Theorem 6.2].
- If $\eta_{n.a.}$ is infinite, $\mathrm{Aut}(Y, \eta_{n.a.})$ has automatic continuity by Theorem 2.13.

The group $\prod_{i=1}^{n} \mathfrak{S}(A_i)$ is finite and thus has automatic continuity. Finally the group $\prod_{j\in J} \mathfrak{S}(B_j)$ is a countable product of groups with ample generics and hence has ample generics. By Kechris and Christian Rosendal [9, Theorem 1.10] it has automatic continuity.

Since any finite product of groups with automatic continuity has automatic continuity, we conclude from (3) that $\mathrm{Aut}(Y, \eta)$ has the automatic continuity property. □

3 The group of non-singular transformations

3.1 Preliminaries

A **standard probability space** is a standard Borel space equipped with a Borel nonatomic probability measure. All such spaces are isomorphic, and we fix from now on such a standard probability space (X, μ).

A Borel bijection T of (X, μ) is called **non-singular** if the pushforward measure $T_*\mu$ is equivalent to μ, that is, if for all Borel $A \subseteq X$, we have $\mu(A) = 0$ if and only if $\mu(T^{-1}(A)) = 0$. Denote by $\mathrm{Aut}^*(X, \mu)$ the group of non-singular Borel bijections of (X, μ), two such bijections being identified if they coincide up to measure zero.

The **strong topology** on $\mathrm{Aut}^*(X, \mu)$ is a metrizable group topology defined by declaring that a sequence (T_n) of elements of $\mathrm{Aut}^*(X, \mu)$ strongly converges to $T \in \mathrm{Aut}^*(X, \mu)$ if for all Borel $A \subseteq X$, one has $\mu(T_n(A) \bigtriangleup T(A)) \to 0$ and

$$\left\| \frac{d(T_{n*}\mu)}{d\mu} - \frac{d(T_*\mu)}{d\mu} \right\|_1 \to 0. \tag{4}$$

We refer the reader to Danilenko and Silva [4] for more on this topology, which is actually a Polish group topology. Our purpose here will be to show that it is the unique Polish group topology one can put on $\mathrm{Aut}^*(X, \mu)$.

For $T \in \mathrm{Aut}^*(X, \mu)$, we define as before its **support** to be the Borel set:

$$\mathrm{supp}\, T := \{x \in X : T(x) \neq x\}$$

Note that we have again the following relation: for all $S, T \in \mathrm{Aut}^*(X, \mu)$, $\mathrm{supp}(STS^{-1}) = S(\mathrm{supp}\, T)$.

We denote by $\mathrm{Aut}(X,\mu)$ the group of measure-preserving transformations of (X,μ), which is a closed subgroup of $\mathrm{Aut}^*(X,\mu)$. Similarly to Proposition 2.2, $\mathrm{Aut}(X,\mu)$ is the group of isometries of the **measure algebra** $\mathrm{MAlg}(X,\mu)$, defined as the set of Borel subsets of (X,μ) up to measure zero and equipped with the metric $d_\mu(A,B)=\mu(A\bigtriangleup B)$.

Finally, we will need the following easy fact: given two subsets $A,B\subseteq X$ of the same measure, there exists $T\in\mathrm{Aut}(X,\mu)$ supported on $A\cup B$ such that $T(A)=B$ up to measure 0.

3.2 Uniqueness of the Polish group topology of $\mathrm{Aut}^*(X,\mu)$

Theorem 3.1 *The strong topology is the unique Polish group topology on the group* $\mathrm{Aut}^*(X,\mu)$.

Proof Let us fix a countable dense subalgebra of $\mathrm{MAlg}(X,\mu)$ and enumerate it as $(A_n)_{n\in\mathbb{N}}$. For $m,n,k\in\mathbb{N}$, we let:

$$\mathbb{B}_{n,m,k}:=\left\{T\in\mathrm{Aut}^*(X,\mu):\mu(T(A_n)\setminus A_m)\leqslant\frac{1}{2^k}\right\}$$

Let us first show that the Borel group structure of $\mathrm{Aut}^*(X,\mu)$ is generated by the subsets $\mathbb{B}_{n,m,k}$.

By definition of the strong topology, we have that $\mathrm{Aut}^*(X,\mu)$ acts continuously on $\mathrm{MAlg}(X,\mu)$, so each $\mathbb{B}_{n,m,k}$ is closed, hence Borel. By density of (A_n) in $\mathrm{MAlg}(X,\mu)$ and faithfulness of the $\mathrm{Aut}^*(X,\mu)$–action on $\mathrm{MAlg}(X,\mu)$, we have that $(\mathbb{B}_{n,m,k})_{n,m,k\in\mathbb{N}}$ is a countable *separating* family of Borel subsets of the standard Borel space $\mathrm{Aut}^*(X,\mu)$. We conclude by Mackey [11, Theorem 3.3] that $(\mathbb{B}_{n,m,k})_{n,m,k\in\mathbb{N}}$ generates the Borel σ–algebra of $\mathrm{Aut}^*(X,\mu)$.

Let now τ be a Polish group topology on $G:=\mathrm{Aut}^*(X,\mu)$. To conclude that τ is the strong topology, our main task is to show that each $\mathbb{B}_{n,m,k}$ is τ–Baire-measurable.

We need a few easy facts from the previous section, adapted to our setup. The reader who already went through the previous section can safely skip them and go directly to Lemma 3.4.

For a Borel subset $A\subseteq X$, we let G_A denote the group of $T\in\mathrm{Aut}^*(X,\mu)$ such that $\operatorname{supp}T\subseteq A$. For a subset $F\subseteq\mathrm{Aut}^*(X,\mu)$, let $\mathcal{C}(F)$ denote its centraliser. We now repeat the short proofs of Lemma 2.7 and 2.8.

Lemma 3.2 *For all $A \subseteq X$ we have $\mathcal{C}(G_A) = G_{X\setminus A}$. In particular G_A is τ–closed.*

Proof We clearly have $G_{X\setminus A} \leqslant \mathcal{C}(G_A)$.

Take $T \notin G_{X\setminus A}$. Then there exists $B \subseteq A$ with $T(B)$ disjoint from B. But clearly T does not commute with any nontrivial element of $\mathrm{Aut}^*(X,\mu)$ supported in B, in particular $T \notin \mathcal{C}(G_A)$. □

Note that for all $T \in \mathrm{Aut}^*(X,\lambda)$ and all $A \subseteq X$, we have again $G_{T(A)} = TG_AT^{-1}$. Denote by $G(A,B)$ the set of $T \in \mathrm{Aut}^*(X,\mu)$ such that $T(A) \subseteq B$.

Lemma 3.3 *For all $A, B \subseteq X$, the set $G(A,B)$ is τ–closed.*

Proof By the equality (1), we have $G(A,B) = \{T \in \mathrm{Aut}^*(X,\mu) : T^{-1}G_AT \subseteq G_B\}$. So by the previous lemma $G(A,B)$ is the set of all $T \in \mathrm{Aut}^*(X,\mu)$ such that for all $U \in G_A$, TUT^{-1} commutes with every element of $G_{X\setminus B}$. This is clearly a τ–closed condition. □

We now make a crucial remark which relies on the automatic continuity property for $\mathrm{Aut}(X,\mu)$.

Lemma 3.4 *For $A \subseteq X$, let $H_A = \{T \in \mathrm{Aut}(X,\mu) : \operatorname{supp} T \subseteq A\}$. Then H_A is a τ–Borel subset of $\mathrm{Aut}^*(X,\mu)$.*

Proof By the automatic continuity property for $\mathrm{Aut}(X,\mu)$ [2, Theorem 6.3], we know that $\mathrm{Aut}(X,\mu)$ has to be a τ–Borel subset of $\mathrm{Aut}^*(X,\mu)$. But $H_A = G_A \cap \mathrm{Aut}(X,\mu)$ and by Lemma 3.2 we have that G_A is closed, so H_A is Borel. □

Let $n, m, k \in \mathbb{N}$; we finally prove that the set $\mathbb{B}_{n,m,k}$ is τ–Baire-measurable. We may assume that $\mu(A_m) < 1 - 1/2^k$ because otherwise $\mathbb{B}_{n,m,k} = \mathrm{Aut}^*(X,\mu)$. Let B a Borel set containing A_m such that $\mu(B) = \mu(A_m) + 1/2^k$.

Claim We have $\mathbb{B}_{n,m,k} = H_{X\setminus A_m} \cdot G(A_n, B)$.

Proof of claim Note that $H_{X\setminus A_m}$ is a group, and that $\mathbb{B}_{n,m,k}$ is left $H_{X\setminus A_m}$–invariant. Moreover since $\mu(B \setminus A_m) = 1/2^k$ we clearly have $G(A_n, B) \subseteq \mathbb{B}_{n,m,k}$ so $H_{X\setminus A_m} \cdot G(A_n, B) \subseteq \mathbb{B}_{n,m,k}$.

For the reverse inclusion, take $T \in \mathbb{B}_{n,m,k}$. Since $\mu(T(A_n)\setminus A_m) \leqslant 1/2^k$ and $\mu(B\setminus A_m) = 1/2^k$ we may find $U \in H_{X\setminus A_m}$ such that $U(T(A_n) \setminus A_m) \subseteq B \setminus A_m$. We conclude that $UT \in G(A_n, B)$ so $T \in H_{X\setminus A_m} \cdot G(A_n, B)$. □

By Lemma 3.3 the set $G(A_n, B)$ is τ–closed, while by Lemma 3.4 the set $H_{X\setminus A_m}$ is τ–Borel. Being the pointwise product of two Borel sets, the set $\mathbb{B}_{n,m,k}$ is analytic, hence Baire-measurable.

We can then conclude the proof in a standard manner: since the sets $\mathbb{B}_{n,m,k}$ generate the σ–algebra of the strong topology s on $\mathrm{Aut}^*(X,\mu)$, the identity map $(\mathrm{Aut}^*(X,\mu),\tau) \to (\mathrm{Aut}^*(X,\mu), s)$ is continuous by the Pettis Lemma [5, Theorem 2.3.2]. Being injective, its inverse is Borel by the Lusin–Suslin Theorem (Kechris [7, Theorem 15.1]), and thus continuous as well by one last application of the Pettis Lemma. □

Remark In the statement of Lemma 3.4, one can replace $\mathrm{Aut}(X,\mu)$ by the full group of any measure-preserving ergodic equivalence relation on (X,μ) and then run the exact same proof to obtain Theorem 1.4. Indeed such a group also has the automatic continuity property by a result of Kittrell and Tsankov [10].

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