Journal of Logic & Analysis 11:FT4 (2019) 1–56
ISSN 1759-9008

Sketches for arithmetic universes[1]

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Abstract: A theory of sketches for arithmetic universes (AUs) is developed, as a base-independent surrogate for suitable geometric theories.

A restricted notion of sketch, called here *context*, is defined with the property that every non-strict model is uniquely isomorphic to a strict model. This allows us to reconcile the syntactic, dealt with strictly using universal algebra, with the semantic, in which non-strict models must be considered.

For any context $\mathbb{T}$, a concrete construction is given of the AU $\mathbf{AU}\langle\mathbb{T}\rangle$ freely generated by it.

A 2-category $\mathfrak{Con}$ of contexts is defined, with a full and faithful 2-functor to the 2-category of AUs and strict AU-functors, given by $\mathbb{T} \mapsto \mathbf{AU}\langle\mathbb{T}\rangle$. It has finite pie limits, and also all pullbacks of a certain class of "extension" maps. Every object, morphism or 2-cell of $\mathfrak{Con}$ is a finite structure.

2010 Mathematics Subject Classification 18C30 (primary); 03G30 (secondary)

Keywords: topos, generalized space, geometric theory

1 Introduction

My 1999 paper "Topical Categories of Domains" [12], which used geometric logic to deal with toposes as generalized spaces, makes the following comment on the logic.

> It seems to us that in the work of the paper the infinities are restricted to those that can be accessed effectively through free algebra constructions. This emboldens us to hope that the full geometric logic is unnecessary, that it suffices to have coherent logic with assorted free algebras, and that [Grothendieck toposes] could be replaced by Joyal's *arithmetic universes* [AUs].

[1]This paper is included in the Proceedings of the Fifth Workshop on Formal Topology, Institut Mittag-Leffler, June 2015 (editors Thierry Coquand, Maria Emilia Maietti and Erik Palmgren).

Published: June 2019 https://doi.org/10.4115/jla.2019.11.FT4

The aim of the present paper is to set out a formal basis for doing this, while a companion paper [16] explores in greater depth the relation with toposes.

Our main accomplishment (Section 8) is to construct a 2-category $\mathfrak{Con}$ that can serve as one of generalized spaces. It is analogous to the 2-category $\mathfrak{Top}$ of Grothendieck toposes and geometric morphisms, or, more generally, to $\mathfrak{BTop}/\mathcal{S}$, where $\mathcal{S}$ is a chosen base elementary topos with nno, and $\mathfrak{BTop}/\mathcal{S}$ is the 2-category of bounded $\mathcal{S}$-toposes (see Johnstone [5, B4.4]).

Although $\mathfrak{Con}$ is based on a semantics using arithmetic universes, its objects are not AUs as such, but finite structures, presentations of AUs in the style of sketches. It is as if we defined Grothendieck toposes to *be* the geometric theories they classify, with no attempt to identify equivalent presentations. Similarly the 1-cells and 2-cells of $\mathfrak{Con}$ are finite structures. We prove (Theorem 51) that a 2-functor from $\mathfrak{Con}$ to AUs and strict AU-functors, mapping presentations $\mathbb{T}$ to the presented AUs $\mathbf{AU}\langle\mathbb{T}\rangle$, is full and faithful. This shows that we have defined the 1- and 2-cells in a sufficiently general manner.

We finish with a proof that the entire construction can be internalized in any AU, as anticipated by Joyal's original work on Gödel's Theorem.

1.1 Generalized spaces via arithmetic universes

The broad goal of the AU programme of Vickers [12] is to use arithmetic universes (AUs) to provide a predicative and base-free surrogate for Grothendieck toposes as generalized spaces, covering also point-free ungeneralized spaces such as locales or formal topologies, and rich enough to include significant mathematics such as the real line.

Briefly, a generalized space is presented by a geometric theory $\mathbb{T}$ that describes – as its models – the points of the space, and then the classifying topos $\mathcal{S}[\mathbb{T}]$ is a presentation-independent representation of the space. In the case of a theory for an ungeneralized space, the topos is the category of sheaves. In general, it embodies (as its internal logic) the "geometric mathematics" of colimits and finite limits that is generated by a generic model of $\mathbb{T}$. In other words, it is the Grothendieck topos presented by $\mathbb{T}$ as a system of generators and relations.

Continuous maps (geometric morphisms) can be expressed as models of one theory in the classifying topos of another – this is the universal property of "classifying topos" – and so this also provides a logical account of continuity. A map from $\mathbb{T}_1$ to $\mathbb{T}_2$ is defined by declaring "Let M be a model of $\mathbb{T}_1$", and then defining a model of $\mathbb{T}_2$

in that context (in other words, in $\mathcal{S}[\mathbb{T}_1]$, with M the generic model) and within the constraints of geometricity. From this point of view one might say that continuity *is* logical geometricity. See Vickers [15, 14] for more detailed accounts of the ideas.

A significant problem in the approach is that the notions of Grothendieck topos and classifying topos are parametrized by the base topos $\mathcal{S}$, whose objects supply the infinities needed for the infinite disjunctions needed in geometric logic, and for the infinite coproducts needed in the category of sheaves, for example to supply a natural numbers object. Technically, Grothendieck toposes (with respect to $\mathcal{S}$) are then elementary toposes equipped with bounded geometric morphisms to $\mathcal{S}$.

The aim of the AU programme is to develop a framework in which spaces, maps and other constructions can be described in a way that does not depend on any choice of base topos. In this "arithmetic" logic, disjunctions would all be finite, but some countable disjunctions could be dealt with by existential quantification over infinite objects (such as $\mathbb{N}$) defined using the list objects of AUs. Thus those infinite disjunctions become an intrinsic part of the logic – albeit a logic with aspects of a type theory – rather than being extrinsically defined by reference to a natural numbers object in a base topos.

Now suppose a geometric theory $\mathbb{T}$ can be expressed in this arithmetic way. We write $\mathbf{AU}\langle\mathbb{T}\rangle$ for its classifying AU, which stands in for the base-dependent classifying topos $\mathcal{S}[\mathbb{T}]$. Models of $\mathbb{T}_0$ in an AU $\mathcal{A}$ correspond to AU-functors from $\mathbf{AU}\langle\mathbb{T}_0\rangle$ to $\mathcal{A}$ (we shall return later to issues of strictness here), and an AU-functor $h\colon \mathbf{AU}\langle\mathbb{T}_1\rangle \to \mathbf{AU}\langle\mathbb{T}_0\rangle$ will, by composition, transform models of $\mathbb{T}_0$ into models of $\mathbb{T}_1$. It is fruitfully thought of as a point-free map between "spaces of models" of the two theories. In particular, for any base topos $\mathcal{S}$ with nno, h will transform the generic model of $\mathbb{T}_0$ in $\mathcal{S}[\mathbb{T}_0]$ into a model of $\mathbb{T}_1$ and so induce a geometric morphism from $\mathcal{S}[\mathbb{T}_0]$ to $\mathcal{S}[\mathbb{T}_1]$. Thus a result expressed using AUs can provide a single statement of a topos result valid over any base topos with nno. This is explored in greater depth in Vickers [16].

It is already known that a range of results proved using geometric logic can in fact be expressed in the setting of AUs. Maietti and Vickers [9] have developed some techniques for dealing with the fact that AUs are not Cartesian closed in general, nor even Heyting pretoposes.

This would be fully predicative, in that it does not at any point rely on the impredicative theory of elementary toposes (with their power objects). Instead of a predicative geometric theory of Grothendieck toposes, parametrized by an impredicative base elementary topos, we have a predicative arithmetic logic of AUs that is itself internalizable in AUs, and so depends on a predicative ambient logic. (This internalizability aspect will be seen in, eg, Section 9, where we give a concrete construction of the AU presented by a context.)

1.2 Outline of paper

Since the objects of our $\mathfrak{Con}$ are to be presentations of AUs, our first task is to show that we do indeed have a suitable notion of presentation of an AU, analogous (at least in its universal characterization, though not in its concrete construction) to the way a geometric theory presents its classifying topos. Section 2, after some initial results about AUs, accomplishes this by showing in detail the known result that there is a Cartesian theory of AUs; specifically we use the approach via quasiequational theories of Palmgren and Vickers [10]. (This algebraic mode of presenting the AU is foundationally much more robust than that used for classifying toposes. The route to sheaves uses presheaves to adjoin "arbitrary" colimits, and depends on a chosen base topos to explain what "arbitrary" means.)

In principle, we could now use the algebraic generators and relations for AUs as "arithmetic" theories for our generalized spaces. However, we find that the universal algebra, with its fundamental reliance on equations, is too "strict". It is difficult to get beyond the semantics in which an algebraic expression for – say – a pullback must, strictly, denote "the canonical" pullback.

In Section 3 we develop an alternative approach using sketches. Traditionally in sketch theory models are defined non-strictly, so that the cone for a pullback may denote *any* suitable pullback. However, it is convenient to consider strict models at the same time and thus get a comparison between them. Our general rule of thumb, more evident in Vickers [16], is that syntax, and algebra more generally, should be strict, while semantics is non-strict.

In Section 4 we gain control over this interaction by giving *extension* rules for building a (finite) sketch out of nothing. The sketches thus obtained are our AU-"contexts", and have the important property (Section 4.2) that every non-strict model of a context has a canonical strict isomorph. A side effect of this restriction is that it is impossible for a context to stipulate equality between objects, which in any case would be categorically unnatural.

At this point we have the object transformation from contexts to AUs, $\mathbb{T} \mapsto \mathbf{AU}\langle\mathbb{T}\rangle$. Although we don't care if different contexts map to isomorphic AUs, we do want a full and faithful 2-functor: so our task is to define the 1-cells and 2-cells in $\mathfrak{Con}$ so that they match exactly the (strict) AU-functors between the AUs $\mathbf{AU}\langle\mathbb{T}\rangle$.

Section 5 shows how the 1-cells between contexts are constructed. The universal characterization of $\mathbf{AU}\langle\mathbb{T}_1\rangle$ tells that morphisms $\mathbb{T}_0 \to \mathbb{T}_1$ interpret the ingredients of $\mathbb{T}_1$ as structure in $\mathbf{AU}\langle\mathbb{T}_0\rangle$, so our task is to describe how that structure is derived

from $\mathbb{T}_0$. Only finitely much structure is needed to interpret the finite sketch $\mathbb{T}_1$, and our notion of *equivalence extension* provides extension rules that adjoin finite amounts of uniquely determined structure. Defining a morphism (a *context map*) as a sketch homomorphism from $\mathbb{T}_1$ to an equivalence extension of $\mathbb{T}_0$ provides enough morphisms to get a full functor to AUs. This stage is a localization, forcing the equivalence extensions to be isomorphisms.

However, we we still require faithfulness. This requires us to compare interpretations and discover equalities required by the algebraic rules of AUs. Section 6 deals with this with its notion of *object equality,* inductive rules to show when the same derivation has been applied twice over.

The next Section 7 contains miscellaneous results about context maps and object equalities (as particular isomorphisms) in preparation for Section 8. Here we put the constructions together, taking context maps modulo object equality, to define our 2-category $\mathfrak{Con}$ and explore its properties.

In Section 9 we finally prove the fullness and faithfulness (Theorem 51), showing (Theorem 49) that $\mathbf{AU}\langle\mathbb{T}\rangle$ is isomorphic to a hom-category of $\mathfrak{Con}$. We finish the section with a proof that the entire construction can be internalized in any AU, as anticipated by Joyal's original work.

Note on notation: Our default order of composition of morphisms is *diagrammatic*. For applicational order we shall always use "$\circ$". For diagrammatic order we shall occasionally show this explicitly using ";".

2 Arithmetic universes

We follow Maietti [8, 9] in defining Joyal's arithmetic universes (AUs) to be *list arithmetic pretoposes*.

More explicitly, as a pretopos an AU $\mathcal{A}$ is a category equipped with finite limits, stable finite disjoint coproducts and stable effective quotients of equivalence relations. (For more detailed discussion, see, eg, Johnstone [5, A1.4.8].)

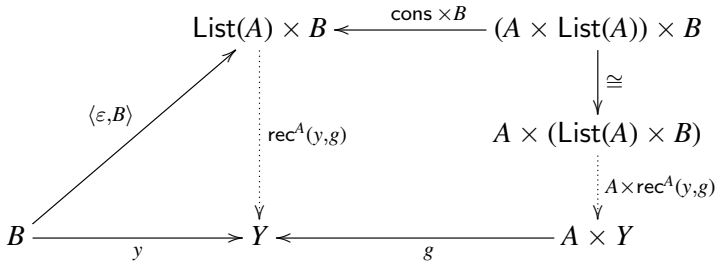
In addition, it has, for each object A, a *parametrized list object* $\mathsf{List}(A)$. It is equipped with morphisms

$$1 \xrightarrow{\varepsilon} \mathsf{List}(A) \xleftarrow{\mathsf{cons}} A \times \mathsf{List}(A)$$

(where $\mathsf{cons}(a,x) = a : x$ is the list x with a appended at the front) and whenever we have the solid part of the following diagram, there is a unique fillin of the dotted parts

to make a commutative diagram.

(1)

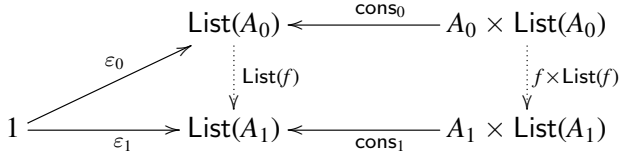


In other words, this recursively defines $r = \mathsf{rec}^A(y, g)$ by:

$$r([], b) = y(b)$$
$$r(a : x, b) = g(a, r(x, b))$$

Note that the use of B rather than 1 corresponds to this being a *parameterized* list object; that is to say, it makes $\mathsf{List}(A) \times B$ a list object in the slice over B.

Remark 1 *For future reference, we note the functoriality of* List*: If* $f : A_0 \to A_1$*, then there is a unique* $\mathsf{List}(f) : \mathsf{List}(A_0) \to \mathsf{List}(A_1)$ *making the following diagram commute.*



To see this, consider the action of A_0 *on* $\mathsf{List}(A_1)$ *by*

$$\mathsf{List}(A_1) \xleftarrow{\mathsf{cons}_1} A_1 \times \mathsf{List}(A_1) \xleftarrow{f \times \mathsf{List}(A_1)} A_0 \times \mathsf{List}(A_1).$$

We assume the AU structure specifies canonical choices of those colimits, limits and list objects. This enables an approach using the universal algebra of Cartesian theories, with (partial) algebraic operators for the canonical choices.

We shall use the quasiequational form of Cartesian theories, Palmgren and Vickers [10]. Our Cartesian theory of AUs will use primitive operators as suggested by the following proposition, although that particular choice of primitives is not critical. Doubtless there are more efficient characterizations, and the techniques in the remainder of the present paper are intended to be equally applicable for other choices.

Proposition 2 *A category* $\mathcal{A}$ *is an arithmetic universe iff the following hold.*

(1) *$\mathcal{A}$ has a terminal object and pullbacks (hence all finite limits).*

(2) *$\mathcal{A}$ has an initial object and pushouts (hence all finite colimits), and they are stable under pullback.*

(3) *Balance (unique choice): if a morphism is both mono and epi, then it is iso.*

(4) *Exactness: any equivalence relation is effective (it is the kernel pair of its own coequalizer).*

(5) *$\mathcal{A}$ has parameterized list objects.*

Proof $\Rightarrow$: (1), (3) and (4) are properties of any pretopos, as is the existence of stable finite coproducts. (5) is a postulate for AUs.

Hence it remains to show the existence of stable coequalizers for all pairs $X \rightrightarrows Y$. First, because as a pretopos $\mathcal{A}$ is regular, we can take the image R of X in $Y \times Y$, a relation on Y. Next, in a pretopos we can find the reflexive-symmetric closure of R. Next, in an AU we can find the free category over any directed graph, and in particular we can find the transitive closure of any relation. We end up with the equivalence relation generated by R, and at each step, we keep the same set of morphisms from Y that compose equally with the two morphisms from X or R. Thus the coequalizer of the equivalence relation, existing because of exactness of $\mathcal{A}$ as pretopos, also serves as a coequalizer for $X \rightrightarrows Y$.

Stability follows from the stability, in a pretopos, of image factorization and of coequalizers of equivalence relations.

$\Leftarrow$: Two properties of pretoposes remain to be proved. First, for binary coproduct, the injections are monic and disjoint. Second, any epi is the coequalizer of its kernel pair.

Consider a coproduct cocone (bottom row here) pulled back along one of the injections. The two squares are pullbacks, Δ is diagonal.

$$\begin{array}{ccccc} K & \xrightarrow{p_2} & X & \xleftarrow{q_2} & L \\ {\scriptstyle p_1}\downarrow & \overset{\Delta}{\curvearrowleft} & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle q_1} \\ X & \xrightarrow[i_1]{} & X+Y & \xleftarrow[i_2]{} & Y \end{array}$$

By stability the top row is a coproduct cocone, and so we can define a copairing morphism $f = [\mathsf{Id}_K, q_2\Delta]\colon X \to K$ and calculate that $f = p_2^{-1}$. Since the kernel pair projection p_2 is an isomorphism, it follows that i_1 is monic.

We can now replace K and the projections by X and identity morphisms, and the coproduct property of the top row can be rephrased as follows: every triangle $L \overset{\curvearrowright}{\underset{q_2}{\rightrightarrows}} X$ commutes.

Now consider $!\colon 0 \to L$. By stability of the initial object, we see that $!$ is mono. It is also epi. For suppose we have two morphisms $f_1, f_2 \colon L \to Z$. Consider the following diagram, where j_1 and j_2 are coproduct injections.

$$\begin{array}{ccc} L & \xrightarrow{q_2} & X \\ {\scriptstyle f_1}\downarrow\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle j_2} \\ Z & \xrightarrow{j_1} & Z+X \end{array}$$

Both squares must commute, and we already know that j_1 is monic, so $f_1 = f_2$. By balance, it follows that $L \cong 0$.

It remains to show that any epi $e\colon X \to Y$ is the coequalizer of its kernel pair. In fact we show something slightly more general, without assuming e is epi. Let K_2 be its kernel pair, with projections p_1 and p_2, and let $e'\colon X \to Y'$ be their coequalizer, with factorization $e = e'e''$. Then we show that e'' is mono. (If e is epi then so too is e'', so e'' is an isomorphism by balance.)

In the following diagram, where the bottom row is pulled back along e, we see that the top row is a split fork and hence a coequalizer.

$$\begin{array}{ccccc} K_3 & \underset{p_{23}}{\overset{p_{13}}{\rightrightarrows}} & K_2 & \xrightarrow{p_2} & X \\ {\scriptstyle p_{12}}\downarrow & \Delta_{122} & \downarrow{\scriptstyle p_1} & \Delta & \downarrow{\scriptstyle e} \\ K_2 & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & X & \xrightarrow{e} & Y \end{array}$$

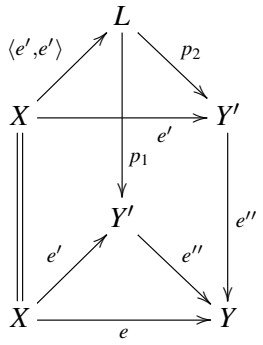
Now consider pulling back the factorization $e'e''$:

$$\begin{array}{ccccc} K_2 & \xrightarrow{f'} & X' & \xrightarrow{f''} & X \\ {\scriptstyle p_1}\downarrow & \Delta & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} \\ X & \xrightarrow[e']{} & Y' & \xrightarrow[e'']{} & Y \end{array}$$

By stability, we see that f' is a coequalizer of p_{13} and p_{23}, and by comparing with the split fork we find $\Delta f' = f''^{-1}$. We deduce that we can replace X' by X, and have a pullback square:

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\ Y' & \xrightarrow[e'']{} & Y \end{array}$$

Now we can use the above pullback square, turned on its side, to pull back the factorization $e'e''$ along e''.



By stability (for e' as coequalizer) we see that p_2 is an isomorphism and so e'' is monic. □

It follows that the theory of AUs is quasiequational as in Palmgren and Vickers [10].

Definition 3 *We present the quasiequational theory of AUs as follows. Some of the operators and axioms are already set out explicitly in [10] or in Maietti and Vickers [9], and we refer back to them for some of the details.*

- *(See [10, Example 4].) The ingredients of the theory of categories: sorts* $\mathrm{obj}, \mathrm{arr}$, *total operators* $\mathsf{d}, \mathsf{c} \colon \mathrm{arr} \to \mathrm{obj}$ *(domain and codomain) and* $\mathsf{id} \colon \mathrm{obj} \to \mathrm{arr}$ *(identity morphisms), and partial* $\circ \colon \mathrm{arr}^2 \to \mathrm{arr}$ *(composition, as binary operator, in applicational order).*
- *(See [10, Section 6].) Ingredients for terminal objects: a constant* $1 \colon \mathrm{obj}$ *and operator* $!^1_{\cdot} \colon \mathrm{obj} \to \mathrm{arr}$ *(unique morphism to terminal) with axioms*

$$\top \vdash\!\!\!\xrightarrow{X} \mathsf{d}(!^1_X) = X \wedge \mathsf{c}(!^1_X) = 1$$

$$\mathsf{c}(u) = 1 \vdash\!\!\!\xrightarrow{u} u = !^1_{\mathsf{d}(u)} \quad \textit{(for uniqueness).}$$

- *(See [10, Section 6.1].) Ingredients for pullbacks:*
  First, operators $\mathsf{p}^1_{\cdot,\cdot}, \mathsf{p}^2_{\cdot,\cdot} \colon \mathrm{arr}^2 \to \mathrm{arr}$ *for pullback projections. If* u_1 *and* u_2 *have a common codomain, then* $\mathsf{p}^1_{u_1,u_2}$ *and* $\mathsf{p}^2_{u_1,u_2}$ *are the two projections from the pullback. We also write* p_{u_1,u_2} *for* $u_1 \circ \mathsf{p}^1_{u_1,u_2}$, *the diagonal of the pullback square, and* P_{u_1,u_2} *for* $\mathsf{d}(\mathsf{p}_{u_1,u_2})$, *the pullback object itself.*

  Next, a pairing operator $\langle \cdot, \cdot \rangle_{\cdot,\cdot} \colon \mathrm{arr}^4 \to \mathrm{arr}$, *with* $\langle v_1, v_2 \rangle_{u_1,u_2}$ *the fillin to the pullback of* u_1 *and* u_2 *for a cone* (v_1, v_2). *It is defined iff the four arrows make*

a commutative square in the obvious way, and it has the expected domain and codomain and commutativities.

For uniqueness of fillins,

$$\mathsf{c}(w) = \mathsf{P}_{u_1,u_2} \overset{w,u_1,u_2}{\longmapsto} w = \left\langle \mathsf{p}^1_{u_1,u_2} \circ w, \mathsf{p}^2_{u_1,u_2} \circ w \right\rangle_{u_1,u_2} .$$

- *We shall also use some derived notation in a self-explanatory way for products* $X \times Y = \mathsf{P}_{!^1_X,!^1_Y}$. *For example, the projections are* $\mathsf{p}^i_{X,Y} = \mathsf{p}^i_{!^1_X,!^1_Y}$, *and the fillins require no subscripts.*

  Also, we shall write $\mathsf{eq}_{u_1,u_2} \colon \mathsf{E}_{u_1,u_2} \to X$ *for the equalizer of* $u_1, u_2 \colon X \to Y$, *defined in a canonical way. Specifically,*

$$\mathsf{eq}_{u_1,u_2} \triangleq \mathsf{p}^1_{\langle \mathsf{id}(X),u_1\rangle,\langle \mathsf{id}(X),u_2\rangle}$$

  (The two projections are equal.)

- *Ingredients for initial objects and pushouts. They are dual to those for terminal objects and pullbacks. (We can also express coproducts and coequalizer, by dualizing the treatment for products and equalizers.)*

  For initial objects we have a constant $0 \colon \mathrm{obj}$, *an operator* $!^0_\cdot \colon \mathrm{obj} \to \mathrm{arr}$, *and a conditional equation that if* $\mathsf{d}(u) = 0$, *then* $u = !^0_{\mathsf{c}(u)}$.

  Operators $\mathsf{q}^1_{\cdot,\cdot}, \mathsf{q}^2_{\cdot,\cdot} \colon \mathrm{arr}^2 \to \mathrm{arr}$ *are for pushout injections. If* u_1 *and* u_2 *have a common domain, then* $\mathsf{q}^1_{u_1,u_2}$ *and* $\mathsf{q}^2_{u_1,u_2}$ *are the two injections to the pushout. We also write* q_{u_1,u_2} *for* $\mathsf{q}^1_{u_1,u_2} \circ u_1$, *the diagonal of the pushout square, and* Q_{u_1,u_2} *for* $\mathsf{c}(\mathsf{q}_{u_1,u_2})$, *the pushout object itself.*

  Next, a copairing operator $[\cdot,\cdot]_{\cdot,\cdot} \colon \mathrm{arr}^4 \to \mathrm{arr}$, *with* $[v_1, v_2]_{u_1,u_2}$ *the fillin from the pushout of* u_1 *and* u_2 *for a cocone* (v_1, v_2).

  For uniqueness of fillins,

$$\mathsf{d}(w) = \mathsf{Q}_{u_1,u_2} \overset{w,u_1,u_2}{\longmapsto} w = \left[w \circ \mathsf{q}^1_{u_1,u_2}, w \circ \mathsf{q}^2_{u_1,u_2} \right]_{u_1,u_2} .$$

- *Ingredients for stability of colimits under pullback.*

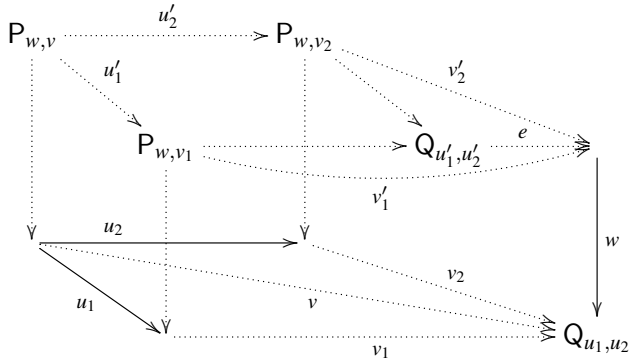
  For stability of the initial object, it suffices to say that any morphism with 0 *for codomain is an isomorphism:*

$$\mathsf{c}(u) = 0 \overset{u}{\longmapsto} !^0_{\mathsf{d}(u)} \circ u = \mathsf{id}(\mathsf{d}(u))$$

  For stability of pushouts, we have an operator $\mathsf{stab}_{\cdot,\cdot}(\cdot) \colon \mathrm{arr}^3 \to \mathrm{arr}$, *with* $\mathsf{stab}_{u_1,u_2}(w)$ *defined iff* $\mathsf{c}(w) = \mathsf{Q}_{u_1,u_2}$. *To express its equations, we define notation as shown in this diagram. Here the base diamond is a pushout, and it is*

pulled back along w. The inner top diamond is also a pushout, with fillin e, and the equations for the operator, when it is defined, are those required to say that $\mathsf{stab}_{u_1,u_2}(w) = e^{-1}$.

(2)



$$
\begin{aligned}
v &= \mathsf{q}_{u_1,u_2} \\
v_i &= \mathsf{q}^i_{u_1,u_2} \\
v_i' &= \mathsf{p}^1_{w,v_i} \\
u_i' &= \left\langle \mathsf{p}^1_{w,v}, u_i \circ \mathsf{p}^2_{w,v} \right\rangle_{w,v_i} \\
e &= \left[v_1', v_2' \right]_{u_1',u_2'}
\end{aligned}
$$

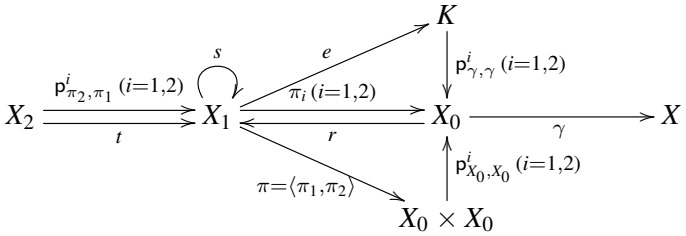
- *Ingredients for balance (unique choice).*

  We have an operator $\mathsf{uc}\colon \mathrm{arr} \to \mathrm{arr}$, *with* $\mathsf{uc}(u)$ *defined if* $\mathsf{p}^1_{u,u} = \mathsf{p}^2_{u,u}$ *and* $\mathsf{q}^1_{u,u} = \mathsf{q}^2_{u,u}$ *(ie u is monic and epi). When it is defined we have* $\mathsf{uc}(u) = u^{-1}$.

- *Ingredients for exactness.*

  We have an operator $\mathsf{ex}\colon \mathrm{arr}^5 \to \mathrm{arr}$, *with* $\mathsf{ex}(\pi_1, \pi_2, r, s, t)$ *defined if* $\langle \pi_1, \pi_2 \rangle$ *describes a binary relation, with r, s, t expressing reflexivity, symmetry and transitivity.*

(3)



We require that π is monic; that $X_2 = \mathsf{P}_{\pi_2,\pi_1}$; *that r, s, t compose correctly with π_1 and π_2; that γ is the canonical coequalizer of π_1 and π_2; that K is the kernel*

pair of γ; and that e is the fillin. Our characterizing equations for ex *are to say*

$$\mathsf{ex}(\pi_1, \pi_2, r, s, t) = e^{-1}.$$

- *Ingredients for list objects.*

  We have total operators $\varepsilon, \mathsf{cons}\colon \mathsf{obj} \to \mathsf{arr}$ *for the principal structure, and we also write* $\mathsf{List}(A)$ *for* $\mathsf{c}(\varepsilon(A))$.

  For the fillins we have a partial operator $\mathsf{rec}^{\cdot}(\cdot,\cdot)\colon \mathsf{obj} \times \mathsf{arr}^2 \to \mathsf{arr}$.

  Let us write, temporarily, the following (see diagram (1)):

$$\begin{aligned}
\phi^A(y,g) &\triangleq \mathsf{c}(y) = \mathsf{c}(g) \wedge \mathsf{d}(g) = A \times \mathsf{c}(g)\\
\psi^A_{y,g}(r) &\triangleq y = r \circ \langle B, \varepsilon\rangle\\
&\qquad \wedge\, g \circ (r \times A) \circ \cong = r \circ (B \times \mathsf{cons}(A))\\
\text{where} \qquad B &\triangleq \mathsf{d}(y)\\
\langle B, \varepsilon\rangle &\triangleq \left\langle \mathsf{id}(B), \varepsilon(A) \circ !^1_B \right\rangle\\
\cong &\triangleq \left\langle \mathsf{p}^1_{A,\mathsf{List}A} \circ \mathsf{p}^1_{A\times\mathsf{List}A,B}, \left\langle \mathsf{p}^2_{A,\mathsf{List}A} \circ \mathsf{p}^1_{A\times\mathsf{List}A,B}, \mathsf{p}^2_{A\times\mathsf{List}A,B}\right\rangle\right\rangle
\end{aligned}$$

  Here ϕ expresses the domain of definition of the fillin $\mathsf{rec}^A(y,g)$, *and ψ is the condition (on r) that it needs to satisfy. The axioms are now:*

$$\begin{aligned}
\top &\vdash^{A} \phi^A(\varepsilon(A), \mathsf{cons}(A)) \wedge \mathsf{d}(\varepsilon(A)) = 1\\
\phi^A(y,g) &\vdash^{A,y,g} \psi^A_{y,g}(\mathsf{rec}^A(y,g))\\
\mathsf{rec}^A(y,g)\!\downarrow &\vdash^{A,y,g} \phi^A(y,g)\\
\psi^A_{y,g}(r) &\vdash^{A,y,g,r} r = \mathsf{rec}^A(y,g)
\end{aligned}$$

Definition 4 *A* strict AU-functor *from one AU to another is a homomorphism for the quasiequational theory of AUs. In other words, it is a functor that preserves terminals, pullbacks, intials, pushouts and list objects* strictly.

An AU-functor *is a functor that preserves those constructions (and hence also all finite limits and finite colimits) up to isomorphism.*

In AUs we have a general ability to construct free algebras. For theories given by finite product (FP) sketches this is described in some detail in Maietti [7]. That paper also alludes to the ability to generalize to finite limit (FL) sketches, in other words to Cartesian theories. Palmgren and Vickers [10] give a general account of the Cartesian construction, and it is valid in AUs.

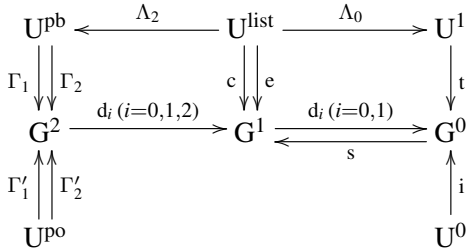
3 AU-sketches

We shall be interested in generators and relations for AUs, but we shall generally not express them directly using the quasiequational algebra. Instead, we borrow the ideas of sketches.

In their most general form (in this section), they are equivalent in expressive power to the quasiequational algebra. In one direction we make this explicit by giving the equations that correspond to ingredients of a sketch. The other direction comes down to the question of how to express the operators in the quasiequational theory of AUs, and this is done using the equivalence extensions of Section 5. The operators for pullbacks and their projections, and analogous operators for other universal constructions, can be captured using the "universals" in a sketch. The operators for fillins, being the unique solutions to certain equational constraints on edges, can be captured with edges constrained by suitable commutativities.

Our main reason for using the sketches is that they give us better control of the important issue of strictness of models (Section 3.1). In Section 4 we shall restrict our attentions from general sketches to "contexts", finite sketches for which non-strict models have canonical strict isomorphs.

Definition 5 *An* AU-sketch *(or just* sketch*) is a structure with sorts and operations as shown in this diagram.*



They are required to satisfy the following equations:

$$\mathrm{sd}_0 = \mathrm{sd}_1 = \mathrm{id}$$

$$\mathrm{d}_0\mathrm{d}_1 = \mathrm{d}_2\mathrm{d}_0 \quad \mathrm{d}_0\mathrm{d}_0 = \mathrm{d}_1\mathrm{d}_0 \quad \mathrm{d}_2\mathrm{d}_1 = \mathrm{d}_1\mathrm{d}_1$$

$$\Gamma_1\mathrm{d}_1 = \Gamma_2\mathrm{d}_1$$

$$\Gamma'_1\mathrm{d}_1 = \Gamma'_2\mathrm{d}_1$$

$$\Lambda_2\Gamma_1\mathrm{d}_1\mathrm{d}_1 = \Lambda_0\mathrm{t}$$

$$\mathrm{ed}_0 = \Lambda_0\mathrm{t} \quad \mathrm{cd}_0 = \Lambda_2\Gamma_1\mathrm{d}_0\mathrm{d}_0$$

$$\mathrm{ed}_1 = \mathrm{cd}_1 = \Lambda_2\Gamma_2\mathrm{d}_0\mathrm{d}_1$$

If $\mathbb{T}_1$ and $\mathbb{T}_2$ are sketches, then a homomorphism *of sketches from $\mathbb{T}_1$ to $\mathbb{T}_2$, written $f\colon \mathbb{T}_1 \ll \mathbb{T}_2$, is defined in the obvious way: a family of carrier functions, one for each sort, preserving the operators.*

However, we shall consider two sketch homomorphisms to be equal *if they agree merely on G^0 and G^1.*

We write $\mathbf{Sk}_{\ll}$ for the category of sketches and sketch homomorphisms.

The structures are a formalization of the sketches well known from eg Barr and Wells [1], but adapted for AUs. We shall describe the parts in more detail below, but as a preliminary let us introduce some language that indicates the connection. The elements of G^0, G^1 and G^2 are referred to as *nodes, edges* and *commutativities*.

The elements of the other sorts are *universals,* and specify universal properties of various kinds for their *subjects*. For example, an element of U^{pb} is a *pullback universal* and corresponds to a cone in a finite limit sketch. Its subjects are the pullback node and the three projection edges of the pullback cone. Similarly, an element of $\mathrm{U}^{\mathrm{list}}$ is a *list universal*. Its subjects are the list object and the two structure maps, for ε and cons. It will also have indirect subjects, since it needs terminal and pullback universals to express the domains of the structure maps.

Any sketch can be used as a system of generators (the nodes and edges) and relations to present an AU. We shall list these implied relations in the general description below. Note that in each case the equations constraining sketches ensure that all the terms used in the relations are defined.

$\mathrm{G}^0, \mathrm{s}, \mathrm{G}^1, \mathrm{d}_0, \mathrm{d}_1$ form the graph (which we take to be reflexive) of nodes and edges, declaring some objects and arrows and specifying their identities, domains and codomains. The elements of G^0 and G^1 are taken as generators of sorts obj and arr. The implied relations are:

$$\mathsf{id}(X) = \mathrm{s}(X) \quad \mathsf{d}(u) = \mathrm{d}_0(u) \quad \mathsf{c}(u) = \mathrm{d}_1(u)$$

G^2, with d_0, d_2 and d_1, comprises the commutativities, stipulating commutative triangles $\overset{\mathrm{d}_1}{\underset{\mathrm{d}_0}{\bullet \rightrightarrows}}\ {}_{\mathrm{d}_2}$. Given a triangle of edges $X \overset{w}{\underset{u}{\rightrightarrows}} Y \underset{v}{\rightarrow} Z$, we shall write $uv \sim_{XYZ} w$ for the existence of a commutativity with that triangle. (Note the *diagrammatic order*.) We shall also write $u \sim_{XY} u'$ for a *unary commutativity*, meaning a commutativity $\mathrm{s}(X)u \sim_{XXY} u'$. We shall omit the node subscripts where convenient.

Equationally, each commutativity ϖ corresponds to a relation

$$\mathrm{d}_2(\varpi) \circ \mathrm{d}_0(\varpi) = \mathrm{d}_1(\varpi).$$

U^1 and U^{pb}, using $\mathrm{t}, \Gamma^1, \Gamma^2$, are universals for finite limits, here terminal objects or pullbacks. For each pullback universal (in U^{pb}) we describe the cone by two commutative triangles (Γ^1, Γ^2), the two halves of the pullback square. For universals $\varpi \in \mathrm{U}^1$ or $\varpi \in \mathrm{U}^{\mathrm{pb}}$, the implied relations are:

$$\mathrm{t}(\varpi) = 1$$

$$\mathrm{d}_0(\Gamma^\lambda(\varpi)) = \mathsf{p}^\lambda_{\mathrm{d}_2(\Gamma^1(\varpi)),\mathrm{d}_2(\Gamma^2(\varpi))} \quad (\lambda = 1, 2)$$

$\mathrm{U}^0, \mathrm{i}, \mathrm{U}^{\mathrm{po}}, \Gamma'^1, \Gamma'^2$ are similar, and dual, for finite colimits:

$$\mathrm{i}(\varpi) = 0$$

$$\mathrm{d}_2(\Gamma'^\lambda(\varpi)) = \mathsf{q}^\lambda_{\mathrm{d}_0(\Gamma'^1(\varpi)),\mathrm{d}_0(\Gamma'^2(\varpi))} \quad (\lambda = 1, 2)$$

$\mathrm{U}^{\mathrm{list}}$, for *list universals*, is novel, but works on similar principles. For a list universal $\varpi \in \mathrm{U}^{\mathrm{list}}$, $\mathrm{e}(\varpi)$ and $\mathrm{c}(\varpi)$ supply the primary structure morphisms ε and cons for $\mathsf{List}(A(\varpi))$, where $A(\varpi) = \mathrm{d}_1(\mathrm{d}_0(\Gamma^1(\Lambda_2(\varpi))))$. The domains of the structure morphisms (1 and $A(\varpi) \times \mathsf{List}(A(\varpi))$) are limits, and Λ_0, Λ_2 supply universals to stipulate them. Note that, since we need a terminal anyway, we might as well reuse it as the terminal needed for a product as special case of pullback. The implied relations, which are in addition to those already implied for $\Lambda_0(\varpi)$ and $\Lambda_2(\varpi)$, are:

$$\varepsilon(A(\varpi)) = \mathrm{e}(\varpi) \quad \mathsf{cons}(A(\varpi)) = \mathrm{c}(\varpi)$$

3.1 Models

Definition 6 *Let $\mathbb{T}$ be a sketch and $\mathcal{A}$ an AU.*

A strict model *of $\mathbb{T}$ in $\mathcal{A}$ is an interpretation of nodes and edges in $\mathbb{T}$ as objects (carriers) and morphisms (operations) in $\mathcal{A}$, in a way that respects all the implied relations of the sketch strictly, ie* up to equality*.*

A model *of $\mathbb{T}$ in $\mathcal{A}$ is an interpretation of nodes and edges in $\mathbb{T}$ as objects and morphisms in $\mathcal{A}$, in a way that respects up to equality all the domains, codomains, identities and commutativities of the sketch, and up to isomorphism all the universals. In other words, the subjects of each universal have to have the appropriate universal property, but do not have to be the canonical construction.*

A homomorphism between models of $\mathbb{T}$ in an AU $\mathcal{A}$ comprises a carrier morphism for each node, together commuting with the operations in the appropriate way. This can be conveniently expressed as a model of $\mathbb{T}$ in the comma category $\mathcal{A} \downarrow \mathcal{A}$, also an AU.

(See Maietti and Vickers [9] for results concerning these comma categories and their AU structure, and also for the related pseudopullback $\mathcal{A} \downarrow_{\cong} \mathcal{A}$.)

*We write $\mathbb{T}$-**Mod**$(\mathcal{A})$ for the category of models of $\mathbb{T}$ in $\mathcal{A}$, and $\mathbb{T}$-**Mod**$_s(\mathcal{A})$ for the full subcategory of strict models.*

If $h\colon \mathcal{A} \to \mathcal{B}$ is an AU-functor, then we obtain a functor

$$\mathbb{T}\text{-}\mathbf{Mod}(h)\colon \mathbb{T}\text{-}\mathbf{Mod}(\mathcal{A}) \to \mathbb{T}\text{-}\mathbf{Mod}(\mathcal{B}).$$

*If h is a strict AU-functor, then $\mathbb{T}$-**Mod**(h) preserves strictness of models.*

As we remarked earlier, any sketch $\mathbb{T}$ can be treated as generators and relations for presenting an arithmetic universe, using the fact that the theory of AUs is Cartesian (see Palmgren and Vickers [10]). We shall write this as $\mathbf{AU}\langle\mathbb{T}\rangle$. It is the AU version of the notion of classifying category, and we shall call it the *classifying AU* for $\mathbb{T}$. It is the analogue of the classifying topos when geometric logic is replaced by an arithmetic form.

The injection of generators provides a strict *generic model* M_G of $\mathbb{T}$ in $\mathbf{AU}\langle\mathbb{T}\rangle$, and then the universal property is that any strict model M of $\mathbb{T}$ in an AU $\mathcal{A}$ extends uniquely to a strict AU-functor $h\colon \mathbf{AU}\langle\mathbb{T}\rangle \to \mathcal{A}$ for which $\mathbb{T}$-**Mod**(h) transforms M_G to M, up to equality. (This is analogous to the universal property for classifying toposes, with strict AU-functors corresponding to the inverse image parts of geometric morphisms, but note that the AU property is stricter.)

Thus strict models of $\mathbb{T}$ are in bijection with strict AU-functors out of $\mathbf{AU}\langle\mathbb{T}\rangle$. We have already seen that a non-strict AU functor out of $\mathbf{AU}\langle\mathbb{T}\rangle$ will also give rise to a non-strict model of $\mathbb{T}$, the non-strict image of the generic model. However, the universal property does not allow us to recover the non-strict AU-functor from the model. Hence the universal algebra is less precise for non-strict models and AU-functors. In Section 4 we restrict the notion of sketch in a way that gives better control over the non-strict models.

Definition 7 *Let $f\colon \mathbb{T}_1 \leqslant \mathbb{T}_0$ be a homomorphism of sketches,*[2] *and M a model of $\mathbb{T}_0$ in $\mathcal{A}$. Then the f-*reduct *of M, written $M|f$, is the model of $\mathbb{T}_1$ whose carriers and operations are got by taking those for M corresponding by f.*

It is a model because the sketch homomorphism transforms all the implied relations of $\mathbb{T}_1$ into implied relations of $\mathbb{T}_0$.

[2] Why this order of 1 and 0? Because in Section 7 we shall think of f as a *map* from the space of models of $\mathbb{T}_0$ to that of $\mathbb{T}_1$, acting by model reduction.

Model reduction is functorial with respect to model homomorphisms, and so the assignment $\mathbb{T} \mapsto \mathbb{T}$-**Mod**$(\mathcal{A})$ is the object part of a contravariant category-valued functor $(-)$-**Mod**$(\mathcal{A})$ on $\mathbf{Sk}_{\leqslant}$, with sketch homomorphisms assigned to model reduction.

Model reduction preserves strictness.

By taking the f-reduct of the generic model in $\mathbf{AU}\langle\mathbb{T}_0\rangle$, we get a strict model of $\mathbb{T}_1$ in $\mathbf{AU}\langle\mathbb{T}_0\rangle$ and hence a strict AU-functor $\mathbf{AU}\langle f\rangle\colon \mathbf{AU}\langle\mathbb{T}_1\rangle \to \mathbf{AU}\langle\mathbb{T}_0\rangle$.

3.2 Examples of sketches

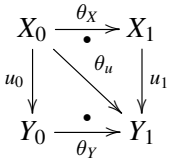
Here are some examples of sketches. Again, the notation is adapted to thinking of the sketch as prescribing a class of models in each AU.

(1) The empty sketch $\mathbb{1}$ has a unique model in any AU.

(2) The sketch $\mathbb{O}$ has a single node and its identity edge and nothing else. Its models in $\mathcal{A}$ are the objects of $\mathcal{A}$.

(3) Let $\mathbb{T}$ and $\mathbb{U}$ be two sketches. Their *disjoint union* is called the *product* sketch $\mathbb{T} \times \mathbb{U}$. Its models are pairs of models of $\mathbb{T}$ and $\mathbb{U}$. We also use notation such as $\mathbb{T}^2$ for $\mathbb{T} \times \mathbb{T}$.

(4) Let $\mathbb{T}$ be a sketch. The *hom sketch* $\mathbb{T}^{\to}$ is made as follows. First, take two disjoint copies of $\mathbb{T}$ as in $\mathbb{T}^2$, distinguished by subscripts 0 and 1. These give two sketch homomorphisms $i_0, i_1\colon \mathbb{T} \to \mathbb{T}^{\to}$. Next, for each node X of $\mathbb{T}$, adjoin an edge $\theta_X\colon X_0 \to X_1$; and, for each edge $u\colon X \to Y$ of $\mathbb{T}$, adjoin an edge $\theta_u\colon X_0 \to Y_1$ together with two commutativities to make a commutative diagram:



Then a model of $\mathbb{T}^{\to}$ comprises a pair M_0, M_1 of models of $\mathbb{T}$, together with a homomorphism $\theta\colon M_0 \to M_1$.

The assignment $\mathbb{T} \mapsto \mathbb{T}^{\to}$ extends functorially to sketch homomorphisms, and then i_0 and i_1 become natural transformations.

(5) We shall also write $\mathbb{T}^{\to\to}$ for the theory of composable pairs of homomorphisms of $\mathbb{T}$-models, and analogously for greater numbers of arrows. In fact, for any finite category $\mathcal{C}$ we can write $\mathbb{T}^{\mathcal{C}}$ for the theory of $\mathcal{C}$-diagrams of models of $\mathbb{T}$. (Actually, finiteness is not important here, as we have not set any finiteness conditions on the sketch $\mathbb{T}$. But it will be important for contexts.)

The existence of $\mathbb{T}^{\to}$ enables us to define *2-cells* in $\mathbf{Sk}_{\ll}$. If $f_0, f_1 : \mathbb{T}_1 \ll \mathbb{T}_0$, then a 2-cell from f_0 to f_1 is a sketch homomorphism $\alpha : \mathbb{T}_1^{\to} \ll \mathbb{T}_0$ such that $i_\lambda \alpha = f_\lambda$ $(\lambda = 0, 1)$. We also say that α is *between* $\mathbb{T}_0$ and $\mathbb{T}_1$.

2-cells cannot yet be composed, either vertically or horizontally, because edges cannot be composed in sketches. However, we do have *whiskering* on both sides, using either αf or $f^{\to}\alpha$, and it has all relevant associativities.

We can also take reducts along 2-cells. If M is a model of $\mathbb{T}_0$ in $\mathcal{A}$, then the homomorphism $M|\gamma : M|f_0 \to M|f_1$ uses the carrier functions of $\mathbb{T}_1^{\to}$ as interpreted in $\mathbb{T}_0$.

4 Extensions, contexts

In this section we define a class of sketches, the *contexts*, for which every non-strict model can be made strict in a canonical way.

What makes this non-trivial is that in general, strictness has the ability to assert *equalities* between sorts by making a single node X the subject of two different universals, for example making it both $A \times B$ and $\mathsf{List}\, C$. In a non-strict model this just requires $A \times B \cong \mathsf{List}\, C$, whereas strictness would require equality; and in an AU it can easily happen that the first holds but not the second. Such equalities are not really the concern of category theory, so better would be to have universals specifying two nodes X_1 and X_2 as $A \times B$ and $\mathsf{List}\, C$ respectively, and then to specify an isomorphism $X_1 \cong X_2$. Strict models of that are unproblematic.

To enforce the latter kind we shall use each universal with a simple definitional effect, defining its subjects fresh from some other ingredients (nodes and edges) defined previously. This leads to our notion of *extension* of sketches. To prepare for this, we introduce a notion of *protoextension*, in which the syntactic notion of freshness is represented using categorical coproducts.

We say that a set is *strongly finite* if it is isomorphic to a finite cardinal $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$. Equivalently, it is Kuratowski finite, has decidable equality, and can be equipped with a decidable total order.

Definition 8 *A sketch homomorphism $i' : \mathbb{U} \ll \mathbb{U}'$ is a* protoextension *if for each sketch sort Ξ, we have that $\mathbb{U}'_\Xi$ can be expressed as a coproduct $\mathbb{U}_\Xi + \delta\Xi$, with i'_Ξ a coproduct injection and $\delta\Xi$ strongly finite.*

Proposition 9 *Let $i' \colon \mathbb{U} \lessdot \mathbb{U}'$ be a sketch homomorphism. Then the following are equivalent.*

(1) *i' is a protoextension.*

(2) *i' is a pushout of some* strongly finite sketch inclusion*, by which we mean a sketch monomorphism $i \colon \mathbb{T} \lessdot \mathbb{T}'$ in which $\mathbb{T}$ and $\mathbb{T}'$ are strongly finite (ie their carriers are).*

Proof (2) $\Rightarrow$ (1): Let $i \colon \mathbb{T} \lessdot \mathbb{T}'$ be a strongly finite sketch inclusion. For each sketch sort Ξ, we can write $\mathbb{T}'_\Xi$ as a coproduct $\mathbb{T}'_\Xi = \mathbb{T}_\Xi + \delta\Xi$. (Informally in such a situation, we shall often write $\mathbb{T}'$ as $\mathbb{T} + \delta\mathbb{T}$, although this is not a coproduct of sketches. $\delta\mathbb{T}$ is not a sketch in its own right, as some of its structure may lie in $\mathbb{T}$.)

Now let $f \colon \mathbb{T} \lessdot \mathbb{U}$ be an arbitrary sketch homomorphism. Then the pushout $i' \colon \mathbb{U} \lessdot \mathbb{U}'$ of i along f can be constructed as follows.

For each sketch sort Ξ, we let $\mathbb{U}'_\Xi = \mathbb{U}_\Xi + \delta\Xi$. For elements of $\mathbb{U}_\Xi$, their structure is determined as in $\mathbb{U}$. Now suppose $\varpi \in \delta\Xi$. In $\mathbb{T} + \delta\mathbb{T}$, each structural element of ϖ (ie the result of applying a sketch operator) is in either $\mathbb{T}$ or $\delta\mathbb{T}$. If the latter, then we keep it there in $\mathbb{U}'$. If the former, then we apply f to get it in $\mathbb{U}$. We obtain a commutative diagram of sketches that is readily verified to be a pushout:

$$\begin{array}{ccc} \mathbb{U} + \delta\mathbb{T} & \xleftarrow{f+\delta\mathbb{T}} & \mathbb{T} + \delta\mathbb{T} \\ {\scriptstyle i'}\uparrow & & \uparrow{\scriptstyle i} \\ \mathbb{U} & \xleftarrow[f]{} & \mathbb{T} \end{array}$$

From the construction, i' is clearly a protoextension.

(1) $\Rightarrow$ (2): Use the elements of the $\delta\Xi$s as generators for a sketch $\mathbb{T}'$, with relations to say that the sketch operations in $\mathbb{U}'$ are preserved *insofar as* they stay in the $\delta\Xi$s. Then $\mathbb{T}'$ is strongly finite, and the inclusion of the $\delta\Xi$s in $\mathbb{U}'$ induces a sketch homomorphism $f' \colon \mathbb{T}' \lessdot \mathbb{U}'$.

Let $\mathbb{T}$ be the pullback of i' and f', with projections i and f. i is monic, because i' is. Also, in a coproduct the images of the injections are decidable subobjects, and it follows that the carriers of $\mathbb{T}$ are decidable subobjects of those of $\mathbb{T}'$, and so $\mathbb{T}$ too is strongly finite.

Applying the construction of (2) $\Rightarrow$ (1), we recover i'. □

It was already clear from the definition that protoextensions are closed under composition. From Proposition 9 it is also clear that protoextensions i' are closed under pushout along any sketch homomorphism g. The pushout is called the *reindexing* of i' along g, and written $g(i')$.

4.1 Extensions: the definition

In the following definition, central to the whole paper, we restrict our proto-extensions by restricting the strongly finite sketch inclusions i of Proposition 9. First we define a finite family of inclusions $i\colon \mathbb{T} \prec \mathbb{T} + \delta\mathbb{T}$ that are generic for *simple extensions*, and then a general *extension* (written $\subset$) is a composite of simple extensions.

For each kind of simple extension, using an inclusion i, the sketch homomorphism $f\colon \mathbb{T} \prec \mathbb{U}$ that we reindex along can be understood as a *data configuration* in $\mathbb{U}$, some tuple of elements satisfying some equations. Thus each kind of simple extension can be understood as a sketch transformation that takes *data* (given by f) and delivers a *delta*, according to Proposition 9.

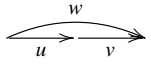
Since any sketch homomorphism will transform extension data to extension data, we see that reindexing (as sketch pushout) is got by applying the same extension to the transformed data. For an extension $c\colon \mathbb{T}_1 \subset \mathbb{T}'_1$, we shall typically write a reindexing square as:

(4)
$$\begin{array}{ccc} \mathbb{T}'_0 & \xleftarrow{\varepsilon} & \mathbb{T}'_1 \\ {\scriptstyle f(c)}\uparrow & & \uparrow{\scriptstyle c} \\ \mathbb{T}_0 & \xleftarrow[f]{} & \mathbb{T}_1 \end{array} .$$

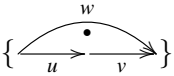
Definition 10 *A* simple extension *is a proto-extension got as a pushout of one of the following strongly finite sketch inclusions $i\colon \mathbb{T} \prec \mathbb{T} + \delta\mathbb{T}$. Where we don't specify $\delta\Xi$, it is empty.*

(1) *(Adding a new* primitive node*) No data (ie $\mathbb{T}$ is $\mathbb{1}$). Deltas:*
$$\delta \mathrm{G}^0 = \{*\}$$
$$\delta \mathrm{G}^1 = \{\mathrm{s}(*)\}$$

(2) *(A simple* functional *extension, by a new* primitive edge*) Data: $(X, Y) \in \mathrm{G}^0 \times \mathrm{G}^0$. Delta:*
$$\delta \mathrm{G}^1 = \{X \dashrightarrow Y\}$$
In other words $\delta \mathrm{G}^1 = 1 = \{*\}, \mathrm{d}_0(*) = X$, $\mathrm{d}_1(*) = Y$. We shall use similar informal notation in the other cases. Note that the "delta" edges are shown dotted.

(3) *(Adding a commutativity) Data:*  . *Delta:*

$$\delta \mathrm{G}^2 = \{ \quad (uv \sim w)$$

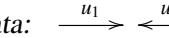


In other words $\delta \mathrm{G}^2 = \{*\}$ *with* $\mathrm{d}_0(*) = u$, $\mathrm{d}_2(*) = v$, $\mathrm{d}_1(*) = w$.

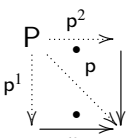
(4) *(Adding a terminal) No data. Deltas:*

$$\begin{aligned} \delta \mathrm{U}^1 &= \{*\} \\ \delta \mathrm{G}^0 &= \{\mathrm{t}(*)\} \\ \delta \mathrm{G}^1 &= \{\mathrm{s}(\mathrm{t}(*))\} \end{aligned}$$

Adding an initial object is similar.

(5) *(Adding a pullback) Data:*  *. Deltas:*

$$\delta \mathrm{U}^{\mathrm{pb}} = \left\{ \quad \right\}$$

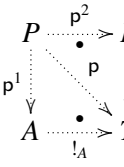


$$\begin{aligned} \delta \mathrm{G}^2 &= \{\mathsf{p}^1 u_1 \sim \mathsf{p}, \mathsf{p}^2 u_2 \sim \mathsf{p}\} \\ \delta \mathrm{G}^1 &= \{\mathsf{p}^1, \mathsf{p}, \mathsf{p}^2, \mathrm{s}(\mathsf{P})\} \\ \delta \mathrm{G}^0 &= \{\mathsf{P}\} \end{aligned}$$

Adding a pushout is similar.

(6) *(Adding a list object) Data:* $A \in \mathrm{G}^0$*. Deltas:*

$$\begin{aligned} \delta \mathrm{U}^{\mathrm{list}} &= \{* = (T \overset{\varepsilon}{\dashrightarrow} L \overset{\mathsf{cons}}{\dashleftarrow} P)\} \\ \delta \mathrm{U}^1 &= \{\Lambda_0(*) = T\} \\ \delta \mathrm{U}^{\mathrm{pb}} &= \left\{ \quad \right\} \end{aligned}$$



$$\begin{aligned} \delta \mathrm{G}^2 &= \{\mathsf{p}^1 !_A \sim \mathsf{p}, \mathsf{p}^2 !_L \sim \mathsf{p}\} \\ \delta \mathrm{G}^1 &= \{\varepsilon, \mathsf{cons}, \mathsf{p}^1, \mathsf{p}, \mathsf{p}^2, !_A, !_L, \mathrm{s}(T), \mathrm{s}(L), \mathrm{s}(P)\} \\ \delta \mathrm{G}^0 &= \{T, L, P\} \end{aligned}$$

An extension *of sketches is a proto-extension that can be expressed as a finite composite of simple extensions. We write* $\mathbb{T}_1 \subset \mathbb{T}_2$.

An AU-context *is an extension of the empty sketch* $\mathbb{1}$.

Proposition 11 *Let* $c\colon \mathbb{T} \subset \mathbb{T}'$ *be an extension of sketches. Then for each node or edge* α *in* $\mathbb{T}'$ *that is not in* $\mathbb{T}$*, we can construct an expression* $w_\alpha(\overrightarrow{n}, \overrightarrow{e})$ *in the quasiequational theory of AUs with the following properties.*

(1) *Its free variables* $\overrightarrow{n}, \overrightarrow{e}$ *are in bijection with some (finitely many) of the nodes and edges of* $\mathbb{T}$*, and the primitive nodes and edges adjoined in the extension.*

(2) *Its definedness can be deduced from the domains and codomains of edges as specified in* $\mathbb{T}'$*. (Recall that the quasiequational theory has partial operators.)*

(3) *Let* M *be a strict model of* $\mathbb{T}'$*. Then the interpretation of* α *in* M *can be found by evaluating* w_α *on the interpretations of* $\overrightarrow{n}$ *and* $\overrightarrow{e}$*.*

Proof By inspecting the cases, we see that for a simple extension each fresh node or edge can be described *uniquely* in one of the ways listed in the following tables. In each case we also give the corresponding w_α in the third column. The free variables (eg u_1 and u_2 for the pullback) correspond to the components of the data for that extension, and the domains and codomains implied by the data suffice to guarantee that w_α is defined. The definition of strict model then tells us that the interpretation of α is got by evaluating w_α.

For nodes: the node is primitive or takes one of the forms

$\mathrm{t}(\varpi)$	$(\varpi \in \mathrm{U}^1)$	1
$\mathrm{i}(\varpi)$	$(\varpi \in \mathrm{U}^0)$	0
$\mathrm{d}_0\mathrm{d}_1\Gamma_1(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{pb}})$	P_{u_1,u_2}
$\mathrm{d}_1\mathrm{d}_1\Gamma'_1(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{po}})$	Q_{u_1,u_2}
$\mathrm{d}_1\mathrm{e}(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{list}})$	$\mathsf{List}(A)$

For edges: the edge is primitive or takes one of the forms

$\mathrm{s}(X)$	$(X \in \mathrm{G}^0)$	$\mathsf{id}(X)$
$\mathrm{d}_1\Gamma_1(\varpi)$ or $\mathrm{d}_0\Gamma_i(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{pb}})$	p_{u_1,u_2} or $\mathsf{p}^i_{u_1,u_2}$
$\mathrm{d}_1\Gamma'_1(\varpi)$ or $\mathrm{d}_2\Gamma'_i(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{po}})$	q_{u_1,u_2} or $\mathsf{q}^i_{u_1,u_2}$
$\mathrm{e}(\varpi)$, $\mathrm{c}(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{list}})$	$\varepsilon(A)$, $\mathsf{cons}(A)$
$\mathrm{d}_2\Gamma_1\Lambda_2(\varpi)$ or $\mathrm{d}_2\Gamma_2\Lambda_2(\varpi)$	$(\varpi \in \mathrm{U}^{\mathrm{list}})$	$!^1(A)$ or $!^1(\mathsf{List}(A))$

These facts are preserved by subsequent simple extensions, since those forms are only introduced for fresh nodes or edges. It follows that the facts remain true for the composite extension.

We can now apply an induction on the number of composed simple extensions, and use the equations for strict models that are imposed by the sketch structure. We look explicitly at universals for pullbacks and list. Other situations are similar or easier.

First, consider a simple extension in the form of a pullback universal ϖ, defined on the configuration $\xrightarrow{u_1} \xleftarrow{u_2}$. The relations for such a universal tell us that the fresh edge $\mathrm{d}_0\Gamma_1(\varpi)$ has to be interpreted as $\mathsf{p}^1_{u_1,u_2}$, and we use induction to find the expressions for u_1 and u_2. (The base case is if they are primitive or in $\mathbb{T}$.) The other fresh edges and the fresh node are dealt with in a similar way. Note that if $\varpi = \Lambda_2(\varpi')$ for some $\varpi' \in \mathrm{U}^{\mathrm{list}}$, then the subjects of ϖ are treated in the same way, but $u_i = \mathrm{d}_2(\Gamma_i(\varpi))$ gets its expression from ϖ'.

Now consider a simple extension in the form of a list universal ϖ, on object A. All the fresh nodes and edges have expressions in terms of A. For $\mathrm{e}(\varpi)$ and $\mathrm{c}(\varpi)$ and their codomain this is clear. Next, from the terminal universal $\Lambda_0(\varpi)$ we have $\mathrm{t}(\Lambda_0(\varpi)) = 1$. Because this appears as a vertex in the pullback square $\Lambda_2(\varpi)$, it follows from the AU axioms that $\mathrm{d}_2(\Gamma_1(\Lambda_2(\varpi))) = !^1_A$ and $\mathrm{d}_2(\Gamma_2(\Lambda_2(\varpi))) = !^1_{\mathsf{List}(A)}$. Since these are u_1 and u_2 in the treatment of the pullback universals, it only remains to deal with the easy case of the identity morphisms. □

Note that a primitive edge can acquire equality with an AU-expression by subsequently added commutativities. We shall use this later for introducing AU operators that have not been mentioned so far in extensions.

4.2 Strictness results

The reason for introducing extensions was for an important property that non-strict interpretations can be reinterpreted strictly in a canonical way. The following definition and lemma will make this precise, in a generality whose usefulness becomes more apparent in Vickers [16].

Definition 12 *Let $\mathbb{T} \subset \mathbb{T}'$ be a sketch extension. A model of $\mathbb{T}'$ is* strict for the extension *if, for each universal, each subject node or edge α is equal to the result of its expression w_α.*

Note that a model of $\mathbb{T}'$ is strict in its own right iff it is strict for the extension and its $\mathbb{T}$-reduct is strict.

Lemma 13 *Suppose, as in the diagram below, an extension $\mathbb{T}_1 \subset \mathbb{T}'_1$ is reindexed along a sketch homomorphism $\mathbb{T}_1 \leqslant \mathbb{T}_0$. Suppose also that in some AU $\mathcal{A}$ we have models M_0 and M'_1 of $\mathbb{T}_0$ and $\mathbb{T}'_1$, with an isomorphism $\phi\colon M_0|\mathbb{T}_1 \cong M'_1|\mathbb{T}_1$.*

$$\begin{array}{lccc} M'_0 & \mathbb{T}'_0 \xleftarrow{\varepsilon} \mathbb{T}'_1 & M'_0|\mathbb{T}'_1 \xrightarrow[\cong]{\phi'} M'_1 \\ & {\scriptstyle f(\subset)}\uparrow \quad \uparrow{\scriptstyle \subset} & \\ M'_0|\mathbb{T}_0 = M_0 & \mathbb{T}_0 \xleftarrow[f]{} \mathbb{T}_1 & M_0|\mathbb{T}_1 \xrightarrow[\cong]{\phi} M'_1|\mathbb{T}_1 \end{array}$$

Then there is a unique model M'_0 of $\mathbb{T}'_0$ and isomorphism $\phi'\colon M'_0|\mathbb{T}'_1 \cong M'_1$ such that

(1) $M'_0|\mathbb{T}_0 = M_0$,

(2) *M'_0 is strict for the extension $\mathbb{T}_0 \subset \mathbb{T}'_0$,*

(3) $\phi'|\mathbb{T}_1 = \phi$, *and*

(4) *ϕ' is equality on all the primitive nodes for the extension $\mathbb{T}_1 \subset \mathbb{T}'_1$.*

Proof It suffices to cover the cases for a simple extension $\mathbb{T}_1 \subset \mathbb{T}'_1$.

If the extension adjoins a primitive node X, then we can and must take its carrier in M'_0 to be equal to its carrier in M'_1, and the carrier function in ϕ' to be the identity.

Suppose the extension adjoins a primitive edge $u\colon X \to Y$. Then ϕ' must equal ϕ, and to preserve the homomorphism property we can and must define the operation for u in M'_0 to be $\phi(X)M'_1(u)\phi^{-1}(Y)$, using the operation in M'_1.

If the extension adjoins a new commutativity, then the morphism equation already holds in $M'_1|\mathbb{T}_1$ and hence in M_0, so we can and must take M'_0 and ϕ' to be given by the same data as M_0 and ϕ.

It remains only to examine the case where the extension adds a universal. We consider the case of a list universal, as the others are similar (and easier). M'_0 has to interpret the new nodes and edges in the canonical way. In particular, T, L and P are 1, $\mathsf{List}(A)$ and $A \times \mathsf{List}(A)$. Then the universal properties (of terminal object, list object and binary product) give canonical isomorphisms between those canonical interpretations in M'_0 and the corresponding interpretations (possibly non-canonical) in M'_1. The corresponding carrier morphisms of ϕ' can be defined to be those canonical isomorphisms, and indeed by the homomorphism properties and uniqueness of fillins they must be so defined. □

By considering the case where $\mathbb{T}_0 = \mathbb{T}_1 = \mathbb{1}$, we obtain:

Corollary 14 *Let $\mathbb{T}$ be a context, $\mathcal{A}$ an AU, and M_1 a model of $\mathbb{T}$. Then there is a unique isomorphism $\phi\colon M_0 \cong M_1$ such that M_0 is a strict model of $\mathbb{T}$ and ϕ is an identity on each primitive node.*

We call M_0 the *canonical strict isomorph* of M_1.

It follows that if $\mathbb{T}$ is a context, and $h\colon \mathcal{A} \to \mathcal{B}$ is a *non*-strict AU-functor, then we get a functor $\mathbb{T}\text{-}\mathbf{Mod}_s(h)\colon \mathbb{T}\text{-}\mathbf{Mod}_s(\mathcal{A}) \to \mathbb{T}\text{-}\mathbf{Mod}_s(\mathcal{B})$. Given a strict model M in $\mathcal{A}$, composing with h gives a non-strict model in $\mathcal{B}$, and we can then take its canonical strict isomorph. An important example, developed in Vickers [16], is when h is the inverse image part of a geometric morphism between two toposes with natural number objects.

4.3 Examples of contexts

Here are some examples of contexts. (cf. Section 3.2.)

(1) The sketches $\mathbb{1}$ and $\mathbb{O}$ are both contexts.

(2) If $\mathbb{T}_0$ and $\mathbb{T}_1$ are both contexts, then so is $\mathbb{T}_0 \times \mathbb{T}_1$. To be specific, we shall adjoin the ingredients of $\mathbb{T}_0$ first, so that $\mathbb{T}_0 \subset \mathbb{T}_0 \times \mathbb{T}_1$ is an extension and for $\mathbb{T}_1$ we just have a homomorphism $\mathbb{T}_1 \lessdot \mathbb{T}_0 \times \mathbb{T}_1$.

(3) If $\mathbb{T}$ is a context, then so is $\mathbb{T}^{\to}$. We take it that $i_0\colon \mathbb{T} \subset \mathbb{T}^{\to}$ is the extension. Similarly, $\mathbb{T}^{\to\to}$ is a context, with extensions $\mathbb{T} \xrightarrow[\subset]{i_0} \mathbb{T}^{\to} \xrightarrow[\subset]{i_{01}} \mathbb{T}^{\to\to}$, where i_{01} is the reindexing $i_1(i_0)$.

More generally, for any strongly finite category $\mathcal{C}$ we have that $\mathbb{T}^{\mathcal{C}}$ can be made a context. The order of simple extensions for it will depend on a total order given to each finite set involved.

(4) If $\mathbb{T}$ is a context, then it has an extension $\mathbb{T}^{ns}$ whose strict models are the *non-strict* models of $\mathbb{T}$. For each non-primitive node X, we adjoin a primitive node X' together with an isomorphism $X' \cong X$.

Note that we do need $\mathbb{T}$ to be a context here, not an arbitrary sketch. A model of $\mathbb{T}^{ns}$ is actually an isomorphic pair of two models, one strict and the other not. We need Corollary 14 to get this pair from any non-strict model.

(5) Without going into details, there is a context $\mathbb{R}$ for the theory of Dedekind sections. It is defined as outlined in Maietti and Vickers [9]. First, the natural numbers $\mathbb{N}$ can be defined as $\mathsf{List}(1)$. Their (decidable) order and arithmetic can be defined using the universal property. Then the rationals $\mathbb{Q}$ can be defined by standard techniques, together with their decidable order and arithmetic. Next, two nodes L and R are adjoined, with edges to $\mathbb{Q}$ and conditions to make them monic. Finally we add axioms to make (L, R) a Dedekind section.

(6) For various kinds of presentation of locales, there are context extensions $\mathbb{T}_0 \subset \mathbb{T}_1$ where a model of $\mathbb{T}_0$ is a presentation, and one of $\mathbb{T}_1$ is a presentation equipped with a point of the corresponding locale.

The same principle also applies in formal topology, with an inductively generated formal topology understood as a presentation.

For example, suppose we take the formal topologies as defined in Coquand et al. [3]. First we declare the base B, a poset. Next, the cover $\lhd_0$ can be adjoined as a node, with an edge to B. A node C is adjoined for a disjoint union of all the covering sets, with an edge to $\lhd_0$. The conditions on these can also be expressed using AU structure in a context $\mathbb{T}_0$. For $\mathbb{T}_1$ we adjoin to $\mathbb{T}_0$ a monic into B, together with conditions to make it a formal point.

Note that we have not attempted here to extract the full cover $\lhd$.

5 Equivalence extensions

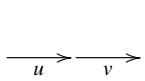
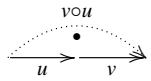
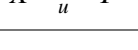
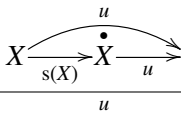
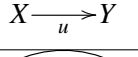
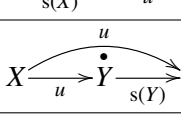
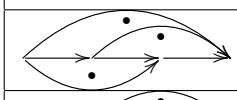
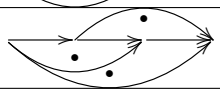
An equivalence extension is an extension, but one in which the simple extension steps are grouped together in a way that guarantees that the fresh ingredients (nodes, edges, properties, equations) introduced in the extension are all already known to exist uniquely. The most intricate parts are for the edges. In an ordinary extension, an unconstrained fresh edge can subsequently be specified uniquely up to equality by commutativities (equations). In an equivalence extension when we introduce an edge we must also document the justification for its existence (as a composite or a fillin; universal structure edges such as limit projections are introduced along with the universal objects). In addition, we must also include steps for proving equations between edges; this is to provide images for commutativities under a sketch morphism. These steps essentially codify the rules for congruences in universal algebra. (The reason this is not needed for nodes is that essentially algebraic theories of categories do not normally have any axioms to imply equations between objects.)

The game now is to describe simple equivalence extensions sufficient to generate all the operators of the the theory of AUs and all the arrow equalities generated by the axioms. (For object equalities see Section 6.)

Definition 15 *A* simple equivalence extension *is a proto-extension of one of the following forms (*or rules*). Note that each is in fact an extension.*

In each case, every node or edge introduced will, in any strict model, become equal to a certain AU expression in terms of the data. For nodes, which are all introduced by simple extensions of universal kind, this has already been covered in Definition 10. For edges the expressions are given in δG^1. Those expressions do indeed satisfy the commutativities listed in δG^2. On the other hand, any edges satisfying them will be equal to the expressions by the AU equations for uniqueness of fillins.

First, there are various rules associated with morphisms and their composition. They are summarized in this table.

Data	*Delta*	
		*composition*
		*left unit law*
		*right unit law*
		*left associativity*
		*right associativity*

Second, for each kind of universal (terminal, pullback, initial, pushout, list), we have three rules. The first will be the simple extension that introduces the corresponding node, the second will introduce fillins by adjoining a primitive edge with the appropriate equations, and the third will introduce equations for the uniqueness of fillins.

We illustrate this for pullbacks and for list objects. The rules for terminals, initials and pushouts follow the same principles as for pullbacks.

For pullbacks:

- *A simple extension for a pullback universal is also an equivalence extension.*

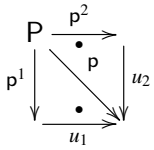
- *Suppose we have a pullback universal* $\varpi \in \mathrm{U}^{\mathrm{pb}}$*, and another cone given as* $\Delta_1, \Delta_2 \in \mathrm{G}^2$*, with*

$$\mathrm{d}_2(\Delta_i) = \mathrm{d}_2(\Gamma_i(\varpi)) = u_i$$

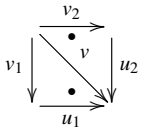
and

$$\mathrm{d}_1(\Delta_1) = \mathrm{d}_1(\Delta_2) = v.$$

ϖ *is*



Δ_1, Δ_2 *are*



Then our equivalence extension has

$$\delta\mathrm{G}^1 = \{w = \langle v_1, v_2\rangle_{u_1,u_2}\}$$

and

$$\delta\mathrm{G}^2 = \{w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2\}.$$

- *Suppose we have a pullback universal* $\varpi \in \mathrm{U}^{\mathrm{pb}}$ *as above, and edges* v_1, v_2, w, w' *with commutativities* $w\mathsf{p}^1 \sim v_1, w\mathsf{p}^2 \sim v_2, w'\mathsf{p}^1 \sim v_1, w'\mathsf{p}^2 \sim v_2$*. Then our equivalence extension has*

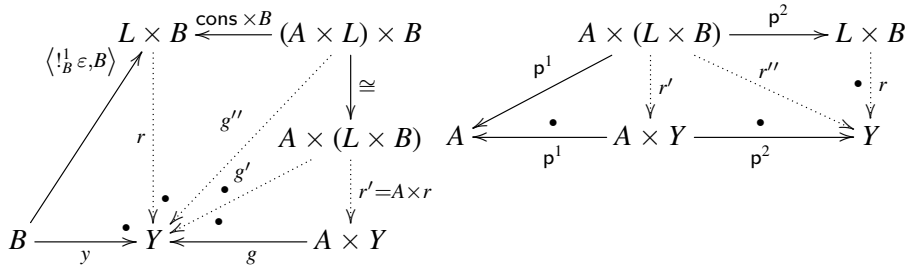
$$\delta\mathrm{G}^2 = \{w \sim w'\}.$$

For list objects:

- *A simple extension for a list universal is also an equivalence extension.*
- *Suppose we have a list universal* $\varpi \in \mathrm{U}^{\mathrm{list}}$ *with* $1 \xrightarrow{\varepsilon} L \xleftarrow{\mathsf{cons}} A \times L$*. Suppose (see diagram (1)) we also have nodes* B, Y*, pullback universals to specify nodes for* $L \times B$*,* $(A \times L) \times B$*,* $A \times (L \times B)$ *and* $A \times Y$*, edges* $B \xrightarrow{y} Y \xleftarrow{g} A \times Y$*, and edges for* $\left\langle !^1_B \varepsilon, B\right\rangle$*,* $\mathsf{cons} \times B$ *and the associativity isomorphism, together with auxiliary edges and commutativities needed to characterize them.*

  Using the notation of the following diagrams, our equivalence extension has $\delta\mathrm{G}^1 = \{r, r', r'', g', g''\}$*, where* $r = \mathsf{rec}^A(y, g)$*, and* $\delta\mathrm{G}^2$ *comprises the seven commutativities shown. The second diagram is what is needed to specify that* $r' = A \times r$*.*

(5)



- *Suppose, given the configuration for the above fillin, we have two solutions with fillins r_1, r_2. Then our equivalence extension has*

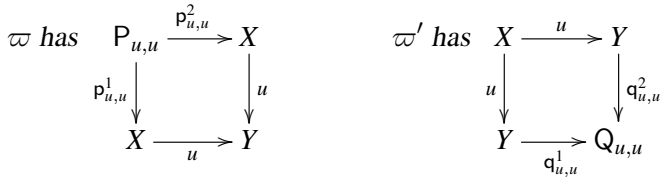
$$\delta \mathrm{G}^2 = \{r_1 \sim r_2\}.$$

(Equivalence of the other edges can then be deduced.)

Finally, we have rules for balance, stability and exactness. In each case, the given configuration contains a particular edge $u\colon X \to Y$ for which the equivalence extension adjoins an inverse. Hence

$$\delta \mathrm{G}^1 = \{u^{-1}\}, \quad \delta \mathrm{G}^2 = \{uu^{-1} \sim \mathrm{s}(X), u^{-1}u \sim \mathrm{s}(Y)\}.$$

- *Rule for balance. Suppose we are given pullback and pushout universals $\varpi \in \mathrm{U}^{\mathrm{pb}}, \varpi' \in \mathrm{U}^{\mathrm{po}}$, expressing the kernel pair and cokernel pair for the same edge $u\colon X \to Y$:*



Suppose we also have commutativities $\mathsf{p}^1_{u,u} \sim \mathsf{p}^2_{u,u}$ (u is monic) and $\mathsf{q}^1_{u,u} \sim \mathsf{q}^2_{u,u}$ (u is epi). Then our equivalence extension has $\delta \mathrm{G}^1 = \{u^{-1} = \mathsf{uc}(u)\}$.

- *Rule for stability of initial objects. Suppose we are given a universal $\varpi \in \mathrm{U}^0$ for an initial object 0, and an edge $u\colon X \to 0$. Then $\delta \mathrm{G}^1 = \{u^{-1} = !^0_X\}$.*

- *Rule for stability of pushouts. Suppose we have data as outlined in diagram (2). This will include two pushout universals (bottom square and inner square on top), three pullback universals for vertical squares (front and right faces, and also one stretching diagonally over v), the extra edge e, and other diagonal edges where necessary. Then the equivalence extension inverts e, $\delta \mathrm{G}^1 = \{e^{-1} = \mathsf{stab}_{u_1,u_2}(w)\}$.*

- *Rule for exactness. Suppose we have data as outlined in diagram (3). This will include pullback universals to specify that $X_0 \times X_0$, X_2 and K are the appropriate limits, pushout universals to specify that γ is a coequalizer, and commutativities to specify that π and e are fillins. Then the equivalence extension inverts e, $\delta \mathrm{G}^1 = \{e^{-1} = \mathsf{ex}(\pi_1, \pi_2, r, s, t)\}$.*

An equivalence extension, *written $\mathbb{T} \Subset \mathbb{T}'$, is a proto-extension that can be expressed as a composite of finitely many simple equivalence extensions.*

Note also that if $\mathbb{T} \Subset \mathbb{T}'$ is an equivalence extension, then so too is its reindexing along any sketch homomorphism.

Definition 16 *If $e_i\colon \mathbb{T} \Subset \mathbb{T}_i$ $(i = 1, 2)$ are two equivalence extensions of a context $\mathbb{T}$, then e_2 is a* refinement *of e_1, by ε, if $\varepsilon\colon \mathbb{T}_1 \to \mathbb{T}_2$ is a homomorphism such that $e_1\varepsilon = e_2$.*

For any two equivalence extensions e_i of $\mathbb{T}$, we can reindex e_2 along e_1 (or vice versa), compose, and thereby get a common refinement of e_1 and e_2.

Equality between morphisms $u, u'\colon X \to Y$ is expressed using unary commutativities $u \sim u'$, defined as $\mathrm{s}(X)u \sim u'$. Since the rules used in equivalence extensions must be capable of supplying proofs of equality, we verify that the standard rules for equality can be derived as composite rules of equivalence extensions. Given these, it will be clear that all proofs of equality of morphisms in the essentially algebraic theory of categories can be represented by commutativities in a suitable equivalence extension.

Proposition 17 *Let $\mathbb{T}$ be a sketch. In the following results we are interested in properties holding in $\mathbb{T}$, and properties derivable from them in the sense that they hold in some equivalence extension of $\mathbb{T}$.*

(1) *For any two nodes X and Y, $\sim_{XY}$ is an equivalence relation on the edges between them. This is in the sense that for each of the three properties for an equivalence relation, if the hypothesis holds in some sketch then the conclusion holds in some equivalence extension.*

(2) *If u, u' are two edges from X to Y, then the commutativities $\mathrm{s}(X)u \sim u'$ and $u'\mathrm{s}(Y) \sim u$ are mutually derivable.*

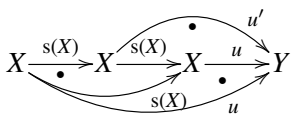
It follows that we have four mutually derivable characterizations of $u \sim u'$, namely $\mathrm{s}(X)u \sim u'$, $\mathrm{s}(X)u' \sim u$, $u\mathrm{s}(Y) \sim u'$ and $u'\mathrm{s}(Y) \sim u$.

(3) *Suppose we have $u \sim_{XY} u'$ and $v \sim_{YZ} v'$, and also $w, w'\colon X \to Z$ with $uv \sim w$. Then the commutativities $u'v' \sim w'$ and $w \sim w'$ are mutually derivable.*

From left to right is congruence. From right to left (with $u' = u$ and $v' = v$) shows that the set of composites uv is the entire congruence class of w.

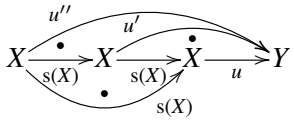
Proof (1) *Reflexivity* is immediate from the left unit law.

For *symmetry,* suppose $u \sim_{XY} u'$. By the left unit law we derive



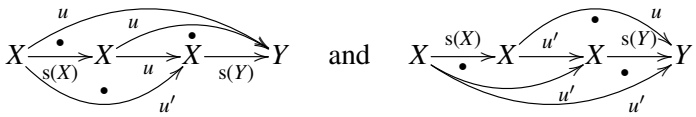
and then right associativity gives $\mathrm{s}(X)u' \sim u$.

For *transitivity,* suppose $u \sim u' \sim u''$. By the left unit law we get

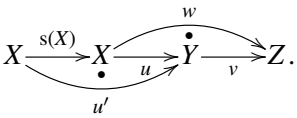


and then $\mathrm{s}(X)u \sim u''$ is derived by left associativity.

(2) The two directions follow by applying associative laws to the two diagrams



(3) First, consider the case when $v = v'$, and the diagram



The two associativities give the two directions we want. A similar proof, but dual (using (2)), deals with the case $u = u'$. Putting these together gives the general result. □

Proposition 18 *Let $\mathbb{T} \Subset \mathbb{T}'$ be an equivalence extension, and let M be a strict model of $\mathbb{T}$ in an AU $\mathcal{A}$.*

Then there is a unique strict model M' of $\mathbb{T}'$ in $\mathcal{A}$ whose restriction to $\mathbb{T}$ is M.

We call this the extension *of M to $\mathbb{T}'$.*

Proof Each node or edge introduced in $\mathbb{T}'$ has a canonical description as an AU-expression in terms of older nodes and edges, and so has a canonical interpretation already in $\mathbf{AU}\langle\mathbb{T}\rangle$. For a node, strictness implies already that we must use this interpretation. For an edge, the commutativities introduced at the same time are enough to force equality in $\mathcal{A}$ between the interpretations of the edge and the canonical description.

It remains to show that all the commutativities $fg \sim h$ in $\mathbb{T}'$ are respected. Let us write $[f]$ for the interpretation of the canonical expression for f in M, and similarly for g and h. We require $[h] = [g] \circ [f]$ in $\mathcal{A}$.

We have to examine the rule that introduces the commutativity, and use induction on the number of simple equational extensions needed.

For the rules that introduce nodes or edges, the commutativities introduced follow directly from quasiequational rules for AUs. It is also clear for unit laws and associativity.

There remain the uniqueness rules for fillins. Suppose we have one expressing $f \sim f'$. Then in $\mathcal{A}$ we have that f and f' are both equal to the fillin, and so equal to each other. □

Proposition 19 *Let $e\colon \mathbb{T} \Subset \mathbb{T}'$ be an equivalence extension. Then the corresponding AU-functor $\mathbf{AU}\langle e\rangle\colon \mathbf{AU}\langle\mathbb{T}\rangle \to \mathbf{AU}\langle\mathbb{T}'\rangle$ is an isomorphism.*

Proof In terms of strict AU-functors, Proposition 18 says that for any strict $M\colon \mathbf{AU}\langle\mathbb{T}\rangle \to \mathcal{A}$ there is a unique strict $M'\colon \mathbf{AU}\langle\mathbb{T}'\rangle \to \mathcal{A}$ such that $\mathbf{AU}\langle e\rangle M' = M$. Applying this with $\mathsf{Id}_{\mathbf{AU}\langle\mathbb{T}\rangle}$ for M gives us an $F\colon \mathbf{AU}\langle\mathbb{T}'\rangle \to \mathbf{AU}\langle\mathbb{T}\rangle$ for M', and then for more general M we see that $M' = FM$. From this we deduce that F is an inverse for $\mathbf{AU}\langle e\rangle$. □

6 Object equalities

The notion of equality between two context homomorphisms (Definition 5) is very strong, and in essence syntactic. The homomorphisms must act equally on the nodes and edges as sketch ingredients. In practice we usually want a more semantic notion that allows us to say when nodes and edges are equal in the sense that they must be interpreted equally in strict models. This will allow us to get faithfulness for a functor that takes $\mathbb{T}$ to $\mathbf{AU}\langle\mathbb{T}\rangle$.

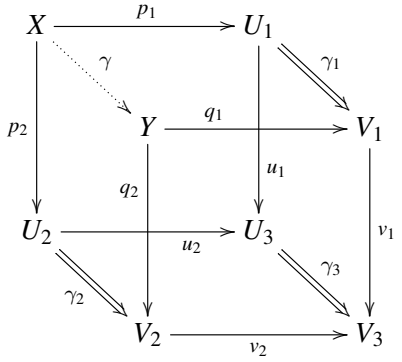
For edges, we already have a machinery for proving equality as morphisms by using commutativities. For nodes we have deliberately avoided anything analogous, beyond equality in the graph. However, semantic equality can still arise when two nodes are declared by universals for two identical constructions from equal data. We define certain kinds of edges as being "object equalities" between their domains and codomains; semantically they must be equal to identity morphisms. We then extend the phrase to apply also to "object equality" between edges or context homomorphisms.

We use the phrase *object equality* for a situation where a context already has the required structure, and *objectively equal*, or *objective equality*, for a situation where an equivalence extension can provide it.

Definition 20 *Let $\mathbb{T}$ be a context, and suppose $\gamma\colon X \to Y$ is an edge in $\mathbb{T}$. Then γ is an* object equality, *written $\gamma\colon X \Rightarrow Y$, if either $X = Y$ as nodes and $\gamma \sim \mathrm{s}(X)$ in $\mathbb{T}$, or γ can be associated with structure in $\mathbb{T}$ in one of the following ways.*

(1) *If X, Y are subjects of terminal universals: no extra structure needed.*

(2) *Suppose X and Y are subjects of pullback universals, for the back and front faces of the following diagram, and we have object equalities $\gamma_i\colon U_i \Rightarrow V_i$. Suppose also that we have sufficient composite edges and commutativities to assert that $u_i\gamma_3 \sim \gamma_i v_i$ and $\gamma q_i \sim p_i\gamma_i$, so that γ is characterized as a fillin $\langle p_1\gamma_1, p_2\gamma_2\rangle_{v_1,v_2}$. Then $\gamma\colon X \Rightarrow Y$ is an object equality.*



(3) *Similarly for initial objects and pushouts.*

(4) *Suppose we have two list universals for $L_i = \mathsf{List}(A_i)$ $(i = 0, 1)$ and an object equality $\gamma_A\colon A_0 \Rightarrow A_1$, and an edge $\gamma_L\colon L_0 \to L_1$ with sufficient data to characterize it as $\mathsf{List}(\gamma_A)$ (see Remark 1). Then γ_L is an object equality.*

Lemma 21 *Let $\mathbb{T}$ be a context.*

(1) *If $\gamma\colon X \Rightarrow Y$ is an object equality, then in $\mathbf{AU}\langle\mathbb{T}\rangle$ we have $X = Y$ and γ is the identity morphism.*

(2) *If $\gamma\colon X \Rightarrow X$ is an object equality, then there is some equivalence extension $\mathbb{T} \Subset \mathbb{T}'$ in which $\mathrm{s}(X) \sim \gamma$.*

(3) *If $\gamma\colon X \Rightarrow Y$ and $\gamma'\colon Y \Rightarrow Z$ are object equalities, then there is some $\mathbb{T} \Subset \mathbb{T}'$ in which we have an object equality $\delta\colon X \Rightarrow Z$ and $\gamma\gamma' \sim \delta$.*

(4) *If $\gamma\colon X \Rightarrow Y$ is an object equality, then there is some $\mathbb{T} \Subset \mathbb{T}'$ in which γ is an isomorphism, and its inverse is also an object equality.*

(5) *If $\gamma, \gamma'\colon X \Rightarrow Y$ are two object equalities, then there is some $\mathbb{T} \Subset \mathbb{T}'$ in which $\gamma \sim \gamma'$.*

Proof (1) is immediate from the definition, bearing in mind that for a list universal the expression for $A \times L$ is defined to be that for the pullback of $!_L$ and $!_A$.

(2) and (3) follow from the uniqueness clauses for fillins.

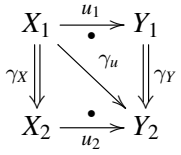
(4) follows because all the cases for object equality are symmetric, and we can then apply (3) and (2).

(5) again follows from the uniqueness clauses for fillins. □

We shall use the phrase "object equality" more generally than just for objects.

Definition 22 *Let $\mathbb{T}$ be a context.*

If $u_i\colon X_i \to Y_i$ $(i = 1, 2)$ are edges in $\mathbb{T}$, then an object equality *from u_1 to u_2 is the data of a commutative diagram*



such that γ_X and γ_Y are object equalities.

Let $f_0, f_1\colon \mathbb{T}_1 \leqslant \mathbb{T}_0$ be two context homomorphisms. Then an object equality *from f_0 to f_1 is a 2-cell γ from f_0 to f_1, for which every carrier edge is an object equality $\gamma_X\colon f_0(X) \Rightarrow f_1(X)$. (It follows that for each edge $u\colon X \to Y$ of $\mathbb{T}_1$, we get an object equality $(\gamma_X, \gamma_u, \gamma_Y)$ from $f_0(u)$ to $f_1(u)$.)*

By taking $\mathbb{T}_1$ as either $\mathbb{O}$ or $\mathbb{O}^{\to}$, we see that object equality for homomorphisms subsumes the cases for nodes and edges.

We say that two context homomorphisms are objectively equal *in $\mathbb{T}$, symbolized $=_o$, if there is some equivalence extension of $\mathbb{T}$ in which they have an object equality.*

Proposition 23 *Objective equality of context homomorphisms is an equivalence relation.*

Proof This is a straightforward extension of Lemma 21. For transitivity $f_0 =_o f_1 =_o f_2$, note that we may have different equivalence extensions for $f_0 =_o f_1$ and for $f_1 =_o f_2$. Work in a common refinement. □

7 Context maps

In Section 8 we shall define a 2-category $\mathfrak{Con}$ whose objects are contexts, and whose morphisms $\mathbb{T}_0 \to \mathbb{T}_1$ are in bijection with strict AU-functors $\mathbf{AU}\langle\mathbb{T}_0\rangle \leftarrow \mathbf{AU}\langle\mathbb{T}_1\rangle$. In fact, its 1-cells will be what we shall define here as *context maps*.

In this section we investigate the 1-category $\mathfrak{Con}_{\Subset\gtrdot}$ of contexts and context maps, from which $\mathfrak{Con}$ is got by factoring out a congruence based on objective equality. To save repetition, we shall exploit the fact that object equalities are a special case of 2-cells, and the present section is really a collection of ad hoc preliminary results about 2-cells in the not-a-2-category $\mathfrak{Con}_{\Subset\gtrdot}$.

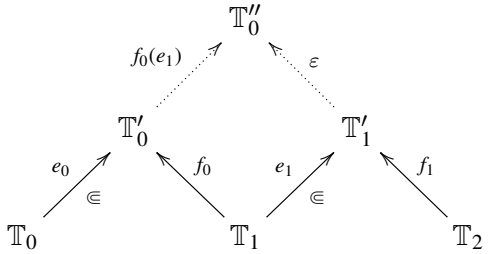
We already have a category $\mathfrak{Con}_{\lessdot}$ of contexts and context homomorphisms (and we shall also write $\mathfrak{Con}_{\gtrdot}$ for its opposite). Recall that we consider two sketch homomorphisms equal if they agree on the nodes and edges. For the present section, all diagrams of contexts are in $\mathfrak{Con}_{\lessdot}$.

Definition 24 *Let $\mathbb{T}_0, \mathbb{T}_1$ be contexts. Then a* context map *from $\mathbb{T}_0$ to $\mathbb{T}_1$ is an opspan (e,f) from $\mathbb{T}_0$ to $\mathbb{T}_1$, where e is an equivalence extension:*

$$\mathbb{T}_0 \xrightarrow[\Subset]{e} \mathbb{T}_0' \xleftarrow{f} \mathbb{T}_1$$

Using reindexing, we can compose context maps.

Definition 25 *Suppose we have context maps as in the bottom two rows of the following diagram, and we reindex e_1 along f_0.*



Then the composite $(e_0,f_0)(e_1,f_1)$ is $(e_0 f_0(e_1), \varepsilon \circ f_1)$.

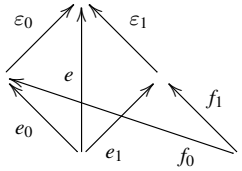
Contexts and context maps form a category $\mathfrak{Con}_{\Subset\gtrdot}$, with composition as defined and identity maps $(\mathsf{Id},\mathsf{Id})$. Note that (e,f) is the composite $(e,\mathsf{Id})(\mathsf{Id},f)$.

Definition 26 *A* 2-cell *in* $\mathfrak{Con}_{\Subset>}$*, between* $\mathbb{T}_0$ *and* $\mathbb{T}_1$*, is a context map* (e,α) *from* $\mathbb{T}_0$ *to* $\mathbb{T}_1^{\to}$*. Its domain and codomain are* $(e,\alpha\circ i_\lambda)$ $(\lambda=0,1)$*.*

An object equality *is a 2-cell* (e,γ) *in which* γ *is an object equality.*

Two context maps (e_i,f_i)*, with the same domain and codomain, are* objectively equal *if* e_0 *and* e_1 *have a common refinement* e *such there is an object equality from* $f_0\varepsilon_0$ *to* $f_1\varepsilon_1$*.*

(6)



From Proposition 23 it is easy to see that objective equality is an equivalence relation on each hom-set of $\mathfrak{Con}_{\Subset>}$.

$\mathfrak{Con}_{\Subset>}$ is not a 2-category; it lacks vertical and horizontal composition. For example, suppose we have two vertically composable 2-cells between $\mathbb{T}_0$ and $\mathbb{T}_1$. To compose them we need to be able to compose the carrier edges in $\mathbb{T}_0$.

For the time being we examine *whiskering*, horizontal composition of 2-cells with 1-cells.

Left whiskering[3] is done by composition of context maps $\mathbb{T}_0\longrightarrow\mathbb{T}_1\longrightarrow\mathbb{T}_2^{\to}$.

Right whiskering by context maps (Id,f) is similar, with a composition

$$\mathbb{T}_0\longrightarrow\mathbb{T}_1^{\to}\xrightarrow{(\mathsf{id},f^{\to})}\mathbb{T}_2^{\to}.$$

For whiskering as defined so far, it is clear that

(1) all possible associativities hold, and

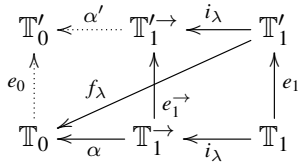
(2) whiskering preserves object equalities.

The remaining case is right whiskering by maps (e,Id). For these we start to need equivalence extensions.

Lemma 27 *Let* $\mathbb{T}_1$*,* $\mathbb{T}_1'$ *and* $\mathbb{T}_0$ *be contexts. Suppose we have an equivalence extension* $e_1\colon\mathbb{T}_1\Subset\mathbb{T}_1'$*, two homomorphisms* $f_0,f_1\colon\mathbb{T}_1'\lessdot\mathbb{T}_0$*, and a 2-cell* $\alpha\colon e_1f_0\to e_1f_1$*. Then:*

[3] It is arguable which is left and which is right. We take it that left whiskering is for when the 1-cell is on the left in diagrammatic order of context maps.

(1) *There is some equivalence extension* $e_0 \colon \mathbb{T}_0 \Subset \mathbb{T}'_0$ *and a 2-cell* $\alpha' \colon f_0 e_0 \to f_1 e_0$ *such that* $\alpha e_0 = e_1^{\to} \alpha'$.



(2) *For any such* e_0 *and* α' *as in (1), suppose we also have (for the same* e_0*)* α'' *satisfying the same conditions as for* α'*. Then* $\alpha' =_o \alpha''$*. (This just means that unary commutativities can be found between the actions of* α' *and* α'' *on edges, since their actions on nodes are already constrained up to equality by* f_0 *and* f_1*.)*

Proof It suffices to consider simple equivalence extensions e_1, and the only non-trivial ones are those that introduce nodes or edges. If e_1 introduces only commutativities, then the action of α' is already explicit in that of α and e_0 just has to introduce the images under f_0 and f_1 of those commutativities. This applies to the unit and associativity rules, and to the rules for the uniqueness of fillins.

We need therefore to find the image under α' uniquely defined for every node and edge in $\mathbb{T}_1^{\to}$.

For everything in the image of i_0 or i_1 (and this includes all the nodes), the image under α' is already in $\mathbb{T}_0$, and given by f_0 or f_1.

It remains only to consider edges θ_X or θ_u, and again the α' image is already in $\mathbb{T}_0$, and given by α, for X or u in the image of e_1: so we must consider fresh X and u in $\mathbb{T}_1$. If $u \colon X \to Y$ then θ_u is determined as a composite as soon as θ_X or θ_Y is known. Then it remains only to adjoin sufficient ingredients to $\mathbb{T}_0$ to justify the commutativities $i_0(u)\theta_Y = \theta_X i_1(u)$.

First, for adjoining a composite $w \sim_{XYZ} uv$, the commutative square for w is got by pasting those already known for u and v. Explicitly, we check algebraically that the appropriate square for w commutes:

$$\begin{aligned} f_0(w)\alpha_Z &= f_0(u)f_0(v)\alpha_Z = f_0(u)\alpha_v = f_0(u)\alpha_Y f_1(v) \\ &= \alpha_u f_1(v) = \alpha_X f_1(u) f_1(v) = \alpha_X f_1(w). \end{aligned}$$

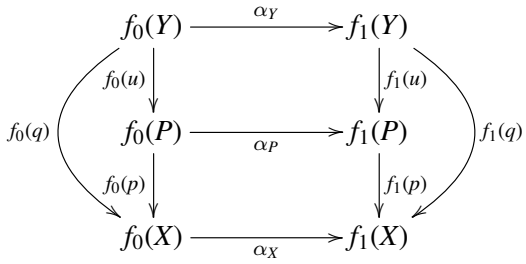
By Proposition 17 we can find an equivalence extension with sufficient edges and commutativities to express this.

A similar argument applies to all those equivalence extensions that adjoin an inverse to a particular edge $u \colon X \to Y$.

We now look at pullbacks.

For a pullback universal we have one fresh node P and three fresh edges for three projections $p\colon P \to X$ (say) in the pullback cone. For each p, θ_p is defined from the known θ_X, and then the three θ_ps provide a cone from $i_0(P)$ to the i_1-copy of the diagram opspan. To achieve the commutative squares for the θ_ps, θ_P can and must be defined as the fillin for the cone.

For pullback fillins, suppose in $\mathbb{T}_1$ we have a pullback P of some opspan, and suppose that $u\colon Y \to P$ fills in for a cone that has, for each projection $p\colon P \to X$, a morphism $q\colon Y \to X$. We need to show $f_0(u)\alpha_P \sim \alpha_Y f_1(u)$, and it suffices to show that when composed with each pullback projection for $f_1(p)$.



The bounding quadrangle, the lower small rectangle and the two side-bows all commute, and so (in some suitable equational extension) we can show $f_0(u)\alpha_P f_1(p) \sim \alpha_Y f_1(u) f_1(p)$.

The same argument applies dually to pushouts, and simpler arguments apply to terminal and initial universals.

For list universals the ideas are similar, but with some extra intricacies that we shall now outline. For a list universal the essential task is to implement the functoriality expressed in Remark 1, for the situation where $f\colon A_0 \to A_1$ is $\theta_A\colon f_0(A) \to f_1(A)$. Over $\mathbb{T}_0$ we can define the A_0-action on $\mathsf{List}(A_1)$, and then we can and must define $\theta_{\mathsf{List}(A)}$ as the fillin. The θs for the other new nodes are determined by the arguments for terminals and pullbacks.

For list fillins the argument is analogous to that for pullbacks, but more intricate. It is left to the reader. □

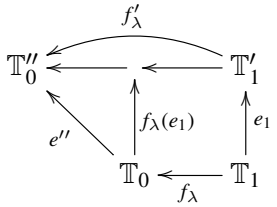
Note that if α is an object equality then so is α'. In other words, we can cancel equivalence extensions e from objective equalities: if there is an object equality from ef_1 to ef_2, then f_1 and f_2 are objectively equal. A particular special case is for refinements of equivalence extensions (Definition 16). If $e_2 = e_1\varepsilon = e_1\varepsilon'$ then ε and ε' are objectively equal.

Definition 28 *Let $\alpha\colon \mathbb{T}_1^{\rightarrow} \lessdot \mathbb{T}_0$ be a 2-cell, with domain and codomain f_0 and f_1, and let $e_1\colon \mathbb{T}_1 \Subset \mathbb{T}'_1$ be an equivalence extension. Then a* right whiskering $(\mathsf{Id}, \alpha)(e_1, \mathsf{Id})$ *is a map (e_0, α') where $e_0\colon \mathbb{T}_0 \Subset \mathbb{T}'_0$ is a common refinement of $f_0(e_1)$ and $f_1(e_1)$, and $\alpha'\colon \mathbb{T}'^{\rightarrow} \to \mathbb{T}'_0$ has $e_1^{\rightarrow}\alpha' = \alpha e_0$.*

Note that, because of the need to use a common refinement of $f_0(e_1)$ and $f_1(e_1)$, the domain of the whiskering is not strictly equal to what it should be at the 1-cell level. However, they are objectively equal. The codomain is similar.

Proposition 29 *Right whiskering $(\mathsf{Id}, \alpha)(e_1, \mathsf{Id})$ exists and is unique up to objective equality.*

Proof First, reindex $f_1(e_1)$ along $f_0(e_1)$ to obtain a common refinement $e''\colon \mathbb{T}_0 \Subset \mathbb{T}''_0$, and define $f'_\lambda\colon \mathbb{T}'_1 \to \mathbb{T}''_0$ by composing the morphisms introduced for reindexing and the refinements.



Now apply Lemma 27 to $\alpha e''$ and the f'_λs to get $e'\colon \mathbb{T}''_0 \Subset \mathbb{T}'_0$ and $\alpha'\colon \mathbb{T}'^{\rightarrow}_1 \to \mathbb{T}'_0$. The required right whiskering is (e_0, α'), where $e_0 = e''e'$.

Uniqueness follows from the uniqueness in Lemma 27. □

General right whiskering can now be defined by

$$(e_0, \alpha_0)(e_1, f_1) = (e_0, \mathsf{Id})((\mathsf{Id}, \alpha)(e_1, \mathsf{Id}))(\mathsf{Id}, f_1).$$

Proposition 30

(1) *Whiskering obeys the usual associative laws up to objective equality.*

(2) *Whiskering preserves object equalities.*

Proof (1) After what we said earlier, the only remaining issue is the associativity of $(\mathsf{Id}, \alpha)(e_0, \mathsf{Id})(e_1, \mathsf{Id})$.

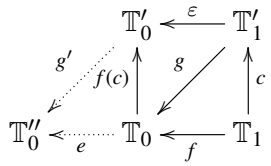
$((\mathsf{Id}, \alpha)(e_0, \mathsf{Id}))(e_1, \mathsf{Id})$ has the property required for $(\mathsf{Id}, \alpha)(e_0 e_1, \mathsf{Id})$, so they are objectively equal.

(2) Clear from the remark after Lemma 27. □

The following lemma has an important application in Lemma 34. Note that if cg is *equal* to f, then e can be trivial, with $\varepsilon g' = g$. With object equalities there is a little more work, and it is embodied in e.

Lemma 31

(1) *Suppose we have the solid parts of the following diagram,*



where c is an extension, the square is the reindexing, and we have an object equality $\gamma\colon f \Rightarrow cg$.

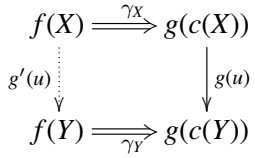
Then we can find an equivalence extension $e\colon \mathbb{T}_0 \Subset \mathbb{T}''_0$ and a homomorphism $g'\colon \mathbb{T}'_0 \to \mathbb{T}''_0$ such that $f(c)g'$ is strictly equal to e and there is an object equality $\gamma'\colon \varepsilon g' \Rightarrow ge$ such that $c\gamma' = \gamma e$.

(2) *Suppose, in the situation above, we have an equivalence extension e and two homomorphisms g'_i with the properties described. Then g'_1 and g'_2 are objectively equal in $\mathbb{T}''_0$.*

Proof (1) By induction we can assume that c is a simple extension.

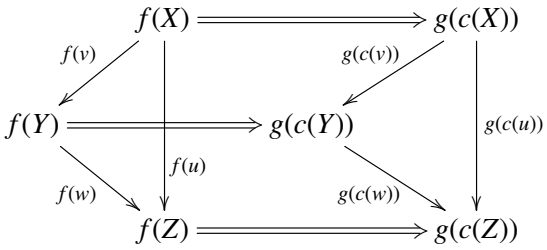
If c adjoins a primitive node X, then we define e as trivial, and $g'(X) = g(X)$.

If c adjoins a primitive edge $u\colon X \to Y$ then in $\mathbb{T}_0$ we have the solid part of



and in a suitable equivalence extension of $\mathbb{T}_0$ we can define $g'(u)$ to make the square commute.

Suppose c adjoins a commutativity $vw \sim_{XYZ} u$. We have:



The square faces all commute because they are object equalities. Once c has made the right-hand triangle commute, in a suitable equivalence extension we can deduce that so does the left-hand one.

If c adjoins a universal, then we let e adjoin the same universal.

(2) Every ingredient of $\mathbb{T}'_0$ is in the image of either $f(c)$ or ε. It therefore suffices to note that $f(c)g'_1$ and $f(c)g'_2$ are strictly equal, while $\varepsilon g'_1$ and $\varepsilon g'_2$ are objectively equal by Proposition 23. □

8 The 2-category of contexts

We now define our 2-category $\mathfrak{Con}$ in which the 0-cells are contexts, and the 1-cells between $\mathbb{T}_0$ and $\mathbb{T}_1$ are in bijection with strict AU-functors from $\mathbf{AU}\langle\mathbb{T}_1\rangle$ to $\mathbf{AU}\langle\mathbb{T}_0\rangle$. At the same time, we shall make the reversal of direction by which a strict AU-functor can be thought of as a transformation of models. Thus we shall think of a 1-cell as a "map" from the "space of models of $\mathbb{T}_0$" to the "space of models of $\mathbb{T}_1$".

8.1 $\mathfrak{Con}$ as a 1-category

Proposition 32 *Objective equality of context maps is a congruence on* $\mathfrak{Con}_{\Subset\gtrdot}$.

Hence contexts and their maps modulo objective equality form a category $\mathfrak{Con}$.

Proof It has already been remarked that objective equality is an equivalence relation on each hom-set. To show that it is a congruence, we show that if two context maps are objectively equal, then their composites with any (e,f) are also objectively equal. On the left, we just reindex everything along f. On the right, we apply right whiskering by (e,f), and use the fact that this preserves objective equality. □

We now have a functor $(\mathsf{Id},-)\colon \mathfrak{Con}_{\gtrdot} \to \mathfrak{Con}$ given by

$$(\mathbb{T}_0 \xleftarrow{f} \mathbb{T}_1) \mapsto (\mathbb{T}_0 \xlongequal{} \mathbb{T}_0 \xleftarrow{f} \mathbb{T}_1).$$

Theorem 33 $\mathfrak{Con}$ *is free over* $\mathfrak{Con}_{\gtrdot}$ *subject to object equalities becoming equalities, and equivalence extensions becoming invertible.*

Proof If $e\colon \mathbb{T}_0 \Subset \mathbb{T}'_0$ is an equivalence extension, then (Id, e) has inverse (e, Id) in $\mathfrak{Con}$.

We have $(e, \mathsf{Id}); (\mathsf{Id}, e) = (e, e)$, and this is objectively equal to $(\mathsf{Id}, \mathsf{Id})$ using e as a refinement of Id.

For the other composite we get $(e(e), \varepsilon)$ by reindexing. Now by the remark preceding Lemma 31, with g as an identity, we get a homomorphism g' with $e(e); g' = \varepsilon; g' = \mathsf{Id}$, showing that $(e(e), \varepsilon)$ is equal to the identity.

It follows that, in $\mathfrak{Con}$, every morphism can be expressed in the form $(\mathsf{Id}, e)^{-1}; (\mathsf{Id}, f)$, where e is an equivalence extension.

Now suppose we have a functor $F\colon \mathfrak{Con}_{\geqslant} \to \mathcal{C}$ with those properties. We must show it factors uniquely via $(\mathsf{Id}, -)$, with $F'\colon \mathfrak{Con} \to \mathcal{C}$. Uniqueness is clear: we must have

$$F'(e,f) = F'(e, \mathsf{Id}); F'(\mathsf{Id}, f) = F(e)^{-1}; F(f).$$

For existence, first we show that F' thus defined transforms objective equality to equality. Suppose (e_i, f_i) $(i = 0, 1)$ are objectively equal, as in diagram (6). Then

$$F(e_i)^{-1}; F(f_i) = F(e)^{-1}; F(\varepsilon_i); F(f_i) = F(e)^{-1}; F(f_i \varepsilon_i)$$

and these are equal for $i = 0, 1$ because F transforms object equality to equality.

It is obvious that F' preserves identities, and for composition it suffices to consider the composite $(\mathsf{Id}, f); (e, \mathsf{Id}) = (f(e), \varepsilon)$. In $\mathcal{C}$ we have

$$\begin{aligned} F(f); F(e)^{-1} &= F(f(e))^{-1}; F(ff(e)); F(e)^{-1} = F(f(e))^{-1}; F(e\varepsilon); F(e)^{-1} \\ &= F(f(e))^{-1}; F(\varepsilon). \end{aligned}$$

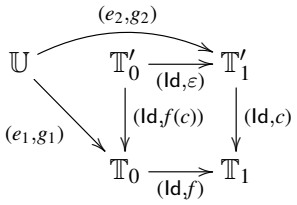
□

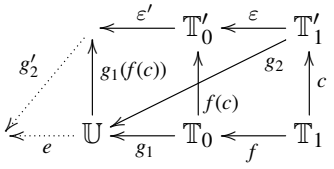
Lemma 34

(1) *Any reindexing square (4) for a context extension becomes a pullback square in* $\mathfrak{Con}$.

(2) *In* $\mathfrak{Con}$, *extension maps (ie those of the form* (Id, c) *where* c *is an extension) can be pulled back along any morphism.*

Proof (1) Consider a diagram as on the left here, with the outer square commuting.





Taking a common refinement of e_1 and e_2, we might as well assume that they are both trivial and that we have an object equality $fg_1 \Rightarrow cg_2$. Now consider the diagram on the right, and apply Lemma 31 with g_2 for g. We obtain e and g_2', with e an equivalence extension, strict equality $g_1(f(c)); g_2' = e$, and an object equality $\varepsilon\varepsilon' g_2' \Rightarrow g_2 e$.

The required fillin is $(e, \varepsilon' g_2')$. It has the correct composites with $(\mathsf{Id}, f(c))$ and $(\mathsf{Id}, \varepsilon)$. Moreover, uniqueness follows by the same argument as in Lemma 31.

(2) After part (1), it suffices to show that (Id, c) can be pulled back along any map (e, Id) where $e\colon \mathbb{T}_0 \Subset \mathbb{T}_1$ is an equivalence extension. This is trivial, because pullbacks along invertible morphisms always exist. □

8.2 $\mathfrak{Con}$ as 2-category

We now develop the 2-categorical structure.

Lemma 35 *Let $\mathbb{T}$ be a context. Then $(\mathbb{T}^{\to})^{\to}$ has an involution (e, τ) such that $(\mathsf{Id}, i_\mu)(e, \tau) = (\mathsf{Id}, i_\mu^{\to})$.*

Proof We shall write $i_{\lambda\mu}$ $(\lambda, \mu = 0, 1)$ for the composite

$$i_\lambda i_\mu = \mathbb{T} \xrightarrow[i_\lambda]{} \mathbb{T}^{\to} \xrightarrow[i_\mu]{} (\mathbb{T}^{\to})^{\to}.$$

In $(\mathbb{T}^{\to})^{\to}$ we write θ for the first level homomorphism, in $\mathbb{T}^{\to}$, represented in $(\mathbb{T}^{\to})^{\to}$ by $i_\mu(\theta)$, and ϕ for the second level homomorphism.

Note that $i_\lambda i_\mu^{\to} = i_\mu i_\lambda$. It follows that any model of $(\mathbb{T}^{\to})^{\to}$ has a square of four models of $\mathbb{T}$, got from the $i_{\lambda\mu}$s, and four homomorphisms between them, got from the i_μs and the $i_\mu^{\to}$s. In fact, the square will commute, because ϕ is homomorphic with respect to the i_μs. Conversely, any such commutative square of homomorphisms gives a model of $(\mathbb{T}^{\to})^{\to}$.

$$\begin{array}{ccc} i_{00}(X) & \xrightarrow{i_0(\theta_X)} & i_{10}(X) \\ {\scriptstyle \phi_{i_0(X)} = i_0^{\to}(\theta_X)}\downarrow & & \downarrow{\scriptstyle \phi_{i_0(X)} = i_1^{\to}(\theta_X)} \\ i_{01}(X) & \xrightarrow[i_1(\theta_X)]{} & i_{11}(X) \end{array}$$

Reflecting the square about its leading diagonal gives another such square, and that is the essential action of τ. The only remaining issue is that in the *context* $(\mathbb{T}^{\to})^{\to}$, we need an equivalence extension e to introduce some composites and associativities; mere commutativity of the squares (of carrier edges) does not explicitly have all the data for a homomorphism between homomorphisms. □

Lemma 36 *Let $f_0, f_1 : \mathbb{T}_1 \lessdot \mathbb{T}_0$ have an object equality γ. Then $f_0^{\to}$ and $f_1^{\to}$ are objectively equal.*

Proof Use $(\mathsf{Id}, \gamma^{\to})(e, \tau)$, where (e, τ) is as in Lemma 35. □

Lemma 37 *Let $e : \mathbb{T}_1 \Subset \mathbb{T}_0$ be an equivalence extension. Then $e^{\to}$ is invertible in $\mathfrak{Con}$.*

Proof The identity on $\mathbb{T}_1^{\to}$ gives the generic 2-cell between $\mathbb{T}_1^{\to}$ and $\mathbb{T}_1$, its domain and codomain being i_0 and i_1. Consider its right whiskering (Definition 28) by (e, Id), giving the right hand square in the diagram below. Then (e', α) is the inverse of $(\mathsf{Id}, e^{\to})$. Immediately we have $(e', \alpha)(\mathsf{Id}, e^{\to}) = (e', e^{\to}\alpha) = (e', e')$. For $(\mathsf{Id}, e^{\to})(e', \alpha)$, consider the reindexing square on the left here.

$$\begin{array}{ccccc}
 & & \xleftarrow{\varepsilon} & \mathbb{U} & \xleftarrow{\alpha} & \mathbb{T}_0^{\to} \\
 & e^{\to}(e') \uparrow & & e' \uparrow & & \uparrow e^{\to} \\
\mathbb{T}_0^{\to} & \xleftarrow[e^{\to}]{} & \mathbb{T}_1^{\to} & = & \mathbb{T}_1^{\to} &
\end{array}$$

$(e^{\to}(e'), \alpha\varepsilon)$ is a right whiskering of $(\mathsf{Id}, e^{\to})$ by (e, Id): for $e^{\to}$ factors via $i_\lambda(e)$, so e' and $e^{\to}(e')$ are refinements of, respectively, $i_\lambda(e)$ and $i_\lambda e^{\to}$. But then so is $(e^{\to}(e'), e^{\to}(e'))$, and so they are objectively equal, and the latter is objectively equal to the identity on $\mathbb{T}_0^{\to}$. □

Theorem 38 *The functor $-^{\to}$ on $\mathfrak{Con}_{\lessdot}$ gives an endofunctor on $\mathfrak{Con}$.*

Proof Theorem 33 reduces this to Lemmas 36 and 37. □

We now define an internal category in the functor category $[\mathfrak{Con}, \mathfrak{Con}]$ in which the object of objects is Id, and the object of morphisms is $-^{\to}$.

The structure operations will be natural transformations. Note that to prove naturality, it suffices to prove it with respect to maps of the form (Id, f), since the rest follows from invertibility of (e, Id).

The domain and codomain, natural transformations from $-^{\to}$ to Id, are given by the maps $\mathsf{dom} = (\mathsf{Id}, i_0)$ and $\mathsf{cod} = (\mathsf{Id}, i_1)$.

The identity $\mathsf{Id} : \mathsf{Id} \to -^{\to}$ is given by maps (e, γ) where $\gamma : \mathbb{T}^{\to} \lessdot \mathbb{T}'$ takes θ to the equality homomorphism on the generic model of $\mathbb{T}$. The equivalence extension $e : \mathbb{T} \Subset \mathbb{T}'$ uses instances of the unit laws to provide the necessary commutativities.

Since i_0 is an extension, we can reindex along i_1, and in fact this gives $\mathbb{T}^{\to\to}$ as a pullback in $\mathfrak{Con}$.

$$\begin{array}{ccc} \mathbb{T}^{\to\to} & \xrightarrow{(\mathsf{Id},\varepsilon)} & \mathbb{T}^{\to} \\ {\scriptstyle (\mathsf{Id},i_1(i_0))}\downarrow & & \downarrow{\scriptstyle \mathsf{dom}=(\mathsf{Id},i_0)} \\ \mathbb{T}^{\to} & \xrightarrow[\mathsf{cod}=(\mathsf{Id},i_1)]{} & \mathbb{T} \end{array}$$

$i_1(i_0)$ maps the ingredients of $\mathbb{T}^{\to}$ to the 0- and 1-copies in $\mathbb{T}^{\to\to}$, and adjoins the 2-copies with the carrier edges from 1 to 2.

In an equivalence extension of $\mathbb{T}^{\to\to}$, the two model homomorphisms can be composed, and this provides composition as a natural transformation from $-^{\to\to}$ to $-^{\to}$. It is vertical composition of the two 2-cells $\mathbb{T}^{\to} \lessdot \mathbb{T}^{\to\to}$.

Thus for each $\mathbb{T}$ we get an internal category $N(\mathbb{T})$ in $\mathfrak{Con}$, on objects $\mathbb{T}$ and morphisms $\mathbb{T}^{\to}$.

Using the category structure of $N(\mathbb{T}_1)$, this makes $\mathfrak{Con}(\mathbb{T}_0,\mathbb{T}_1)$ into a category, with objects and morphisms the 1-cells and 2-cells between $\mathbb{T}_0$ and $\mathbb{T}_1$.

We already have vertical composition of 2-cells. (We shall compose from top to bottom, so the codomain of the upper 2-cell must equal the domain of the lower.)

We deal with horizontal composition by whiskering. Using the functor $-^{\to}$, we can make $\mathfrak{Con}(-,-^{\to})$ into a profunctor from $\mathfrak{Con}$ to $\mathfrak{Con}$, and this provides whiskering on both sides. The proof of Lemma 37 shows that this agrees with the whiskering we already have.

Horizontal composition can now be defined as:

$$\alpha\beta = \frac{\alpha\,\mathsf{dom}(\beta)}{\mathsf{cod}(\alpha)\beta}$$

The interchange law follows from:

Lemma 39

$$\frac{\alpha\,\mathsf{dom}(\beta)}{\mathsf{cod}(\alpha)\beta} = \frac{\mathsf{dom}(\alpha)\beta}{\alpha\,\mathsf{cod}(\beta)}$$

Proof Suppose we have the following:

$$\mathbb{T}_0 \;\underset{}{\overset{}{\Downarrow(e,\alpha)}}\; \mathbb{T}_1 \;\underset{}{\overset{}{\Downarrow(e',\beta)}}\; \mathbb{T}_2$$

By whiskering $(e,\alpha)(e',\mathsf{Id})$ we might as well assume that e and e' are both identities. In $\mathfrak{Con}_{\ll}$ we now have:

$$\alpha\circ\beta^{\to}\colon\ \mathbb{T}_0 \xleftarrow{\alpha} \mathbb{T}_1^{\to} \xleftarrow{\beta^{\to}} (\mathbb{T}_2^{\to})^{\to}$$

The two vertical composites in the statement are the images in $\mathbb{T}_0$ of the two routes round the square of homomorphisms in $(\mathbb{T}_2^{\to})^{\to}$ (see Lemma 35) and so are equal. □

Putting together the properties proved so far, we can deduce:

Theorem 40 $\mathfrak{Con}$ *is a 2-category.*

8.3 Limits in $\mathfrak{Con}$

We have two main results here. The first (Theorem 48) is that $\mathfrak{Con}$ has finite PIE-limits (products, inserters and equifiers; Power and Robinson [11]).

This is a large class of finite weighted limits, but a notable lack is equalizers and pullbacks. Although by universal algebra $\mathbf{AU}_s$ has all pushouts and $\mathbf{AU}_s^{op}$ has all pullbacks, in general we cannot replicate this in contexts. For example, suppose we have two context homomorphisms $f_i\colon \mathbb{T}_0 \ll \mathbb{T}_i$ where $\mathbb{T}_0$ has just a single node, and the f_is map it to nodes introduced by two different kinds of universals. Then the pushout must specify an equality between those two different nodes, and that cannot be done with a context.

The second main result (Theorem 42) is that, nonetheless, pullbacks of extension maps do exist, essentially by reindexing. In fact this has already been addressed in Lemma 34. All that remains here is to show that they are 2-categorical conical limits (in other words, they take proper account of 2-cells between fillins).

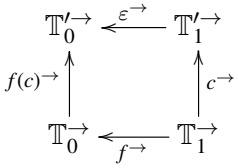
Note that all our weighted limits are *strict*, with strict cones, as in [11]. We do not follow the convention in Johnstone [5, page 244] of interpreting them in a "pseudo" or "bi" sense.

Also note that we do not claim to have constructed the limits in a canonical way, at least not those – such as pullbacks, inserters and equifiers – that depend on maps. This is because the construction will depend on the representatives (e,f) of the maps.

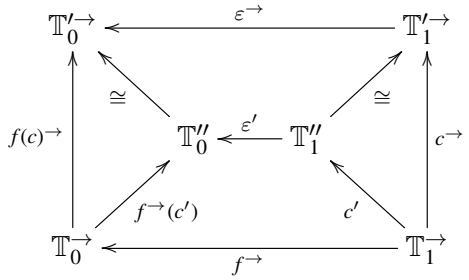
Pullbacks and products

Lemma 41 *Consider a context reindexing square (4). Then the following square in* $\mathfrak{Con}_{\lessdot}$ *becomes a pullback in* $\mathfrak{Con}$.

(7)



Proof If $c^{\rightarrow}$ were an extension, then we could apply Lemma 34. In fact it is not, but only for bureaucratic reasons based on the concrete definition of coproduct "+" (see Section 9). The issue is that the steps constructing $\mathbb{T}'^{\rightarrow}_1$ are applied in an order that does not start off with all those for $\mathbb{T}^{\rightarrow}_1$. Those steps can be reordered to give an extension $c' \colon \mathbb{T}^{\rightarrow}_1 \subset \mathbb{T}''_1$ isomorphic to $c^{\rightarrow}$, and moreover that reordering can be reindexed along $f^{\rightarrow}$ to get a reindexing square isomorphic to (7):



By Lemma 34 the reindexing square is a pullback in $\mathfrak{Con}$, and it follows that so too is (7). □

Theorem 42 $\mathfrak{Con}$ *has pullbacks of extension maps along any map.*

Proof Lemma 34 has already shown the 1-categorical form of this. It remains to show that we also have 2-cell fillins, and the ability to do this follows from Lemma 41. □

Lemma 43 $\mathfrak{Con}$ *has all finite products.*

Proof The empty theory $\mathbb{1}$ is initial in $\mathfrak{Con}_{\lessdot}$. After that one easily shows that it is terminal in $\mathfrak{Con}$.

The case for binary products follows from Theorem 42, since the unique homomorphism $\mathbb{1} \lessdot \mathbb{T}$ is an extension. □

Inserters

First, we work in $\mathfrak{Con}_{\ll}$ (or, dually, in $\mathfrak{Con}_{\gg}$).

Definition 44 *Let $f_\lambda\colon \mathbb{T}_1 \ll \mathbb{T}_0$ ($\lambda = 0,1$) be two context homomorphisms. Then we define an extension $c\colon \mathbb{T}_0 \subset \mathsf{Ins}(f_0,f_1)$ by adjoining:*

- *for every node Y in $\mathbb{T}_1$, an edge $\theta_Y\colon f_0(Y) \to f_1(Y)$; and*
- *for every edge $u\colon Y \to Y'$ in $\mathbb{T}_1$, an edge θ_u and two commutativities*

$$\begin{array}{ccc} f_0(Y) & \xrightarrow[\bullet]{\theta_Y} & f_1(Y) \\ {\scriptstyle f_0(u)}\downarrow & \searrow^{\theta_u} & \downarrow{\scriptstyle f_1(u)} \\ f_0(Y') & \xrightarrow[\theta_{Y'}]{\bullet} & f_1(Y') \end{array}$$

Obviously this generalizes the construction of $\mathbb{T}_1^{\to}$ out of $\mathbb{T}_1^2$, and gives $\bar{f}\colon \mathbb{T}_1^{\to} \to \mathsf{Ins}(f_0,f_1)$ with two strictly commutative squares:

$$\begin{array}{ccc} \mathsf{Ins}(f_0,f_1) & \xleftarrow{\bar{f}} & \mathbb{T}_1^{\to} \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle i_\lambda} \\ \mathbb{T}_0 & \xleftarrow[f_\lambda]{} & \mathbb{T}_1 \end{array}$$

In fact $\mathsf{Ins}(f_0,f_1)$ is their joint pushout in $\mathfrak{Con}_{\ll}$.

To put this another way, whiskering induces a bijection between

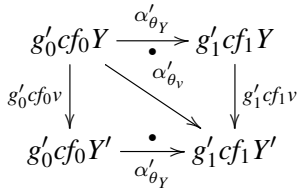
(1) context homomorphisms $g'\colon \mathsf{Ins}(f_0,f_1) \ll \mathbb{U}$, and

(2) pairs (g,θ) where $g\colon \mathbb{T}_0 \ll \mathbb{U}$ is a context homomorphism, and $\theta\colon f_0 g \to f_1 g$ is a 2-cell.

This very nearly also works at the level of 2-cells. Consider two sketch homomorphisms $g'_\mu\colon \mathsf{Ins}(f_0,f_1) \ll \mathbb{U}$ ($\mu = 0,1$), corresponding to pairs $(g_\mu = cg'_\mu, \theta_\mu = \bar{f}g'_\mu)$ as above, and suppose we have a 2-cell $\alpha'\colon g'_0 \to g'_1$. Considering the nodes and edges of $\mathsf{Ins}(f_0,f_1)$, we see that the edge data needed for α' comprises edges of the form α'_{cX} and α'_{cu}, for nodes and edges in $\mathbb{T}_0$, and α'_{θ_Y} and α'_{θ_v}, for nodes and edges in $\mathbb{T}_1$. The first two kinds come along with commutativitites that make the whiskered 2-cell $g_0 \to g_1$. The last two kinds have commutativities:

$$\begin{array}{ccc} g'_0cf_0Y & \xrightarrow[\bullet]{\alpha'_{cf_0Y}} & g'_1cf_0Y \\ {\scriptstyle g'_0\theta_Y}\downarrow & \searrow^{\alpha'_{\theta_Y}} & \downarrow{\scriptstyle g'_1\theta_Y} \\ g'_0cf_1Y & \xrightarrow[\alpha'_{cf_1Y}]{\bullet} & g'_1cf_1Y \end{array} \qquad \begin{array}{ccc} g'_0cf_0Y & \xrightarrow[\bullet]{\alpha'_{cf_0Y}} & g'_1cf_0Y \\ {\scriptstyle g'_0\theta_v}\downarrow & \searrow^{\alpha'_{\theta_v}} & \downarrow{\scriptstyle g'_1\theta_v} \\ g'_0cf_1Y' & \xrightarrow[\alpha'_{cf_1Y'}]{\bullet} & g'_1cf_1Y' \end{array}$$

The first of these expresses that the α'_{θ_Y}s give the correct carrier edges for the horizontal composition of α' and θ. The second is equivalent to saying that the α'_{θ_v}s give the correct naturality diagonals for this horizontal composition, in other words

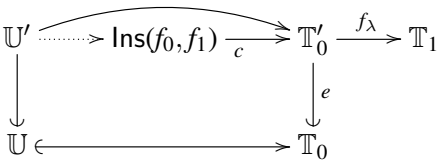


but only modulo applications of associativity laws.

Lemma 45 $\mathfrak{Con}$ *has inserters.*

Proof Suppose we have two maps from $\mathbb{T}_0$ to $\mathbb{T}_1$. We can represent them as homomorphisms f_0 and f_1 into a single equivalence extension $\mathbb{T}'_0$ of $\mathbb{T}_0$. We shall show that $\mathbb{T}_0 \Subset \mathbb{T}'_0 \subset \mathsf{Ins}(f_0,f_1)$ provides the inserter in $\mathfrak{Con}$.

In the following diagram we use arrows $\xleftarrow{(e,\mathsf{Id})}, \xrightarrow{(\mathsf{Id},f)}, \xleftarrow{(e,f)}$ for maps of the forms indicated.



The map from $\mathsf{Ins}(f_0,f_1)$ to $\mathbb{T}_0$ is got by inverting the equivalence extension $e\colon \mathbb{T}_0 \Subset \mathbb{T}'_0$.

Suppose we have a map from $\mathbb{U}$ to $\mathbb{T}_0$ and a 2-cell between its composites with the f_λs. By replacing $\mathbb{U}$ by a suitable equivalence extension $\mathbb{U}'$, we may assume that the 2-cell, between maps from $\mathbb{U}'$ to $\mathbb{T}_1$, is entirely in $\mathfrak{Con}_\geqslant$ as in the above diagram, and we get a unique factorization $\mathbb{U}' \to \mathsf{Ins}(f_0,f_1)$ in $\mathfrak{Con}_\geqslant$. This then gives us a unique factorization in $\mathfrak{Con}$.

The remarks before the lemma now enable us to extend this to 2-cells in the manner required for a weighted limit. (Now we need an equivalence extension of $\mathbb{U}'$ for the associativities needed.) □

Equifiers

Again, we start off in $\mathfrak{Con}_\geqslant$.

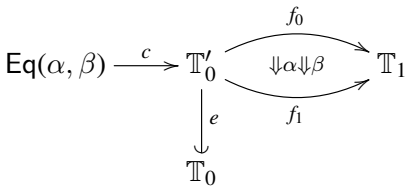
Definition 46 *Suppose we have two homomorphisms $\alpha, \beta\colon \mathbb{T}_1^{\to} \lessdot \mathbb{T}_0$ that, as 2-cells, have the same domain and codomain: $f_\lambda = i_\lambda\alpha = i_\lambda\beta$ ($\lambda = 0, 1$). (Equality is in the sense of agreeing on nodes and edges.) Then we define an extension $c\colon \mathbb{T}_0 \subset \mathsf{Eq}(\alpha, \beta)$ that adjoins unary commutativities $\alpha_Y \sim \beta_Y$ and $\alpha_v \sim \beta_v$ for the nodes Y and edges v in $\mathbb{T}_1$.*

A homomorphism $g'\colon \mathsf{Eq}(\alpha, \beta) \lessdot \mathbb{U}$ is equivalent to a homomorphism $g\colon \mathbb{T}_0 \lessdot \mathbb{U}$ such that αg and βg are equal in the sense that there are unary commutativities in $\mathbb{U}$ equating the images under g of the θ_Ys and the θ_vs.

We can extend this precisely to 2-cells in $\mathfrak{Con}_{\geqslant}$. If g'_λ are two homomorphisms from $\mathsf{Eq}(\alpha, \beta)$ to $\mathbb{U}$, then a 2-cell from g'_0 to g'_1 is equivalent to a 2-cell from cg'_0 to cg'_1.

Lemma 47 $\mathfrak{Con}$ *has equifiers.*

Proof Suppose in $\mathfrak{Con}$ we have two 2-cells between $\mathbb{T}_0$ and $\mathbb{T}_1$ with equal domain and codomain. Then by taking common refinements, and vertically composing one of the 2-cells with object equalities, we can suppose without loss of generality that our 2-cells are given by a suitable equivalence extension $e\colon \mathbb{T}_0 \Subset \mathbb{T}'_0$ and, entirely in $\mathfrak{Con}_{\geqslant}$, two 2-cells between $\mathbb{T}'_0$ and $\mathbb{T}_1$ with equal domain and codomain. Then $\mathsf{Eq}(\alpha, \beta)$, mapped through to $\mathbb{T}_0$ using (Id, e), provides the equifier we seek.



The rest is similar to Lemma 45. □

Theorem 48 $\mathfrak{Con}$ *has finite pie limits.*

Proof This is the combined content of Lemmas 43, 45 and 47. □

We emphasize again that the inserters and equifiers do not have canonical constructions, because they depend on the representations of 1-cells.

9 A concrete construction of $\mathbf{AU}\langle\mathbb{T}\rangle$

We can define a 2-functor $\mathbf{AU}\langle-\rangle\colon \mathfrak{Con} \to \mathbf{AU}_s^{op}$, acting on objects as $\mathbb{T} \mapsto \mathbf{AU}\langle\mathbb{T}\rangle$. (At the 1-category level this is immediate from Theorem 33, using Proposition 19 and Lemma 21.)

The main result of this section, Theorem 49, is that this 2-functor is representable, with $\mathbf{AU}\langle\mathbb{T}\rangle$ isomorphic to $\mathfrak{Con}(\mathbb{T},\mathbb{O})$. We also show, Theorem 51, that it is full and faithful: thus *all* strict AU-functors between AUs of the form $\mathbf{AU}\langle\mathbb{T}\rangle$, with $\mathbb{T}$ a context, can be got by the finitary means of constructions in $\mathfrak{Con}$.

Finally we shall show how the construction itself can be conducted entirely within the logic of AUs. This is in the spirit of the idea that AU constructions should be internalizable within AUs, the idea that inspired Joyal's original use of them with regard to Gödel's Theorem.

For the 2-cells, first note that $\mathbf{AU}\langle\mathbb{T}^{\to}\rangle$ is a tensor $\mathbf{2} \otimes \mathbf{AU}\langle\mathbb{T}\rangle$ in $\mathbf{AU}_s$. This is because a strict AU-functor $\mathbf{AU}\langle\mathbb{T}^{\to}\rangle \to \mathcal{A}$ is equivalent to a strict model of $\mathbb{T}^{\to}$ in $\mathcal{A}$, which is equivalent to a strict model of $\mathbb{T}$ in $\mathcal{A}\downarrow\mathcal{A}$, which is equivalent to a strict AU-functor $\mathbf{AU}\langle\mathbb{T}\rangle \to \mathcal{A}\downarrow\mathcal{A}$, which is equivalent to a 2-cell between $\mathbf{AU}\langle\mathbb{T}\rangle$ and $\mathcal{A}$ with domain and codomain both strict.

Hence $\mathbf{AU}\langle\mathbb{T}^{\to}\rangle$ is a cotensor $\mathbf{2} \pitchfork \mathbf{AU}\langle\mathbb{T}\rangle$ in $\mathbf{AU}_s^{op}$. Thus we find that 2-cells in $\mathfrak{Con}$, which are 1-cells to some $\mathbb{T}^{\to}$, are mapped to 2-cells in $\mathbf{AU}_s^{op}$, and this preserves vertical and horizontal composition.

We next investigate the categories $\mathfrak{Con}(\mathbb{T},\mathbb{O})$. The objects and morphisms of this are the nodes and edges of equivalence extensions of $\mathbb{T}$, all modulo objective equality.

Theorem 49 *Let $\mathbb{T}$ be a context. Then $\mathfrak{Con}(\mathbb{T},\mathbb{O})$ is an AU freely presented by $\mathbb{T}$, in other words $\mathbf{AU}\langle\mathbb{T}\rangle \cong \mathfrak{Con}(\mathbb{T},\mathbb{O})$.*

Proof All the AU constructions can be captured by equivalence extensions, and have the necessary properties. The rules of object equalities (for nodes) and fillin uniqueness (for edges) ensure that the constructions yield equals when applied to equals, and so have canonical representatives. Thus $\mathfrak{Con}(\mathbb{T},\mathbb{O})$ is an AU.

To show its freeness property, let M be a strict model of $\mathbb{T}$ in $\mathcal{A}$. Then any object or morphism in $\mathfrak{Con}(\mathbb{T},\mathbb{O})$ gets a unique interpretation in $\mathcal{A}$ by model extension along the equivalence extension used. This respects objective equality, and so yields a well defined interpretation of the object or morphism. □

Proposition 50 *Let $\mathbb{T}, \mathbb{T}_0$ be contexts. If (e,f) is a context map from $\mathbb{T}$ to $\mathbb{T}_0$, then the nodes and edges of $\mathbb{T}_0$, translated along f, give a strict model of $\mathbb{T}_0$ in $\mathfrak{Con}(\mathbb{T},\mathbb{O})$. This induces a bijection between*

- *context maps from $\mathbb{T}$ to $\mathbb{T}_0$ (modulo objective equality), and*
- *strict models of $\mathbb{T}_0$ in $\mathfrak{Con}(\mathbb{T},\mathbb{O})$.*

Proof Objective equality of the context maps is determined solely by objective equalities for their nodes and edges, which is equality of the models in $\mathfrak{Con}(\mathbb{T},\mathbb{O})$. Hence we have injectivity.

For surjectivity, each piece of data for a strict model of $\mathbb{T}_0$ is expressed in an equivalence extension of $\mathbb{T}$. There are only finitely many of these, so they have a common refinement e, say, and then the strict model can be expressed as a context map (e,f). □

Theorem 51 *The 2-functor $\mathbf{AU}\langle -\rangle$ is full and faithful on 1-cells and 2-cells.*

Proof Let $\mathbb{T}_0$ and $\mathbb{T}_1$ be contexts. Strict AU-functors $\mathbf{AU}\langle\mathbb{T}_1\rangle \to \mathbf{AU}\langle\mathbb{T}_0\rangle$ are equivalent to strict models of $\mathbb{T}_1$ in $\mathbf{AU}\langle\mathbb{T}_0\rangle \cong \mathfrak{Con}(\mathbb{T},\mathbb{O})$, and these are equivalent to 1-cells in $\mathfrak{Con}$.

The result for 2-cells follows by considering maps to arrow contexts $\mathbb{T}_1^{\to}$. □

We now look at the concrete construction in AU logic.

Each kind σ of simple extension or simple equivalence extension takes some given *data*, and produces a *delta*. The possible data are given by a functor Dat_σ from sketches to sets. More carefully, an element of $\mathsf{Dat}_\sigma(\mathbb{T})$ is some finite tuple of elements of carriers in $\mathbb{T}$, subject to some equations. Hence Dat_σ can be understood as an object of the Cartesian classifying category for the unary theory of sketches, and for any sketch $\mathbb{T}$ in a Cartesian category $\mathcal{C}$, $\mathsf{Dat}_\sigma(\mathbb{T})$ is an object of $\mathcal{C}$. If the sketch $\mathbb{T}$ is in an AU, then, for each element of $\mathsf{Dat}_\sigma(\mathbb{T})$, the delta now gives us a proto-extension $\mathbb{T} \lessdot \mathbb{T}'$.

Since there are only finitely many kinds of simple extension or simple equivalence extension, in an AU we can sum over them and get:

$$\mathsf{Dat}_{\mathrm{s}\subset} \triangleq \sum\{\mathsf{Dat}_\sigma \mid \sigma \text{ a kind of simple extension}\}$$
$$\mathsf{Dat}_{\mathrm{s}\Subset} \triangleq \sum\{\mathsf{Dat}_\sigma \mid \sigma \text{ a kind of simple equivalence extension}\}$$

Let us now restrict ourselves to strongly finite sketches, in other words, sketches in the category Fin whose objects are natural numbers and whose morphisms are functions

between the corresponding finite cardinals. This can be defined internally in any AU. We obtain an internal graph $\mathsf{Sk}_{s\subset}$ whose nodes are strongly finite sketches $\mathbb{T}$, and whose edges are pairs $(\mathbb{T}, e \in \mathsf{Dat}_{s\subset}(\mathbb{T}))$: the source is $\mathbb{T}$, the target is the corresponding simple extension $\mathbb{T}'$. Note that we can, and shall, choose the deltas in such a way that, for every carrier, the corresponding carrier function for the extension is the natural inclusion for some natural numbers $m \leq n$. We write $\mathsf{Sk}_{\subset}$ for the path category of $\mathsf{Sk}_{s\subset}$, its morphisms being the composable tuples of edges. (Note that two different paths could still give the same extension.)

We can now take the contexts to be the targets of extensions whose domains are the empty sketch $\mathbb{1}$.

Next we do the same with equivalence extensions, to obtain a graph $\mathsf{Sk}_{s\Subset}$ and its path category $\mathsf{Sk}_{\Subset}$.

Note that if $f\colon \mathbb{T}_1 \prec \mathbb{T}_2$ then f extends to a function $\mathsf{Dat}_{s\subset}(\mathbb{T}_1) \to \mathsf{Dat}_{s\subset}(\mathbb{T}_2)$, and so transforms any extension c of $\mathbb{T}_1$ into one $f(c)$ of $\mathbb{T}_2$. This is the reindexing, and it applies similarly to equivalence extensions.

From these ingredients we can now, internally in any AU, define the 2-category $\mathfrak{Con}$ and also, from any internal context $\mathbb{T}$, define $\mathfrak{Con}(\mathbb{T}, \mathbb{O})$ and hence $\mathbf{AU}\langle\mathbb{T}\rangle$.

10 Conclusion

The technical achievement of the present paper, of providing a finitary means for dealing with arbitrary strict AU-functors between certain finitely presented AUs, was a necessary first step in pursuing the programme set out in Vickers [12], with its goal of using AUs to provide a uniform, base-independent setting for geometric reasoning about toposes as generalized spaces.

That programme is developed further in Vickers [16]. It shows in particular how extension maps $U\colon \mathbb{T}_1 \supset \mathbb{T}_0$ in $\mathfrak{Con}$ can be understood as *bundles:* for each model M of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$ with nno, the fibre over M is a certain bounded $\mathcal{S}$-topos, the generalized space over $\mathcal{S}$ of $\mathbb{T}_1$-models N for which $U(N) = M$. Hazratpour and Vickers [4] show furthermore that if U is an (op)fibration in $\mathfrak{Con}$ then the bounded geometric morphism for each M is an (op)fibration in the 2-category of elementary toposes with nno.

Experience with geometric reasoning has shown that many of the constructions can be conducted within the "arithmetic" AU constraints, so an obvious direction of

investigation is to attempt to express them within the finitary formalism developed in the present paper. At the same time, it is also necessary to clarify the connection with the type theory for AUs as set out in Maietti and Vickers [6].

Another pressing need is for a coherent account of the "geometricity" properties of point-free hyperspaces and related constructions. Existing accounts such as that of Vickers [13] prove that the constructions are preserved up to isomorphism by pullback of bundles, but do not express any coherence properties of those isomorphisms. It is to be hoped that that will become clearer in the arithmetic account when bundles are understood as extensions.

A big question, requiring much further work, is that of whether the AU formalization is capable of providing a satisfactory substitute for Grothendieck toposes as an alternative account of generalized spaces. Already it seems clear that they can support much of arithmetic, algebra, real analysis, and logic. The critical question is whether they can also recapture the invariants, such as cohomology, for which Grothendieck invented toposes as generalized spaces. Many of the established techniques rely on a classical ambient set theory that is alien to the spirit of the finitary constructions with AU-contexts.

Many of the technical details in the present paper are open to change. Our axiomatization of AUs is surely not the final word on the subject. Nonetheless, the broad approach of sketches, using the novel notions of extensions, contexts, equivalence extensions and object equalities, would seem to have very general usefulness. They would apply not only to modified definitions of AU but also to other logics. A simple and important fragment of the AU theory would be a sketch account of Cartesian logic. Following the ideas of Palmgren and Vickers [10], we would expect there to be a rich interplay between that and the AU theory, because the AU logic is what is needed to construct initial models for Cartesian theories.

11 Acknowledgements

I am grateful to the organizers of the 5th Workshop on Formal Topology, held at the Institute Mittag-Leffler, Stockholm, on 8-10 June 2015, for the opportunity to outline the ideas of this paper there.

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Received: 19 July 2018 Revised: 27 May 2019