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# A point-free characterisation of Bishop locally compact metric spaces

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*Abstract*: We give a characterisation of Bishop locally compact metric spaces in terms of formal topology. To this end, we introduce the notion of inhabited enumerably locally compact regular formal topology, and show that the category of Bishop locally compact metric spaces is equivalent to the full subcategory of formal topologies consisting of those objects which are isomorphic to some inhabited enumerably locally compact regular formal topology.

In the course of obtaining the above equivalence, we show a couple of point-free results which are of independent interest. First, we show that any overt enumerably locally compact regular formal topology admits a one-point compactification, i.e. it can be embedded into a compact overt enumerably completely regular formal topology as the open complement of a formal point. Second, we characterise the class of enumerably completely regular formal topologies as the subtopolgies of the product of countably many copies of the formal unit interval.

We work in Aczel's constructive set theory CZF with Regular Extension Axiom and Dependent Choice.

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## 1 Introduction

In locale theory (Johnstone [13]), the standard adjunction between the category of topological spaces and that of locales restricts to an equivalence between the category of sober spaces and that of spatial locales. The equivalence allows us to transfer results between general topology and locale theory.

Aczel [1] showed that the adjunction is constructively valid by replacing the notion of locale with Sambin's notion of formal topology [19]. As was stressed by Palmgren [18], however, the adjunction is of little practical use constructively since some of the important examples of formal topologies cannot be shown to be spatial. In particular, as shown by Fourman and Grayson [11, Theorem 4.10] the spatiality of the formal reals is equivalent to the compactness of the unit interval, and a proof of the latter requires the Fan theorem. Since the Fan theorem is not acceptable in Bishop constructive mathematics [3], the current situation prevents us from applying the results in formal topology to Bishop's theory of metric spaces [3, Chapter 4].

To overcome this difficulty, Palmgren [17] constructed another embedding, a full and faithful functor $\mathcal{M}\colon \mathbf{LCM} \to \mathbf{FTop}$, from the category of locally compact metric spaces **LCM** into that of formal topologies **FTop**, using the localic completion of generalized metric spaces due to Vickers [21]. Unlike the standard adjunction, the embedding $\mathcal{M}$ has important properties that a metric space $X$ is compact if and only if $\mathcal{M}(X)$ is compact and that $\mathcal{M}(X)$ is locally compact whenever $X$ is locally compact.

In our previous work [14, Chapter 4], we characterised the image of the category of compact metric spaces under the embedding $\mathcal{M}$ using the notion of compact overt enumerably completely regular formal topology. This means that the category of compact metric spaces is equivalent to the full subcategory of **FTop** consisting of those formal topologies which are isomorphic to some compact overt enumerably completely regular formal topology.

In the present paper, we extend the characterisation to the class of Bishop locally compact metric spaces. We introduce the notion of inhabited enumerably locally compact regular formal topology and show that the class of inhabited enumerably locally compact regular formal topologies characterises the image of Bishop locally compact metric spaces under the embedding $\mathcal{M}$ up to isomorphism. Specifically, we show that the category of Bishop locally compact metric spaces is equivalent to the full subcategory of formal topologies consisting of those objects which are isomorphic to some inhabited enumerably locally compact regular formal topology (Theorem 7.6).

In the course of obtaining the above equivalence, we show a couple of new results which are of independent interest. First, we show that any overt enumerably locally compact regular formal topology admits a one-point compactification (Theorem 6.5). Second, we characterise the class of enumerably completely regular formal topologies as the subtopolgies of the product of countably many copies of the formal unit interval (Proposition 5.4).

The paper is organised as follows. Section 2 and Section 3 contain background on formal topologies and the embedding of locally compact metric spaces into formal topologies by Palmgren [17], respectively. The rest of the paper consists of our original contributions.

In Section 4 we give a sufficient condition under which a formal topology is isomorphic to the localic completion of a Bishop locally compact metric space (Corollary 4.15). To this end, we introduce the notion of the open complement of a located subtopology. In Section 5 we characterise enumerably completely regular formal topologies by the subtopologies of the countable product of the formal unit interval (Proposition 5.4). In Section 6 we construct a one-point compactification of an overt enumerably locally compact regular formal topology (Theorem 6.5). In Section 7 we show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space up to isomorphism (Theorem 7.6).

We work informally in Aczel's constructive set theory CZF (Aczel and Rathjen [2]) extended with the Regular Extension Axiom (REA) and Dependent Choice (DC). Our previous work [14, Chapter 4] on which this paper depends was carried out in the same system. The axiom REA is needed to define the notion of inductively generated formal topology (see Section 2.1).

For the background on Bishop metric spaces, the reader is referred to Bishop and Bridges [4, Chapter 4].

**Notation 1** We define some terms and notations which we frequently use in this paper.

First, when we say that $A$ is a set, it mean that $A$ forms a set in CZF, and when we say that $A$ is a class, it means that $A$ is a definable class of CZF, i.e. its member can be specified using a formula of CZF.

Let $S$ be a set. Then $\mathrm{Pow}(S)$ denotes the class of subsets of $S$. Note that since CZF is predicative, $\mathrm{Pow}(S)$ cannot be shown to be a set unless $S = \emptyset$. $\mathrm{Fin}(S)$ denotes the *set* of finitely enumerable subsets of $S$, where a set $A$ is *finitely enumerable* if there exists a surjection $f\colon \{0, \ldots, n-1\} \to A$ for some $n \in \mathbb{N}$. For subsets $U, V \subseteq S$, we define

$$U \between V \overset{\text{def}}{\iff} (\exists a \in S)\, a \in U \cap V.$$

The complement of a subset $U \subseteq S$ is denoted by $\neg U$:

$$\neg U \overset{\text{def}}{=} \{a \in S \mid a \notin U\}.$$

If $r \subseteq X \times S$ is a relation between sets $X$ and $S$, we define

$$rD \overset{\text{def}}{=} \{a \in S \mid (\exists x \in D)\, x\, r\, a\},$$
$$r^- U \overset{\text{def}}{=} \{x \in X \mid (\exists a \in U)\, x\, r\, a\},$$
$$r^{-*} D \overset{\text{def}}{=} \left\{a \in S \mid r^- \{a\} \subseteq D\right\}$$

for any subsets $D \subseteq X$ and $U \subseteq S$. We often write $r^- a$ for $r^- \{a\}$.

## 2 Formal topologies

We recall the relevant facts about formal topology. See Sambin [20] and Fox [12] for further details.

**Definition 2.1** A *formal topology* $\mathcal{S}$ is a triple $(S, \lhd, \leq)$ where $(S, \leq)$ is a preordered set and $\lhd$ is a relation between $S$ and $\mathrm{Pow}(S)$ such that

$$\mathcal{A}\,U \stackrel{\text{def}}{=} \{a \in S \mid a \lhd U\}$$

is a set for each $U \subseteq S$ and that

(1) $U \lhd U$,

(2) $a \lhd U \;\&\; U \lhd V \implies a \lhd V$,

(3) $a \lhd U \;\&\; a \lhd V \implies a \lhd U \downarrow V$,

(4) $a \leq b \implies a \lhd b$

for all $a, b \in S$ and $U, V \subseteq S$, where

$$U \lhd V \stackrel{\text{def}}{\iff} (\forall a \in U)\, a \lhd V,$$
$$U \downarrow V \stackrel{\text{def}}{=} \{c \in S \mid (\exists a \in U)(\exists b \in V)\, c \leq a \;\&\; c \leq b\}.$$

We write $a \downarrow U$ for $\{a\} \downarrow U$ and $U \lhd a$ for $U \lhd \{a\}$. The set $S$ is called the *base* of $\mathcal{S}$, and the relation $\lhd$ is called a *cover* on $(S, \leq)$, or the cover of $\mathcal{S}$. For any $U, V \subseteq S$ we define

$$U =_{\mathcal{S}} V \stackrel{\text{def}}{\iff} \mathcal{A}\,U = \mathcal{A}\,V.$$

**Notation 2** In this paper, the letters $\mathcal{S}, \mathcal{S}', \mathcal{T}, \ldots$ denote formal topologies. If $\mathcal{S}$ is a formal topology, the symbols $S$, $\lhd$ and $\leq$ denote the base, the cover and the preorder of $\mathcal{S}$ respectively. Subscripts or superscripts are sometimes added to those symbols for clarity. For example, the base, the cover and the preorder of a formal topology $\mathcal{S}'$ will be denoted by $S'$, $\lhd'$ and $\leq'$ respectively.

**Definition 2.2** Let $\mathcal{S}$ and $\mathcal{S}'$ be formal topologies. A relation $r \subseteq S \times S'$ is called a *formal topology map* from $\mathcal{S}$ to $\mathcal{S}'$ if

(FTM1) $S \lhd r^{-}S'$,

(FTM2) $r^{-}a \downarrow r^{-}b \lhd r^{-}(a \downarrow' b)$,

(FTM3) $a \lhd' U \implies r^{-}a \lhd r^{-}U$

for all $a, b \in S'$ and $U \subseteq S'$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be formal topologies. Two formal topology maps $r, s \colon \mathcal{S} \to \mathcal{S}'$ are defined to be *equal*, denoted by $r = s$, if

$$r^- a =_{\mathcal{S}} s^- a$$

for all $a \in S'$.

The formal topologies and formal topology maps between them form a category **FTop**. The composition of two formal topology maps is the composition of the underlying relations of these maps. The identity morphism on a formal topology is the identity relation on its base.

The formal topology $\mathbf{1} \overset{\text{def}}{=} \left(\{*\}, \in, =\right)$ is a terminal object in **FTop**. A formal topology map $r : \mathbf{1} \to \mathcal{S}$ is equivalent to the following notion.

**Definition 2.3** Let $\mathcal{S}$ be a formal topology. A subset $\alpha \subseteq S$ is called a *formal point* of $\mathcal{S}$ if

(P1) $S \between \alpha$,

(P2) $a, b \in \alpha \implies \alpha \between (a \downarrow b)$,

(P3) $a \in \alpha \;\&\; a \lhd U \implies \alpha \between U$

for all $a, b \in S$ and $U \subseteq S$. The class of formal points of $\mathcal{S}$ is denoted by $\mathrm{Pt}(\mathcal{S})$.

A formal topology often comes equipped with a positivity predicate.

**Definition 2.4** Let $\mathcal{S}$ be a formal topology. A subset $V \subseteq S$ is said to be *splitting* if

$$a \in V \;\&\; a \lhd U \implies V \between U$$

for all $a \in S$ and $U \subseteq S$.

**Definition 2.5** A *positivity predicate* (or just a positivity) on a formal topology $\mathcal{S}$ is a splitting subset $\mathrm{Pos} \subseteq S$ which satisfies

(Pos) $$a \lhd \{x \in S \mid x = a \;\&\; \mathrm{Pos}(a)\}$$

for all $a \in S$, where we write $\mathrm{Pos}(a)$ if $a \in \mathrm{Pos}$.

A formal topology is *overt* if it is equipped with a positivity predicate. A formal topology is *inhabited* if it is overt and its positivity is inhabited.

Let $\mathcal{S}$ be a formal topology. By the condition (Pos), a positivity predicate on $\mathcal{S}$, if it exists, is the largest splitting subset of $\mathcal{S}$. Thus $\mathcal{S}$ admits at most one positivity predicate. Note also that every formal point of $\mathcal{S}$ is a splitting subset of $\mathcal{S}$. Hence, if $\mathcal{S}$ is overt with a positivity predicate Pos, then $\alpha \subseteq \mathrm{Pos}$ for any formal point $\alpha \in \mathrm{Pt}(\mathcal{S})$.

## 2.1 Inductively generated formal topologies

The notion of inductively generated formal topology by Coquand et al. [7] gives us a convenient method to define formal topologies.

**Definition 2.6** Let $S$ be a set. An *axiom-set* on $S$ is a pair $(I, C)$ where $(I(a))_{a \in S}$ is a family of sets indexed by $S$, and $C$ is a family $(C(a,i))_{a \in S, i \in I(a)}$ of subsets of $S$ indexed by $\sum_{a \in S} I(a)$. For each $a \in S$ and $i \in I(a)$, the pair $(a, C(a,i))$ is called an *axiom* of $(I, C)$.

We recall the main result of the work by Coquand et al. [7, Theorem 3.3].

**Theorem 2.7** *Let $(S, \leq)$ be a preordered set, and let $(I, C)$ be an axiom-set on $S$. Let $\triangleleft_{I,C}$ be the relation between $S$ and $\mathrm{Pow}(S)$ generated by the following rules:*

$$\frac{a \in U}{a \triangleleft_{I,C} U} \text{ (reflexivity)}, \qquad \frac{a \leq b \quad b \triangleleft_{I,C} U}{a \triangleleft_{I,C} U} \text{ ($\leq$-left)},$$

$$\frac{a \leq b \quad i \in I(b) \quad a \downarrow C(b,i) \triangleleft_{I,C} U}{a \triangleleft_{I,C} U} \text{ ($\leq$-infinity)}.$$

*Then $\triangleleft_{I,C}$ is the least cover on $(S, \leq)$ such that $a \triangleleft_{I,C} C(a,i)$ for all $a \in S$ and $i \in I(a)$.*

A formal topology $\mathcal{S} = (S, \triangleleft, \leq)$ is *inductively generated* if it is equipped with an axiom-set $(I, C)$ on $S$ such that $\triangleleft = \triangleleft_{I,C}$.

**Remark 2.8** In Definition 2.2 of a formal topology map, if the formal topology $\mathcal{S}'$ is inductively generated by an axiom-set $(I, C)$ on $S'$, then the condition (FTM3) is equivalent to the following conditions under the condition (FTM2).

(FTM3a) $\quad a \leq' b \implies r^- a \triangleleft r^- b,$

(FTM3b) $\quad r^- a \triangleleft r^- C(a,i)$

for all $a, b \in S'$ and $i \in I(a)$.

Similarly, in Definition 2.3 of a formal point, if the formal topology $\mathcal{S}$ is inductively generated by an axiom-set $(I, C)$ on $S$, then the condition (P3) is equivalent to the following conditions:

(P3a) $\quad a \leq b \;\&\; a \in \alpha \implies b \in \alpha,$

(P3b) $\quad a \in \alpha \implies \alpha \between C(a,i)$

for all $a, b \in S$ and $i \in I(a)$. See Fox [12, Section 4.1.2] for details.

**Example 2.9** Let $\mathbb{Q}$ be the set of rationals, and let

$$S_{\mathcal{R}} \stackrel{\text{def}}{=} \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}.$$

Define a preorder $\leq_{\mathcal{R}}$ and a transitive relation $<_{\mathcal{R}}$ on $S_{\mathcal{R}}$ by

$$(p,q) \leq_{\mathcal{R}} (r,s) \stackrel{\text{def}}{\iff} r \leq p \;\&\; q \leq s,$$
$$(p,q) <_{\mathcal{R}} (r,s) \stackrel{\text{def}}{\iff} r < p \;\&\; q < s$$

for all $(p,q),(r,s) \in S_{\mathcal{R}}$. The *formal reals* $\mathcal{R}$ is a formal topology $(S_{\mathcal{R}}, \lhd_{\mathcal{R}}, \leq_{\mathcal{R}})$ inductively generated by an axiom-set on $S_{\mathcal{R}}$ consisting of the following axioms for each $(p,q) \in S_{\mathcal{R}}$:

(R1) $(p,q) \lhd_{\mathcal{R}} \{a \in S_{\mathcal{R}} \mid a <_{\mathcal{R}} (p,q)\}$,

(R2) $(p,q) \lhd_{\mathcal{R}} \{(p,s),(r,q)\}$ for each $(r,s) \in S_{\mathcal{R}}$ such that $(r,s) <_{\mathcal{R}} (p,q)$.

It is well known that the class of formal points of $\mathcal{R}$ is isomorphic to the Dedekind cuts. See Fourman and Grayson [11], Negri and Soravia [16] and Coquand et al. [7] for further details.

### 2.1.1 Products

We recall the construction of a product of a family of inductively generated formal topologies by Vickers [22][1]. Let $(\mathcal{S}_i)_{i \in I}$ be a set-indexed family of inductively generated formal topologies each of which is of the form $\mathcal{S}_i = (S_i, \lhd_i, \leq_i)$ and generated by an axiom-set $(K_i, C_i)$ on $S_i$. Define a preorder $(S_\Pi, \leq_\Pi)$ by

$$S_\Pi \stackrel{\text{def}}{=} \mathrm{Fin}\Big(\sum_{i \in I} S_i\Big),$$
$$A \leq_\Pi B \stackrel{\text{def}}{\iff} \big(\forall (i,b) \in B\big)\big(\exists (j,a) \in A\big)\; i = j \;\&\; a \leq_i b$$

for all $A, B \in S_\Pi$. Define an axiom-set on $S_\Pi$ as follows:

(S1) $A \lhd_\Pi \{\{(i,a)\} \in S_\Pi \mid a \in S_i\}$ for each $A \in S_\Pi$ and $i \in I$,

(S2) $\{(i,a),(i,b)\} \lhd_\Pi \{\{(i,c)\} \in S_\Pi \mid c \leq_i a \;\&\; c \leq_i b\}$ for each $i \in I$ and $a, b \in S_i$,

[1] In order to construct a product of a family of formal topologies predicatively, we need to know that each member the family is inductively generated. Whether this requirement is really necessary is not known. Of course, for the empty and singleton families, the construction of their product is trivial. Impredicatively, the construction of products of locales is well known (see Johnstone [13, Chapter II, Proposition 2.12]).

(S3) $\{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in C_i(a,k)\}$ for each $i \in I$, $a \in S_i$ and $k \in K_i(a)$.

Let $\prod_{i\in I} \mathcal{S}_i = (S_\Pi, \lhd_\Pi, \leq_\Pi)$ be the formal topology inductively generated by (S1), (S2) and (S3).

For each $i \in I$, the projection $p_i \colon \prod_{i\in I} \mathcal{S}_i \to \mathcal{S}_i$ is defined by

$$A \; p_i \; a \overset{\text{def}}{\iff} A = \{(i,a)\}$$

for all $A \in S_\Pi$ and $a \in S_i$. By the definition of $\prod_{i\in I} \mathcal{S}_i$, the relation $p_i$ is a formal topology map. Then the family $\left(p_i \colon \prod_{i\in I} \mathcal{S}_i \to \mathcal{S}_i\right)_{i\in I}$ is a product of $(\mathcal{S}_i)_{i\in I}$. In particular, given any family $(r_i \colon \mathcal{S} \to \mathcal{S}_i)_{i\in I}$ of formal topology maps, there exists a unique formal topology map $r \colon \mathcal{S} \to \prod_{i\in I} \mathcal{S}_i$ such that $r_i = p_i \circ r$ for all $i \in I$. The formal topology map $r$ is defined by

$$a \; r \; A \overset{\text{def}}{\iff} \left(\forall (i,b) \in A\right) a \lhd r_i^{-} b$$

for all $a \in S$ and $A \in S_\Pi$.

For later use, we note the following facts.

**Lemma 2.10** *Let $i \in I$. Then*

$$a \lhd_i U \implies \{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in U\}$$

*for all $a \in S_i$ and $U \subseteq S_i$.*

**Proof** This follows from the definition of the projection $p_i \colon \prod_{i\in I} \mathcal{S}_i \to \mathcal{S}_i$ and the condition (FTM3) for a formal topology map. □

**Corollary 2.11** *Let $\{i_0,\dots,i_{n-1}\} \in \mathrm{Fin}(I)$, and for each $k < n$ let $a_k \in S_{i_k}$ and $U_k \subseteq S_{i_k}$ such that $a_k \lhd_{i_k} U_k$. Then*

$$\{(i_0,a_0),\dots,(i_{n-1},a_{n-1})\} \lhd_\Pi \{\{(i_0,b_0),\dots,(i_{n-1},b_{n-1})\} \in S_\Pi \mid \left(\forall k < n\right) b_k \in U_k\}.$$

## 2.2 Open subtopologies and closed subtopologies

**Definition 2.12** A *subtopology* of a formal topology $\mathcal{S} = (S, \lhd, \leq)$ is a formal topology $\mathcal{T} = (S, \lhd^{\mathcal{T}}, \leq)$ such that

$$a \lhd U \implies a \lhd^{\mathcal{T}} U$$

for all $a \in S$ and $U \subseteq S$. If $\mathcal{T}$ is a subtopology of $\mathcal{S}$, we write $\mathcal{T} \sqsubseteq \mathcal{S}$.

Given a formal topology map $r\colon \mathcal{S} \to \mathcal{S}'$, the relation $\lhd_r$ between $S'$ and $\mathrm{Pow}(S')$ defined by

$$a \lhd_r U \overset{\text{def}}{\iff} r^- a \lhd r^- U$$

is a cover on $(S', \leq')$. The formal topology $S_r = (S', \lhd_r, \leq')$ is called the *image* of $\mathcal{S}$ under $r$.

A formal topology map $r\colon \mathcal{S} \to \mathcal{S}'$ is an *embedding* if $r$ restricts to an isomorphism between $\mathcal{S}$ and its image $\mathcal{S}_r$.

By the condition (FTM3) for a formal topology map, we have $\mathcal{S}_r \sqsubseteq \mathcal{S}'$ for any formal topology map $r\colon \mathcal{S} \to \mathcal{S}'$. If $\mathcal{T}$ is a subtopology of $\mathcal{S} = (S, \lhd, \leq)$, then the identity relation $id_S$ on $S$ is an embedding $id_S\colon \mathcal{T} \to \mathcal{S}$. Hence the notion of embedding is essentially equivalent to that of subtopology.

It can be shown that $r\colon \mathcal{S} \to \mathcal{S}'$ is an embedding if and only if

$$a \lhd r^- r^{-*} \mathcal{A}\{a\}$$

for all $a \in S$. See Fox [12, Proposition 3.5.2].

The following is well known.

**Lemma 2.13** *Let $\mathcal{S}$ be an overt formal topology with a positivity* Pos, *and let $r\colon \mathcal{S} \to \mathcal{S}'$ be a formal topology map. Then the image $\mathcal{S}_r$ of $\mathcal{S}$ under $r$ is overt with the positivity*

$$r\,\mathrm{Pos} \overset{\text{def}}{=} \left\{a \in S' \mid (\exists b \in \mathrm{Pos})\, b\ r\ a\right\}.$$

**Proof** It is straightforward to show that $r\,\mathrm{Pos}$ is a splitting subset of $\mathcal{S}_r$. To see that $r\,\mathrm{Pos}$ satisfies the condition (Pos), let $\lhd_r$ be the cover of $\mathcal{S}_r$, and let $a \in S'$. We must show that $a \lhd_r \{a\} \cap r\,\mathrm{Pos}$. Let $b \in r^- a$, and suppose that $b \in \mathrm{Pos}$. Then $a \in r\,\mathrm{Pos}$, so that $b \in r^-(\{a\} \cap r\,\mathrm{Pos})$. Hence $r^- a \lhd r^-(\{a\} \cap r\,\mathrm{Pos})$, and thus $a \lhd_r \{a\} \cap r\,\mathrm{Pos}$. □

**Definition 2.14** Let $\mathcal{S}$ be a formal topology and let $V \subseteq S$. The *open subtopology* of $\mathcal{S}$ determined by $V$ is a subtopology $\mathcal{S}_V$ of $\mathcal{S}$ with the cover $\lhd_V$ given by

$$a \lhd_V U \overset{\text{def}}{\iff} a \downarrow V \lhd U$$

for all $a \in S$ and $U \subseteq S$.

**Lemma 2.15** *Let $\mathcal{S}$ be a formal topology, and let $\mathcal{S}_V$ be the open subtopology of $\mathcal{S}$ determined by $V \subseteq S$.*

(1) *$\mathcal{S}_V$ is the largest subtopology $\mathcal{S}'$ of $\mathcal{S}$ such that $S \lhd' V$.*

(2) *If $\mathcal{S}$ is overt with a positivity* Pos*, then $\mathcal{S}_V$ is overt with the positivity* $\mathrm{Pos}_V$ *given by*

$$\mathrm{Pos}_V \stackrel{\text{def}}{=} \{a \in S \mid \mathrm{Pos} \between (a \downarrow V)\}.$$

**Proof** (1) Since $S \downarrow V \lhd V$ we have $S \lhd_V V$. Let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$ such that $S \lhd' V$. Suppose that $a \lhd_V U$. Then $a \downarrow V \lhd U$, and thus $a \downarrow V \lhd' U$. Hence $a \lhd' a \downarrow S \lhd' a \downarrow V \lhd' U$. Therefore $\mathcal{S}' \sqsubseteq \mathcal{S}_V$.

(2) Suppose that $\mathcal{S}$ is overt with a positivity Pos, and let $\mathrm{Pos}_V$ be the subset of $S$ as defined above. Suppose that $a \lhd_V U$ and $a \in \mathrm{Pos}_V$, that is $a \downarrow V \lhd U$ and $\mathrm{Pos} \between (a \downarrow V)$. Then $a \downarrow V \lhd U \downarrow V$ and thus $\mathrm{Pos} \between (U \downarrow V)$, that is $\mathrm{Pos}_V \between U$. Hence $\mathrm{Pos}_V$ is a splitting subset of $\mathcal{S}_V$. Moreover, for any $a \in S$ we have $a \downarrow V \lhd (a \downarrow V) \cap \mathrm{Pos}$ by the property (Pos) of Pos. Thus $a \lhd_V (a \downarrow V) \cap \mathrm{Pos} \lhd_V \{a\} \cap \mathrm{Pos}_V$. Therefore $\mathrm{Pos}_V$ satisfies (Pos). □

**Definition 2.16** Let $\mathcal{S}$ be a formal topology and let $V \subseteq S$. The *closed subtopology* of $\mathcal{S}$ determined by $V$ is a subtopology $\mathcal{S}^{\mathcal{S}-V}$ of $\mathcal{S}$ with the cover $\lhd^{\mathcal{S}-V}$ given by

$$a \lhd^{\mathcal{S}-V} U \stackrel{\text{def}}{\iff} a \lhd V \cup U$$

for all $a \in S$ and $U \subseteq S$.

**Lemma 2.17** *Let $\mathcal{S}$ be a formal topology and let $V \subseteq S$. Then the closed subtopology $\mathcal{S}^{\mathcal{S}-V}$ is the largest subtopology $\mathcal{S}'$ of $\mathcal{S}$ such that $V \lhd' \emptyset$.*

**Proof** The proof is analogous to that of Lemma 2.15 (1). □

**Definition 2.18** Let $\mathcal{S}$ be a formal topology and let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$. Then the *closure* of $\mathcal{S}'$ in $\mathcal{S}$ is the closed subtopology $\mathcal{S}^{\mathcal{S}-Z}$ of $\mathcal{S}$ determined by the subset

$$Z \stackrel{\text{def}}{=} \left\{a \in S \mid a \lhd' \emptyset\right\}. \tag{1}$$

The closure of a formal topology has an expected property.

**Proposition 2.19** *Let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$. Then the closure of $\mathcal{S}'$ in $\mathcal{S}$ is the smallest closed subtopology of $\mathcal{S}$ that is larger than $\mathcal{S}'$.*

**Proof** Let $\mathcal{S}^{\mathcal{S}-Z}$ be the closure of $\mathcal{S}'$, where $Z \subseteq S$ is defined as in (1). By Lemma 2.17 we have $\mathcal{S}' \sqsubseteq \mathcal{S}^{\mathcal{S}-Z}$. Let $V \subseteq S$ and suppose that $\mathcal{S}' \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$. Then $V \lhd' \emptyset$, so that $V \subseteq Z$. Hence $\mathcal{S}^{\mathcal{S}-Z} \sqsubseteq \mathcal{S}^{\mathcal{S}-V}$. □

**Lemma 2.20** *Let $\mathcal{S}'$ be an overt subtopology of $\mathcal{S}$ with a positivity* Pos*. Then the closure of $\mathcal{S}'$ in $\mathcal{S}$ is the closed subtopology $\mathcal{S}^{\mathcal{S}-\neg \mathrm{Pos}}$.*

**Proof** Let $Z = \{a \in S \mid a \lhd' \emptyset\}$. It suffices to show that $\neg\,\mathrm{Pos} = Z$. Since Pos is the positivity of $\mathcal{S}'$, we have $\neg\,\mathrm{Pos} \lhd' \emptyset$, and thus $\neg\,\mathrm{Pos} \subseteq Z$. Conversely, if $a \lhd' \emptyset$ and $a \in \mathrm{Pos}$ then $\mathrm{Pos} \between \emptyset$, a contradiction. Hence $Z \subseteq \neg\,\mathrm{Pos}$. □

**Example 2.21** (See also Example 2.9) Let $\mathcal{R}$ be the formal reals. The formal unit interval $\mathcal{I}[0,1]$ is the closed subtopology of $\mathcal{R}$ determined by the subset

$$\{(p,q) \in S_{\mathcal{R}} \mid p \geq 1 \vee q \leq 0\}\,.$$

Equivalently, $\mathcal{I}[0,1]$ can be defined as a formal topology $(S_{\mathcal{R}}, \lhd_{\mathcal{I}[0,1]}, \leq_{\mathcal{R}})$ inductively generated by the axioms of $\mathcal{R}$ together with the following axiom for each $(p,q) \in S_{\mathcal{R}}$:

$$(p,q) \lhd_{\mathcal{I}[0,1]} \{(p,q) \mid p < 1 \;\&\; 0 < q\}\,. \tag{2}$$

## 2.3 Regularity, compactness and local compactness

Let $\mathcal{S}$ be a formal topology. For each $a \in S$ define

$$a^* \stackrel{\text{def}}{=} \{b \in S \mid b \downarrow a \lhd \emptyset\}\,,$$

and for each $a, b \in S$ define

$$a \lll b \stackrel{\text{def}}{\iff} S \lhd a^* \cup \{b\}\,.$$

We extend the relation $\lll$ to the subsets of $S$ by defining

$$U \lll V \stackrel{\text{def}}{\iff} S \lhd U^* \cup V$$

for all $U, V \subseteq S$, where $U^* \stackrel{\text{def}}{=} \bigcap_{a \in U} a^*$. We write $a \lll U$ for $\{a\} \lll U$ and $U \lll a$ for $U \lll \{a\}$. By Lemma 2.15, we have that $U \lll V$ if and only if the closure of $\mathcal{S}_U$ is a subtopology of $\mathcal{S}_V$.

It is easy to see that $U \lll V$ implies $U \lhd V$ and that $U' \lhd U \lll V \lhd V'$ implies $U' \lll V'$. Moreover, if $r\colon \mathcal{S} \to \mathcal{S}'$ is a formal topology map, then $U \lll' V$ implies $r^- U \lll r^- V$ for any $U, V \subseteq S'$.

**Definition 2.22** A formal topology $\mathcal{S}$ is *regular* if it is equipped with a function $\mathsf{wc}\colon S \to \mathrm{Pow}(S)$ such that

(1) $\big(\forall b \in \mathsf{wc}(a)\big)\; b \lll a$,

(2) $a \lhd \mathsf{wc}(a)$

for all $a \in S$.

**Remark 2.23** A formal topology $\mathcal{S}$ is regular if and only if

$$a \lhd \{b \in S \mid b \lll a\} \tag{3}$$

for all $a \in S$. Indeed, if $\mathcal{S}$ is regular with a function $\mathsf{wc}\colon S \to \mathrm{Pow}(S)$, then

$$a \lhd \mathsf{wc}(a) \subseteq \{b \in S \mid b \lll a\}\,.$$

Conversely, if $\mathcal{S}$ satisfies (3), we define

$$\mathsf{wc}(a) \stackrel{\text{def}}{=} \{b \in S \mid b \lll a\}\,. \tag{4}$$

Thus, if $\mathcal{S}$ is regular, we always have a canonical choice of the function $\mathsf{wc}\colon S \to \mathrm{Pow}(S)$ that is given by (4).

**Definition 2.24** A formal topology $\mathcal{S}$ is *compact* if

$$S \lhd U \implies \big(\exists U_0 \in \mathrm{Fin}(U)\big)\, S \lhd U_0$$

for all $U \subseteq S$.

The following are well known in locale theory (see Johnstone [13, Chapter III, Section 1]).

**Proposition 2.25**

(1) *A subtopology of a regular formal topology is regular.*

(2) *A closed subtopology of a compact formal topology is compact.*

(3) *A compact subtopology of a regular formal topology is closed.*

**Proof** (1) If $\mathcal{S}$ is regular and $\mathcal{S}'$ is a subtopology of $\mathcal{S}$, then $a \lll b$ in $\mathcal{S}$ implies $a \lll b$ in $\mathcal{S}'$, from which the conclusion follows.

(2) Let $\mathcal{S}$ be a compact formal topology, and let $\mathcal{S}^{\mathcal{S}-V}$ be the closed subtopology of $\mathcal{S}$ determined by a subset $V \subseteq S$. Let $U \subseteq S$ and suppose that $S \lhd^{\mathcal{S}-V} U$. Then $S \lhd V \cup U$. Since $\mathcal{S}$ is compact, there exists $U_0 \in \mathrm{Fin}(U)$ such that $S \lhd V \cup U_0$, that is $S \lhd^{\mathcal{S}-V} U_0$.

(3) See Curi [9, Proposition 2.3]. □

**Proposition 2.26**

(1) *A product of inductively generated regular formal topologies is regular.*

(2) *A product of inductively generated compact formal topologies is compact.*

**Proof** (1) Let $(\mathcal{S}_i)_{i\in I}$ be a family of inductively generated regular formal topologies, and let $(\mathsf{wc}_i)_{i\in I}$ be a family such that for each $i \in I$, $\mathsf{wc}_i \colon S_i \to \mathrm{Pow}(S_i)$ is a function which makes $\mathcal{S}_i$ regular. Let $\prod_{i\in I}\mathcal{S}_i = (S_\Pi, \lhd_\Pi, \leq_\Pi)$ be the product of $(\mathcal{S}_i)_{i\in I}$. Define a function $\mathsf{wc}_\Pi \colon S_\Pi \to \mathrm{Pow}(S_\Pi)$ by

$$\mathsf{wc}_\Pi(A) \overset{\text{def}}{=} \{\{(i_0,b_0),\dots,(i_{n-1},b_{n-1})\} \in S_\Pi \mid (\forall k<n)\ b_k \in \mathsf{wc}_{i_k}(a_k)\}$$

for each $A = \{(i_0,a_0),\dots,(i_{n-1},a_{n-1})\} \in S_\Pi$. Then $A \lhd_\Pi \mathsf{wc}_\Pi(A)$ for all $A \in S_\Pi$ by Corollary 2.11. Let $A, B \in S_\Pi$ such that $B \in \mathsf{wc}_\Pi(A)$. Then $A$ and $B$ are of the forms

$$\begin{aligned} A &= \{(i_0,a_0),\dots,(i_{n-1},a_{n-1})\}, \\ B &= \{(i_0,b_0),\dots,(i_{n-1},b_{n-1})\} \end{aligned}$$

such that $b_k \in \mathsf{wc}_{i_k}(a_k)$ for all $k<n$. Then for each $k<n$, since $S_{i_k} \lhd_{i_k} b_k^* \cup \{a_k\}$, we have $S_\Pi \lhd_\Pi \{\{(i_k,c)\} \in S_\Pi \mid c \in b_k^* \cup \{a_k\}\}$. Thus

$$S_\Pi \lhd_\Pi \{\{(i_0,c_0),\dots,(i_{n-1},c_{n-1})\} \in S_\Pi \mid (\forall k<n)\, c_k \in b_k^* \cup \{a_k\}\}.$$

Let $C = \{(i_0,c_0),\dots,(i_{n-1},c_{n-1})\}$ be an element of $S_\Pi$ such that $c_k \in b_k^* \cup \{a_k\}$ for all $k<n$. Then, either $c_k = a_k$ for all $k<n$ or $c_k \in b_k^*$ for some $k<n$. In the former case we have $C = A$. In the latter case, there exists $k<n$ such that $c_k \in b_k^*$. Then

$$\begin{aligned} C \downarrow B &\lhd_\Pi C \cup B \\ &\lhd_\Pi \{(i_k,c_k),(i_k,b_k)\} \\ &\lhd_\Pi \{\{(i_k,d)\} \in S_\Pi \mid d \in c_k \downarrow b_k\} \\ &\lhd_\Pi \{\{(i_k,d)\} \in S_\Pi \mid d \in \emptyset\} \lhd_\Pi \emptyset, \end{aligned}$$

and so $C \in B^*$. Thus $S_\Pi \lhd_\Pi B^* \cup \{A\}$, that is $B \lll A$. Therefore, the function $\mathsf{wc}_\Pi$ makes $\prod_{i\in I}\mathcal{S}_i$ regular.

(2) See Vickers [22, Theorem 14.6]. □

Let $\mathcal{S}$ be a formal topology. For each $a, b \in S$ define

$$a \ll b \overset{\text{def}}{\iff} \big(\forall U \in \mathrm{Pow}(S)\big)\left[\, b \lhd U \implies \big(\exists U_0 \in \mathrm{Fin}(U)\big)\ a \lhd U_0 \right].$$

Note that $\ll$ is a proper class in general. The class relation $\ll$ is extended to the subsets of $S$ in an obvious way. For any $a \in S$ and $U \subseteq S$, we define $a \ll U \overset{\text{def}}{\iff} \{a\} \ll U$ and $U \ll a \overset{\text{def}}{\iff} U \ll \{a\}$.

**Definition 2.27** A formal topology $\mathcal{S}$ is *locally compact* if it is equipped with a function $\mathsf{wb}\colon S \to \mathrm{Pow}(S)$ such that

(1) $\big(\forall b \in \mathsf{wb}(a)\big)\ b \ll a$,

(2) $a \lhd \mathsf{wb}(a)$

for all $a \in S$.

**Remark 2.28** Since the relation $\ll$ is a proper class in general, the existence of a function $\mathsf{wb}\colon S \to \mathrm{Pow}(S)$ is indispensable for the predicative definition of locally compact formal topologies.

Note, however, that once we know that $\mathcal{S}$ is locally compact with an associated function $\mathsf{wb}\colon S \to \mathrm{Pow}(S)$, we have that

$$a \ll b \iff \big(\exists U \in \mathrm{Fin}\big(\mathsf{wb}(b)\big)\big)\ a \lhd U$$

for all $a, b \in S$. Indeed, the direction $\Rightarrow$ is immediate from the condition (2) on $\mathsf{wb}$. For the opposite direction, suppose that we have a finitely enumerable subset $\{c_0, \cdots, c_{n-1}\} \subseteq \mathsf{wb}(b)$ such that $a \lhd \{c_0, \cdots, c_{n-1}\}$. Let $U \subseteq S$, and suppose that $b \lhd U$. Then, for each $i < n$, there exists $U_i \in \mathrm{Fin}(U)$ such that $c_i \lhd U_i$. Hence, $a \lhd U_0 \cup \cdots \cup U_{n-1}$. Since a finite union of finitely enumerable subsets is again finitely enumerable, we have $a \ll b$.

In summary, a formal topology $\mathcal{S}$ is locally compact if and only if the relation $\ll$ is a set and

$$a \lhd \{b \in S \mid b \ll a\}$$

for all $a \in S$.

We import the notion of boundedness to formal topology from locale theory (see Escardó [10, Definition 4.1]).

**Definition 2.29** Let $\mathcal{S}$ be a formal topology. A subset $U \subseteq S$ is *bounded* if $U \ll S$. A subtopology $\mathcal{S}'$ of $\mathcal{S}$ is *bounded* if there exists a bounded subset $U \subseteq S$ such that $\mathcal{S}' \sqsubseteq \mathcal{S}_U$.

The following seems to be new.

**Proposition 2.30** *Let $\mathcal{S}$ be a locally compact regular formal topology. Then a subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ is compact if and only if $\mathcal{S}'$ is closed and bounded.*

**Proof** Suppose that $\mathcal{S}'$ is compact. Since $\mathcal{S}$ is regular, $\mathcal{S}'$ is closed by Proposition 2.25 (3), and since $S \lhd' \{a \in S \mid (\exists b \in S)\, a \ll b\}$, there exists $U \in \mathrm{Fin}(S)$ such that $S \lhd' U$ and $U \ll S$. Then $\mathcal{S}' \sqsubseteq \mathcal{S}_U$, and so $\mathcal{S}'$ is bounded.

Conversely, suppose that $\mathcal{S}'$ is closed and bounded. Then there exist a subset $V \subseteq S$ and a bounded subset $U \subseteq S$ such that $\mathcal{S}' = \mathcal{S}^{\mathcal{S}-V} \sqsubseteq \mathcal{S}_U$. Let $W \subseteq S$ and suppose that $S \lhd' W$. Then $S \lhd V \cup W$. Since $U \ll S$, there exists $W_0 \in \mathrm{Fin}(W)$ such that $U \lhd' W_0$, and since $\mathcal{S}' \sqsubseteq \mathcal{S}_U$, we have $S \lhd' W_0$. Therefore $\mathcal{S}'$ is compact. □

We note some connections between the relations $\lll$ and $\ll$. The following is due to Escardó [10, Lemma 4.2].

**Lemma 2.31** *Let $\mathcal{S}$ be a formal topology. For any $U, V \subseteq S$ we have*

$$U \ll S \;\&\; U \lll V \implies U \ll V.$$

**Proof** Let $U, V \subseteq S$ and suppose that $U \ll S$ and $U \lll V$. Let $W$ be a subset of $S$ such that $V \lhd W$. Then $S \lhd U^* \cup V \lhd U^* \cup W$. Thus there exists $W_0 \in \mathrm{Fin}(W)$ such that $U \lhd U^* \cup W_0$. Hence $U \lhd (U^* \cup W_0) \downarrow U \lhd (U^* \downarrow U) \cup (W_0 \downarrow U) \lhd W_0$. Therefore $U \ll V$. □

Note that a formal topology $\mathcal{S}$ is compact if and only if $S \ll S$. Thus we have the following, which is well known in locale theory (see Johnstone [13, Chapter VII, Lemma 3.5 (i)]).

**Corollary 2.32** *Let $\mathcal{S}$ be a compact formal topology. For any $U, V \subseteq S$ we have*

$$U \lll V \implies U \ll V.$$

The converse of Corollary 2.32 holds for regular formal topologies (see Johnstone [13, Chapter VII, Lemma 3.5 (ii)]).

**Lemma 2.33** *Let $\mathcal{S}$ be a regular formal topology. For any $U, V \subseteq S$ we have*

$$U \ll V \implies U \lll V.$$

**Proof** Let $U, V \subseteq S$ and suppose that $U \ll V$. Since $\mathcal{S}$ is regular we have

$$V \lhd \{a \in S \mid (\exists b \in V)\, a \lll b\}.$$

Then there exists $W = \{a_0, \dots, a_{n-1}\} \in \mathrm{Fin}(S)$ such that $U \lhd W$ and $a_i \lll V$ for each $i < n$. Thus $W \lll V$, and so $U \lll V$. □

As a corollary we obtain a well known fact (see Johnstone [13, Chapter VII, Corollary 3.5]).

**Proposition 2.34** *Let $\mathcal{S}$ be a compact regular formal topology. Then $\mathcal{S}$ is locally compact, and the relations $\ll$ and $\lll$ coincide.*

**Example 2.35** (See also Example 2.9) The formal reals $\mathcal{R}$ is regular and locally compact. To see that $\mathcal{R}$ is regular, we first show that axiom (R2) of $\mathcal{R}$ is equivalent to the following axiom:

(R2') $(p,q) \lhd_{\mathcal{R}'} \left\{(r,s) \in S_{\mathcal{R}} \mid s - r = 2^{-k}\right\}$ for each $k \in \mathbb{N}$.

Let $\lhd_{\mathcal{R}'}$ be the cover generated by (R2'). Let $(p,q),(r,s) \in S_{\mathcal{R}}$ and suppose that $(r,s) <_{\mathcal{R}} (p,q)$. By choosing $k \in \mathbb{N}$ such that $2^{-k} < s - r$, we have

$$(p,q) \lhd_{\mathcal{R}'} \left\{(p',q') \in S_{\mathcal{R}} \mid q' - p' = 2^{-k}\right\} \downarrow (p,q) \lhd_{\mathcal{R}'} \{(p,s),(r,q)\}\,.$$

Hence $\lhd_{\mathcal{R}'}$ satisfies (R2). Conversely, we have

$$(p,q) \lhd_{\mathcal{R}} \left\{(r,s) \in S_{\mathcal{R}} \mid s - r = (2/3)^{-n}(q-p)\ \&\ (r,s) \leq_{\mathcal{R}} (p,q)\right\}$$

for each $(p,q) \in S_{\mathcal{R}}$ and $n \in \mathbb{N}$. Thus $\lhd_{\mathcal{R}}$ clearly satisfies (R2').

Now, it readily follows from (R2') that

$$a <_{\mathcal{R}} b \implies a \lll b$$

for all $a,b \in S_{\mathcal{R}}$. Hence by the axiom (R1), $\mathcal{R}$ is regular with the function $\mathsf{wc}_{\mathcal{R}} \colon S_{\mathcal{R}} \to \mathrm{Pow}(S_{\mathcal{R}})$ given by

$$\mathsf{wc}_{\mathcal{R}}(a) \stackrel{\text{def}}{=} \{b \in S_{\mathcal{R}} \mid b <_{\mathcal{R}} a\}\,. \tag{5}$$

To see that $\mathcal{R}$ is locally compact, we first observe that

$$a \lhd_{\mathcal{R}} U \implies \left(\forall b <_{\mathcal{R}} a\right)\left(\exists U_0 \in \mathrm{Fin}(U)\right) b \lhd_{\mathcal{R}} U_0$$

for all $a \in S_{\mathcal{R}}$ and $U \subseteq S_{\mathcal{R}}$. This can be proved by straightforward induction on $\lhd_{\mathcal{R}}$. Hence

$$a <_{\mathcal{R}} b \implies a \ll b$$

for all $a,b \in S_{\mathcal{R}}$. Thus $\mathcal{R}$ is locally compact with the function $\mathsf{wc}_{\mathcal{R}} \colon S_{\mathcal{R}} \to \mathrm{Pow}(S_{\mathcal{R}})$ defined by (5).

**Example 2.36** (See also Example 2.21) The formal unit interval $\mathcal{I}[0,1]$ is compact. A direct proof was given by Cederquist and Negri [5]. The following argument is due to Fourman and Grayson [11, Lemma 4.8].

Since $\mathcal{I}[0,1]$ is a closed subtopology of $\mathcal{R}$, it suffices to show that $\mathcal{I}[0,1]$ is bounded. But for any $(p,q) \in S_{\mathcal{R}}$ such that $p < 0$ and $1 < q$, the subset $\{(p,q)\}$ is clearly bounded. Moreover, since $(0,1) \lll (p,q)$ we have $\mathcal{I}[0,1] \sqsubseteq \mathcal{R}_{\{(p,q)\}}$ . Hence $\mathcal{I}[0,1]$ is compact by Proposition 2.30.

$\mathcal{I}[0,1]$ is also regular by Proposition 2.25 (1) and the function $\mathsf{wc}_{\mathcal{R}}$ defined by (5) in Example 2.35 makes $\mathcal{I}[0,1]$ regular.

## 3 Localic completion of metric spaces

In this section, we recall the embedding of the category of locally compact metric spaces into that of formal topologies by Palmgren [17]. The reader is referred to Palmgren [17] for further details.

The embedding is based on the representation of complete metric spaces by formal topologies, called localic completion, due to Vickers [21].

**Definition 3.1** Let $X = (X,d)$ be a metric space with a metric $d$ on $X$, and let $\mathbb{Q}^{>0}$ be the set of positive rationals. Define

$$M_X \stackrel{\text{def}}{=} X \times \mathbb{Q}^{>0}.$$

An element $(x,\varepsilon)$ of $M_X$ will be denoted by $\mathsf{b}(x,\varepsilon)$. Define an order $\leq_X$ and a transitive relation $<_X$ on $M_X$ by

$$\mathsf{b}(x,\varepsilon) \leq_X \mathsf{b}(y,\delta) \stackrel{\text{def}}{\iff} d(x,y) + \varepsilon \leq \delta,$$
$$\mathsf{b}(x,\varepsilon) <_X \mathsf{b}(y,\delta) \stackrel{\text{def}}{\iff} d(x,y) + \varepsilon < \delta.$$

The *localic completion* of $X$ is a formal topology $\mathcal{M}(X) = (M_X, \lhd_X, \leq_X)$ inductively generated by the axiom-set on $M_X$ consisting of the following axioms:

(M1) $a \lhd_X \{b \in M_X \mid b <_X a\}$,

(M2) $a \lhd_X \mathcal{C}_\varepsilon$ for each $\varepsilon \in \mathbb{Q}^{>0}$

for each $a \in M_X$, where $\mathcal{C}_\varepsilon \stackrel{\text{def}}{=} \{\mathsf{b}(x,\varepsilon) \in M_X \mid x \in X\}$.

For any metric space $X$, its localic completion $\mathcal{M}(X)$ is always overt and its positivity is the whole of $M_X$. Moreover, we have

$$a <_X b \implies a \lll b$$

for any $a, b \in M_X$, and so $\mathcal{M}(X)$ is regular by the axiom (M1). The class $\mathrm{Pt}(\mathcal{M}(X))$ admits a metric $\rho\colon \mathrm{Pt}(\mathcal{M}(X)) \times \mathrm{Pt}(\mathcal{M}(X)) \to \mathbb{R}^{\geq 0}$ which can be defined using upper Dedekind cuts:

$$\rho(\alpha, \beta) \stackrel{\text{def}}{=} \left\{ q \in \mathbb{Q}^{>0} \mid (\exists\, \mathsf{b}(x, \varepsilon) \in \alpha)\, (\exists\, \mathsf{b}(y, \delta) \in \beta)\ d(x, y) + \varepsilon + \delta < q \right\}$$

for each $\alpha, \beta \in \mathrm{Pt}(\mathcal{M}(X))$.[2] Furthermore, the function $j_X\colon X \to \mathrm{Pt}(\mathcal{M}(X))$ defined by

$$j_X(x) \stackrel{\text{def}}{=} \{\mathsf{b}(y, \varepsilon) \in M_X \mid d(x, y) < \varepsilon\}$$

is a metric completion of $X$. Thus $j_X$ is a metric isomorphism if and only if $X$ is complete. Note that since $j_X$ is a metric completion, the class $\mathrm{Pt}(\mathcal{M}(X))$ is actually a set which is isomorphic to the usual construction of completion of $X$, i.e. the set of Cauchy sequences on $X$ with a suitable equivalence relation.

For each $\mathsf{b}(x, \varepsilon) \in M_X$, we write $\mathsf{b}(x, \varepsilon)_*$ for the open ball associated with $\mathsf{b}(x, \varepsilon)$:

$$\mathsf{b}(x, \varepsilon)_* \stackrel{\text{def}}{=} B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}\,.$$

We extend the notation $(-)_*$ to the subsets of $M_X$ by defining $U_* \stackrel{\text{def}}{=} \bigcup_{a \in U} a_*$ for each $U \subseteq M_X$. Dually, each point $x \in X$ is associated with the set $\Diamond x$ of open neighbourhoods of $x$ given by

$$\Diamond x \stackrel{\text{def}}{=} \{a \in M_X \mid x \in a_*\}\,.$$

Note that $j_X(x) = \Diamond x$ for all $x \in X$. We extend the notation $\Diamond(-)$ to the subsets of $X$ by defining $\Diamond Y \stackrel{\text{def}}{=} \bigcup_{y \in Y} \Diamond y$ for each $Y \subseteq X$.

The following is crucial to the main result of the present paper.

**Theorem 3.2** (Palmgren [17, Theorem 2.7]) *Let $X$ be a metric space, and let $Y$ be a dense subset of $X$. Then $\mathcal{M}(Y) \cong \mathcal{M}(X)$.*

Next, we recall the definition of the category of locally compact metric spaces.

[2] In fact, Palmgren defined a metric on $\mathrm{Pt}(\mathcal{M}(X))$ using Cauchy reals [17, Section 2], but it is not difficult to show that the metric $\rho$ is equivalent to the one defined by Palmgren using the correspondence between Dedekind cuts and Cauchy reals.

**Definition 3.3** A metric space $X$ is *totally bounded* if for any $\varepsilon \in \mathbb{Q}^{>0}$, there exists $Y = \{x_0, \ldots, x_{n-1}\} \in \mathrm{Fin}(X)$ such that $X \subseteq \bigcup_{i<n} B(x_i, \varepsilon)$. The set $Y$ is called an *$\varepsilon$–net* to $X$. A metric space is *compact* if it is complete and totally bounded.

A metric space $X$ is *locally compact* if each open ball $B(x, \varepsilon)$ is contained in a compact subset of $X$. A locally compact metric space is *Bishop locally compact* if it is inhabited. If $X$ and $Y$ are locally compact metric spaces, a function $f\colon X \to Y$ is said to be *continuous* if $f$ is uniformly continuous on each compact subset of $X$.

The locally compact metric spaces and continuous functions between them form a category **LCM**.

Note that any compact metric space is locally compact. Moreover, a locally compact metric space is complete, and a Bishop locally compact metric space is separable.

If $X$ is a locally compact metric space, we have

$$a <_X b \implies a \ll b$$

for all $a, b \in M_X$. Hence, $\mathcal{M}(X)$ is a locally compact formal topology with a function $\mathsf{wb}\colon S \to \mathrm{Pow}(S)$ given by $\mathsf{wb}(a) \stackrel{\mathrm{def}}{=} \{b \in M_X \mid b <_X a\}$. If $X$ is a compact metric space, then it can be shown that $\mathcal{M}(X)$ is a compact formal topology.

Palmgren [17, Section 5] extended the construction $\mathcal{M}$ to a full and faithful functor $\mathcal{M}\colon \mathbf{LCM} \to \mathbf{FTop}$. By an abuse of terminology, we call this functor $\mathcal{M}$ the localic completion. One of the aims of this paper is to characterise the image of Bishop locally compact metric spaces under the localic completion up to isomorphism.

# 4 Open complements of located subtopologies

We give a sufficient condition under which a formal topology is isomorphic to the localic completion of a Bishop locally compact metric space. We exploit the category **OLCM** of open complements of locally compact metric spaces by Palmgren [18].

**Definition 4.1** The category **OLCM** consists of the following data. An object of **OLCM** is a pair $(X, U)$ where $X$ is a locally compact metric space and $U$ is an open subset of $X$. A morphism $f\colon (X, U) \to (Y, V)$ of **OLCM** is a function $f\colon U \to V$ such that for any inhabited compact subset $K \subseteq X$ with $K \Subset U$, we have

(1) $f$ is uniformly continuous on $K$,

(2) $f[K] \Subset V$,

where the relation $\Subset$ is given by

$$K \Subset U \overset{\text{def}}{\iff} \left(\exists r \in \mathbb{Q}^{>0}\right) K_r \subseteq U,$$
$$K_r \overset{\text{def}}{=} \{x \in X \mid d(x,K) \leq r\}$$

for any located subset $K$. Here, a subset $A$ of a metric space $(X,d)$ is *located* if the distance

$$d(x,A) \overset{\text{def}}{=} \inf\{d(x,a) \mid a \in A\}$$

exists for every $x \in X$.

Note that an inhabited totally bounded subset of a metric space is located and that the image of a totally bounded subset under a uniformly continuous function is totally bounded. Hence, the second condition for a morphism is well-defined.

Palmgren [18] showed that **OLCM** can be embedded into **FTop** via a full and faithful functor $\mathcal{OM}\colon \mathbf{OLCM} \to \mathbf{FTop}$. The functor $\mathcal{OM}$ assigns to each object $(X,U)$ of **OLCM** the open subtopology $\mathcal{M}(X)_{H(U)}$ of $\mathcal{M}(X)$ determined by the subset

$$H(U) \overset{\text{def}}{=} \{\mathsf{b}(x,\varepsilon) \in M_X \mid B(x,\varepsilon) \subseteq U\}.$$

The category **LCM** is embedded into **OLCM** via the inclusion $X \mapsto (X,X)$. Note that $\mathcal{OM}((X,X)) = \mathcal{M}(X)$ for any locally compact metric space $X$.

We recall the notion of located subtopology and a characterisation thereof from our previous work [14, Chapter 4, Section 1].

**Definition 4.2** Let $\mathcal{S}$ be a locally compact formal topology. A subset $V \subseteq S$ is *located* if it is a splitting subset of $\mathcal{S}$, and moreover satisfies

$$a \ll b \implies a \in \neg V \vee b \in V$$

for all $a,b \in S$.

A subtopology $\mathcal{S}'$ of $\mathcal{S}$ is *located* if $\mathcal{S}'$ is the closed subtopology $\mathcal{S}^{\mathcal{S}-\neg V}$ determined by the complement $\neg V$ of a located subset $V$ of $\mathcal{S}$.

If $\mathsf{wb}\colon S \to \mathrm{Pow}(S)$ is a function which makes $\mathcal{S}$ locally compact, then it can be shown that a splitting subset $V$ of $\mathcal{S}$ is located if and only if

$$a \in \mathsf{wb}(b) \implies a \in \neg V \vee b \in V$$

for all $a,b \in S$.

**Lemma 4.3** *Let $\mathcal{S}$ be a locally compact formal topology. Then the assignment*

$$V \mapsto \mathcal{S}^{\mathcal{S}-\neg V} \tag{6}$$

*is a bijection between the located subsets of $\mathcal{S}$ and the overt closed subtopologies of $\mathcal{S}$.*

**Proof** Let $V$ be a located subset of $\mathcal{S}$. Then for any $a \in S$, we have

$$a \lhd \{a \in S \mid b \ll a\} \lhd^{\mathcal{S}-\neg V} \{a\} \cap V.$$

Hence $V$ satisfies the condition (Pos), so that $V$ is the positivity of $\mathcal{S}^{\mathcal{S}-\neg V}$.

Conversely, suppose that $\mathcal{S}^{\mathcal{S}-\neg V}$ is the overt closed subtopology of $\mathcal{S}$ determined by a subset $V \subseteq S$, and let Pos be the positivity of $\mathcal{S}^{\mathcal{S}-\neg V}$. Let $a, b \in S$, and suppose that $a \ll b$. Since $b \lhd^{\mathcal{S}-\neg V} \{b\} \cap \mathrm{Pos}$, there exists $U \in \mathrm{Fin}(\{b\} \cap \mathrm{Pos})$ such that $a \lhd^{\mathcal{S}-\neg V} U$. If $U$ is inhabited, then $b \in \mathrm{Pos}$. If $U$ is empty, then $a \in \mathrm{Pos}$ implies $\mathrm{Pos} \between \emptyset$, a contradiction. Hence Pos is a located subset of $\mathcal{S}$.

The fact that the assignment (6) is a bijection follows from Lemma 2.20 and uniqueness of positivity predicates. □

By Proposition 2.25 and Proposition 2.34, we obtain the following.

**Corollary 4.4** *Let $\mathcal{S}$ be a compact regular formal topology. Then a subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ is located if and only if $\mathcal{S}'$ is compact overt.*

For later use, we note a special case of the result by Coquand, Palmgren and Spitters [6, Lemma 3.2].

**Lemma 4.5** *Let $X$ be a locally compact metric space, and let $V$ be a located subset of $\mathcal{M}(X)$. Then for any $a \in V$ there exists a formal point $\alpha \in \mathrm{Pt}(\mathcal{M}(X))$ such that $a \in \alpha \subseteq V$.*

**Proof** See Coquand, Palmgren and Spitters [6, Lemma 3.2]. The proof requires Dependent Choice. □

**Definition 4.6** Let $A$ be a located subset of a metric space $X$. The *metric complement* of $A$ is the open subset $X - A$ of $X$ given by

$$X - A \overset{\text{def}}{=} \{x \in X \mid d(x, A) > 0\}.$$

A corresponding point-free notion is the following.

**Definition 4.7** Let $\mathcal{S}$ be a locally compact formal topology, and let $V$ be a located subset of $\mathcal{S}$. The *open complement* of the located subtopology $\mathcal{S}^{\mathcal{S}-\neg V}$ is the open subtopology $\mathcal{S}_{\neg V}$ determined by $\neg V$.

Let $X$ be a locally compact metric space. In our previous work [14, Theorem 4.1.9], we showed that there exists a bijection $\varphi\colon \mathrm{Cl}^{+}(X) \to \mathrm{Loc}^{+}(\mathcal{M}(X))$ between the class $\mathrm{Cl}^{+}(X)$ of inhabited closed located subsets of $X$ and the class $\mathrm{Loc}^{+}(\mathcal{M}(X))$ of inhabited located subsets of $\mathcal{M}(X)$. Specifically, $\varphi$ and its inverse $\varphi^{-1}$ are defined by

$$\begin{aligned} \varphi(A) &\stackrel{\text{def}}{=} \Diamond A, \\ \varphi^{-1}(V) &\stackrel{\text{def}}{=} \{x \in X \mid \Diamond x \subseteq V\} \end{aligned} \tag{7}$$

for any $A \in \mathrm{Cl}^{+}(X)$ and $V \in \mathrm{Loc}^{+}(\mathcal{M}(X))$.

The embedding $\mathcal{OM}\colon \mathbf{OLCM} \to \mathbf{FTop}$ preserves metric complements and open complements of located subtopologies in the following sense.

**Proposition 4.8** *Let $X = (X, d)$ be a locally compact metric space, and let $\varphi\colon \mathrm{Cl}^{+}(X) \to \mathrm{Loc}^{+}(\mathcal{M}(X))$ be the bijection given by (7). Then, for any $A \in \mathrm{Cl}^{+}(X)$ we have*

$$H(X - A) = \neg\varphi(A). \tag{8}$$

*Dually, for any $V \in \mathrm{Loc}^{+}(\mathcal{M}(X))$ we have*

$$(\neg V)_{*} = X - \varphi^{-1}(V). \tag{9}$$

*The assignments $U \mapsto H(U)$ and $W \mapsto W_{*}$ restrict to a bijective correspondence between the metric complements of inhabited closed located subsets of $X$ and the open complements of inhabited located subtopologies of $\mathcal{M}(X)$.*

**Proof** (8) Let $A \in \mathrm{Cl}^{+}(X)$. Let $\mathsf{b}(x,\varepsilon) \in H(X - A)$ and suppose that $\mathsf{b}(x,\varepsilon) \in \Diamond A$. Then $B(x,\varepsilon) \subseteq X - A$ and $B(x,\varepsilon) \between A$, a contradiction. Hence $\mathsf{b}(x,\varepsilon) \in \neg\varphi(A)$.

Conversely, let $\mathsf{b}(x,\varepsilon) \in \neg\varphi(A)$ and $x' \in B(x,\varepsilon)$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,x') + \theta < \varepsilon$, and suppose that $d(x',A) < \theta$. Then there exists $y \in A$ such that $d(x',y) < \theta$, and so $d(x,y) < \varepsilon$. Thus $\mathsf{b}(x,\varepsilon) \in \varphi(A)$, a contradiction. Hence $d(x',A) \geq \theta$, and therefore $\mathsf{b}(x,\varepsilon) \in H(X - A)$.

(9) Let $V \in \mathrm{Loc}^{+}(\mathcal{M}(X))$. Let $\mathsf{b}(y,\delta) \in \neg V$ and $x \in \mathsf{b}(y,\delta)_{*}$. Choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,y) + \theta < \delta$. Suppose that $d(x,\varphi^{-1}(V)) < \theta$. Then there exists $x' \in \varphi^{-1}(V)$ such that $d(x,x') < \theta$, so that $\mathsf{b}(x,\theta) \in \Diamond x' \subseteq V$. Since $\mathsf{b}(x,\theta) <_{X} \mathsf{b}(y,\delta)$, we have

$\mathsf{b}(y,\delta) \in V$, a contradiction. Thus $d(x,\varphi^{-1}(V)) \geq \theta$, and hence $x \in X - \varphi^{-1}(V)$. Therefore $(\neg V)_* \subseteq X - \varphi^{-1}(V)$.

Conversely, let $x \in X - \varphi^{-1}(V)$ and choose $\theta \in \mathbb{Q}^{>0}$ such that $d(x,\varphi^{-1}(V)) > \theta$. Suppose that $\mathsf{b}(x,\theta) \in V$. Then there exists $\alpha \in \mathrm{Pt}(\mathcal{M}(X))$ such that $\mathsf{b}(x,\theta) \in \alpha \subseteq V$ by Lemma 4.5. Since $X$ is complete, there exists $x' \in X$ such that $\diamond x' = j_X(x') = \alpha$. Thus $d(x,x') < \theta$ and $x' \in \varphi^{-1}(V)$, contradicting $d(x,\varphi^{-1}(V)) > \theta$. Hence $\mathsf{b}(x,\theta) \in \neg V$, and so $x \in (\neg V)_*$.

Lastly, for any $A \in \mathrm{Cl}^+(X)$ we have

$$X - A = X - \varphi^{-1}(\varphi(A)) = (\neg\varphi(A))_* = (H(X-A))_*.$$

Conversely, for any $V \in \mathrm{Loc}^+(\mathcal{M}(X))$ we have

$$\neg V = \neg\left(\varphi(\varphi^{-1}(V))\right) = H(X - \varphi^{-1}(V)) = H\left((\neg V)_*\right). \qquad \square$$

Let $X$ be a compact metric space, and let $A$ be a compact subset of $X$. We extend the definition of $X - A$ as follows:

$$X - A \overset{\text{def}}{=} \begin{cases} X & \text{if } A = \emptyset, \\ \{x \in X \mid d(x,A) > 0\} & \text{if } A \text{ is inhabited.} \end{cases}$$

Note that since any compact metric space is totally bounded, we can decide whether a given compact metric space is empty or inhabited.

If $X$ is a compact metric space, the bijection defined by (7) extends to a bijection between the compact subsets of $X$ and the located subsets of $\mathcal{M}(X)$. This follows from the fact that a subset $A$ of a compact metric space is compact if and only if either $A$ is empty or $A$ is closed and located.

**Corollary 4.9** *Let $X$ be a compact metric space. For any located subset $V$ of $\mathcal{M}(X)$, there exists a unique compact subset $A \subseteq X$ such that $\mathcal{OM}\left((X, X - A)\right) = \mathcal{M}(X)_{\neg V}$.*

**Proof** Let $V$ be a located subset of $\mathcal{M}(X)$. By Corollary 4.4, the located subtopology $\mathcal{M}(X)^{\mathcal{M}(X)-\neg V}$ is compact overt with the positivity $V$. Thus, $V$ is either empty or inhabited. In the former case, we put $A = \emptyset$. Then $\mathcal{OM}\left((X, X - A)\right) = \mathcal{M}_{H(X)} = \mathcal{M}_{\neg\emptyset}$. In the latter case, the desired conclusion follows from Proposition 4.8. $\square$

**Lemma 4.10** *Let $X$ be a compact metric space, and let $V$ be a located subset of $\mathcal{M}(X)$. Then the open complement $\mathcal{M}(X)_{\neg V}$ is inhabited if and only if $(\neg V)_*$ is inhabited.*

**Proof** Straightforward. $\square$

**Corollary 4.11** *Let $X$ be a compact metric space, and let $V$ be a located subset of $\mathcal{M}(X)$ such that $\mathcal{M}(X)_{\neg V}$ is inhabited. Then there exists a unique compact subset $A \subseteq X$ such that $X - A$ is inhabited and that $\mathcal{OM}((X, X - A)) = \mathcal{M}(X)_{\neg V}$.*

The following lemma is essentially due to Palmgren [18, Lemma 2.2].[3]

**Lemma 4.12** *Let $X$ be a locally compact metric space. Then for any $x \in X$ and $\varepsilon, \delta \in \mathbb{Q}^{>0}$ such that $\varepsilon < \delta$, there exists a compact subset $K \subseteq X$ such that*

$$B(x, \varepsilon) \subseteq K \subseteq B(x, \delta).$$

**Proof** First, note that since $X$ is locally compact, we have

$$a <_X b \implies a \ll b$$

for all $a, b \in M_X$. Let $x \in X$ and $\varepsilon, \delta \in \mathbb{Q}^{>0}$, and suppose that $\varepsilon < \delta$. Choose $N \in \mathbb{N}$ such that $\varepsilon + 2^{-N} < \delta$. For each $n \in \mathbb{N}$, define

$$a_n \stackrel{\text{def}}{=} \mathsf{b}(x, \varepsilon + 2^{-(N+n)}).$$

Then for each $n \in \mathbb{N}$, since $a_{n+1} <_X a_n$, there exists $V_n \in \mathrm{Fin}(a_n \downarrow \mathcal{C}_{2^{-n}})$ such that $a_{n+1} \lhd_X V_n$. By Countable Choice, we obtain a sequence $(V_n)_{n \in \mathbb{N}} \colon \mathbb{N} \to \mathrm{Fin}(M_X)$ such that

(1) $a_{n+1} \lhd_X V_n \lhd_X a_n$,

(2) $\left(\forall\, \mathsf{b}(z, \gamma) \in V_n\right) \gamma \leq 2^{-n}$

for all $n \in \mathbb{N}$. Let

$$A \stackrel{\text{def}}{=} \left\{ y \in X \mid \left(\exists n \in \mathbb{N}\right)\left(\exists \gamma \in \mathbb{Q}^{>0}\right) \mathsf{b}(y, \gamma) \in V_n \right\}.$$

Then $A$ is clearly totally bounded, so that the closure $K$ of $A$ is compact. Moreover we have $B(x, \varepsilon) \subseteq K \subseteq B(x, \delta)$. Thus $K$ is a desired compact subset of $X$. □

**Proposition 4.13** *Let $X = (X, d)$ be a compact metric space, and let $A$ be a compact subset of $X$. Then there exists a locally compact metric space $Y$ such that $(Y, Y)$ is isomorphic to $(X, X - A)$ in* **OLCM**.

*Moreover if $X - A$ is inhabited, then there exists a Bishop locally compact metric space $Y$ such that $(Y, Y)$ is isomorphic to $(X, X - A)$ in* **OLCM**.

[3] The proof of Palmgren [18, Lemma 2.2] seems to be incomplete. Nevertheless, the argument used in the proof of Palmgren [17, Proposition 4.8] provides a correct proof of Lemma 4.12, which we recall here.

**Proof** If $A = \emptyset$, we define $Y \stackrel{\text{def}}{=} X$. Suppose that $A$ is inhabited. Let $Y \stackrel{\text{def}}{=} X - A$, and define a new metric $d^*$ on $Y$ by

$$d^*(x, y) \stackrel{\text{def}}{=} d(x, y) + \left| \frac{1}{d(x, A)} - \frac{1}{d(y, A)} \right|$$

for all $x, y \in Y$. It is straightforward to show that $d^*$ is a metric on $Y$. We show that the metric space $Y = (Y, d^*)$ has the required properties. Since $d(x, y) \leq d^*(x, y)$ for all $x, y \in Y$, the inclusion $i_Y \colon Y \hookrightarrow (X - A)$ is uniformly continuous. Let $K$ be an inhabited $d^*$–compact subset $Y$, where $K$ is *$d^*$–compact* if $K$ is compact with respect to $d^*$. Then $K$ is contained in some open ball $B^*(y, \varepsilon) \stackrel{\text{def}}{=} \{y' \in Y \mid d^*(y', y) < \varepsilon\}$ of $Y$. By the proof of local compactness of $Y$ which is to be given below, there exists a $d$–compact subset $L$ of $X$ such that $B^*(y, \varepsilon) \subseteq L \Subset X - A$. Hence $i_Y$ is a morphism from $(Y, Y)$ to $(X, X - A)$ in **OLCM**. Moreover $i_Y$ is injective. To see this, suppose that $d^*(x, y) > 0$, and choose $r \in \mathbb{Q}^{>0}$ such that $d^*(x, y) > r$. Let $c = \min\{d(x, A), d(y, A)\}$. Since $d^*(x, y) \leq \left(1 + 1/c^2\right) d(x, y)$, we have $d(x, y) \geq r/(1 + 1/c^2)$. Hence $i_Y$ is injective.

Next, we show that the inverse $j \colon (X - A) \to Y$ of $i_Y$ is uniformly continuous on each inhabited $d$–compact subset $K$ of $X$ such that $K \Subset X - A$. Let $K \Subset X - A$ be an inhabited $d$–compact subset of $X$. Then there exists $r \in \mathbb{Q}^{>0}$ such that $K_r \subseteq X - A$, and so $d(x, A) \geq r$ for all $x \in K$. Hence $d^*(x, y) \leq \left(1 + 1/r^2\right) d(x, y)$ for all $x, y \in K$. Uniform continuity of $j \colon K \to Y$ now follows.

It remains to be shown that $Y$ is a locally compact metric space. Let $y \in Y$ and $\varepsilon \in \mathbb{Q}^{>0}$. We must find a $d^*$–compact subset $K \subseteq Y$ such that $B^*(y, \varepsilon) \subseteq K$. To this end, it suffices to find a $d$–compact subset $K \Subset X - A$ such that $B^*(y, \varepsilon) \subseteq K$; for if such $K$ exists, then $i_Y \colon Y \to (X - A)$ and $j \colon (X - A) \to Y$ restrict to uniform isomorphisms on $K$.

To find such a $d$–compact subset of $X$, notice that for any $x \in B^*(y, \varepsilon)$, we have $d(x, A) > 1/\left(\varepsilon + 1/d(y, A)\right)$. Thus $B^*(y, \varepsilon) \subseteq U_{A,r}$, where

$$r \stackrel{\text{def}}{=} 1/\left(\varepsilon + 1/d(y, A)\right),$$
$$U_{A,r} \stackrel{\text{def}}{=} \{x \in X \mid d(x, A) \geq r\}.$$

Choose $\theta \in \mathbb{Q}^{>0}$ such that $7\theta < r$, and let $X_\theta = \{x_0, \ldots, x_{n-1}\}$ be a $\theta$–net to $X$. For each $i < n$, we have either $5\theta < d(x_i, A)$ or $d(x_i, A) < 6\theta$. Split $X_\theta$ into two finitely enumerable subsets $X_\theta^+$ and $X_\theta^-$ such that $X_\theta = X_\theta^+ \cup X_\theta^-$ and that

(1) $x \in X_\theta^+ \implies 5\theta < d(x, A)$,

(2) $x \in X_\theta^- \implies d(x, A) < 6\theta$.

Write $X_\theta^+ = \{z_0, \ldots, z_{m-1}\}$. Let $x \in U_{A,r}$. Then there exists $i < n$ such that $d(x, x_i) < \theta$. If $x_i \in X_\theta^-$, we have $d(x, A) \leq 7\theta < r$, contradicting $x \in U_{A,r}$. Thus $x_i \in X_\theta^+$, and hence $U_{A,r} \subseteq \bigcup_{j<m} B(z_j, \theta)$. For each $j < m$, there exists a compact subset $K_j \subseteq X$ such that $B(z_j, \theta) \subseteq K_j \subseteq B(z_j, 2\theta)$ by Lemma 4.12. Let $K = \bigcup_{j<m} K_j$. Then $K$ is inhabited and totally bounded, and so it is located. Let $x \in K_\theta = \{x' \in X \mid d(x', K) \leq \theta\}$, and suppose that $d(x, A) < \theta$. Then there exist $y \in A$ and $w \in K$ such that $d(x, y) < \theta$ and $d(x, w) < 2\theta$. Thus there exists $j < m$ such that $d(w, z_j) < 2\theta$, so that

$$d(y, z_j) \leq d(y, x) + d(x, w) + d(w, z_j) \leq \theta + 2\theta + 2\theta \leq 5\theta,$$

contradicting $z_j \in X_\theta^+$. Thus $d(x, A) \geq \theta$, and so $K_\theta \subseteq X - A$. Hence $K \Subset X - A$. Then $L \Subset X - A$, where $L$ is the closure of $K$. Therefore $L$ is a desired $d$–compact subset of $X$.

The second statement is obvious. □

**Proposition 4.14** *Let $X$ be a compact metric space, and let $V$ be a located subset of $\mathcal{M}(X)$. Then there exists a locally compact metric space $Y$ such that $\mathcal{M}(Y) \cong \mathcal{M}(X)_{\neg V}$.*

**Proof** By Lemma 4.9, there exists a unique compact subset $A$ of $X$ such that

$$\mathcal{OM}\left((X, X - A)\right) = \mathcal{M}(X)_{\neg V}.$$

Then there exists a locally compact metric space $Y$ such that $(Y, Y) \cong (X, X - A)$ in **OLCM** by Proposition 4.13. Since every functor preserves isomorphisms, we have

$$\mathcal{M}(Y) = \mathcal{OM}\left((Y, Y)\right) \cong \mathcal{OM}\left((X, X - A)\right) = \mathcal{M}(X)_{\neg V}. \qquad \square$$

**Corollary 4.15** *Let $X$ be a compact metric space, and let $V$ be a located subset of $\mathcal{M}(X)$ such that the open complement $\mathcal{M}(X)_{\neg V}$ is inhabited. Then there exists a Bishop locally compact metric space $Y$ such that $\mathcal{M}(Y) \cong \mathcal{M}(X)_{\neg V}$.*

# 5 Enumerably completely regular formal topologies

We characterise enumerably completely regular formal topologies by the subtopologies of the countable product of the formal unit interval. Except for the definition of enumerably completely regular formal topology, which is due to Curi [8, Section 2.2], the results in this section appear to be new.

**Definition 5.1** Let $\mathbb{I} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. Given a formal topology $\mathcal{S}$ and subsets $U, V \subseteq S$, a *scale* from $U$ to $V$ is a family $\left(U_q\right)_{q \in \mathbb{I}}$ of subsets of $S$ such that

(1) $U \lhd U_0$ and $U_1 \lhd V$,

(2) $\left(\forall p, q \in \mathbb{I}\right) p < q \implies U_p \lll U_q$.

**Definition 5.2** A formal topology $\mathcal{S}$ is *enumerably completely regular* if it is equipped with a function $\mathsf{wc}\colon S \to \mathrm{Pow}(S)$ such that

(1) $a \lhd \mathsf{wc}(a)$ for each $a \in S$,

(2) the relation $\overline{\mathsf{wc}} = \{(b, a) \in S \times S \mid b \in \mathsf{wc}(a)\}$ is countable, i.e. there exists a surjection $f\colon \mathbb{N} \to \overline{\mathsf{wc}}$,

(3) there exists a function $\mathsf{sc} \in \prod_{(b,a) \in \overline{\mathsf{wc}}} \mathrm{Sc}_{\lll}\left(\{b\}, \{a\}\right)$, called a *choice of scale* for $\mathsf{wc}$,

where $\mathrm{Sc}_{\lll}\left(\{b\}, \{a\}\right)$ is the class of scales from $\{b\}$ to $\{a\}$.[4]

Let $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1] = (S_\Pi, \lhd_\Pi, \leq)$ be the product of countably many copies of the formal unit interval $\mathcal{I}[0,1]$. According to Section 2.1.1, the preorder $(S_\Pi, \leq)$ is given by

$$S_\Pi \overset{\text{def}}{=} \mathrm{Fin}(\mathbb{N} \times S_{\mathcal{R}}),$$
$$A \leq B \overset{\text{def}}{\iff} \left(\forall (n, b) \in B\right)\left(\exists (m, a) \in A\right) m = n \;\&\; a \leq_{\mathcal{R}} b$$

for all $A, B \in S_\Pi$. Here $(S_{\mathcal{R}}, \leq_{\mathcal{R}})$ is the underlying preorder of the formal reals $\mathcal{R}$ as defined in Example 2.9. The cover $\lhd_\Pi$ is generated by the axioms (S1), (S2) and (S3) for a product, where (S3) is derived from the axioms (R1) and (R2) of $\mathcal{R}$ and the axiom (2) of $\mathcal{I}[0,1]$.

Since $\mathcal{I}[0,1]$ is regular with the function $\mathsf{wc}_{\mathcal{R}}$ defined by (5) in Example 2.35, the proof of Proposition 2.26 (1) shows that $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$ is regular with the function $\mathsf{wc}_\Pi\colon S_\Pi \to \mathrm{Pow}(S_\Pi)$ given by

(10) $$\mathsf{wc}_\Pi(A) \overset{\text{def}}{=} \left\{\{(m_0, b_0), \ldots, (m_{n-1}, b_{n-1})\} \in S_\Pi \mid (\forall i < n)\ b_i <_{\mathcal{R}} a_i\right\}$$

for each $A = \{(m_0, a_0), \ldots, (m_{n-1}, a_{n-1})\} \in S_\Pi$.

**Lemma 5.3** $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$ *is enumerably completely regular.*

[4] In fact, Curi [8] did not require an existence of a choice of scale for an enumerably completely regular formal topology, but only an existence of a scale from $\{b\}$ to $\{a\}$ for each element $(b, a) \in \overline{\mathsf{wc}}$. With Countable Choice, however, a choice of scale can always be chosen.

**Proof** Let $\overline{\mathsf{wc}_\Pi} \overset{\text{def}}{=} \{(B,A) \in S_\Pi \times S_\Pi \mid B \in \mathsf{wc}_\Pi(A)\}$. We show that $\overline{\mathsf{wc}_\Pi}$ is countable and define a choice of scale for $\mathsf{wc}_\Pi$.

First, the set $S_\Pi$ is countable since it is the set of finitely enumerable subsets of a countable set, and for each $A \in S_\Pi$ the set $\mathsf{wc}_\Pi(A)$ is countable since it is a finite product of countable sets. Thus $\overline{\mathsf{wc}_\Pi}$ is countable.

Next, we define a choice of scale for $\mathsf{wc}_\Pi$. Let $(B,A) \in \overline{\mathsf{wc}_\Pi}$, so that $A$ and $B$ are of the forms

$$A = \{(m_0,(p_0,q_0)),\dots,(m_{n-1},(p_{n-1},q_{n-1}))\},$$
$$B = \left\{(m_0,(p'_0,q'_0)),\dots,(m_{n-1},(p'_{n-1},q'_{n-1}))\right\}$$

such that $(p'_i,q'_i) <_{\mathcal{R}} (p_i,q_i)$ for each $i < n$. Then for each $i < n$, we can define an order reversing bijection $\varphi_i \colon \mathbb{I} \to \left[p_i,p'_i\right] \cap \mathbb{Q}$ and an order preserving bijection $\psi_i \colon \mathbb{I} \to \left[q'_i,q_i\right] \cap \mathbb{Q}$. For each $q \in \mathbb{I}$, define

$$B_q \overset{\text{def}}{=} \{(m_0,(\varphi_0(q),\psi_0(q))),\dots,(m_{n-1},(\varphi_{n-1}(q),\psi_{n-1}(q)))\}.$$

Then the family $(\{B_q\})_{q\in\mathbb{I}}$ is a scale from $\{B\}$ to $\{A\}$. Thus, we can define a function $\mathsf{sc} \in \prod_{(B,A)\in\overline{\mathsf{wc}_\Pi}} \mathrm{Sc}_{\lll}\big(\{B\},\{A\}\big)$ which assigns to each $(B,A) \in \overline{\mathsf{wc}_\Pi}$ the scale from $\{B\}$ to $\{A\}$ as described above. □

Let $\mathcal{S}$ be a formal topology and let $U,V \subseteq S$. Then any scale $\left(U_q\right)_{q\in\mathbb{I}}$ from $U$ to $V$ determines a formal topology map $r \colon \mathcal{S} \to \mathcal{I}[0,1]$ such that

(1) $r^-(0,\infty) \downarrow U \lhd \emptyset$,

(2) $r^-(-\infty,1) \lhd V$,

where for each $q \in \mathbb{Q}$ we define

$$(q,\infty) \overset{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid r \geq q\},$$
$$(-\infty,q) \overset{\text{def}}{=} \{(r,s) \in S_{\mathcal{R}} \mid s \leq q\}.$$

The formal topology map $r$ is defined by

$$a\ r\ (p,q) \overset{\text{def}}{\iff} \left(\exists (p',q') \in S_{\mathcal{R}}\right) p < p' < q' < q \,\&\, a \lhd U^*_{p'} \downarrow U_{q'} \tag{11}$$

for all $a \in S$ and $(p,q) \in S_{\mathcal{R}}$, where we define $U_q = \emptyset$ if $q < 0$ and $U_q = S$ if $q > 1$. See Johnstone [13, Chapter IV, Proposition 1.4] for details.

The following characterisation of enumerably completely regular formal topology is a special case of Tychonoff's embedding theorem for completely regular locales by Johnstone [13, Chapter IV, Theorem 1.7], which characterises a completely regular locale as a sublocale of a product of copies of $\mathcal{I}[0,1]$. For the convenience of the reader, we give a proof in the language of formal topology (in contrast to the localic language), although our proof is quite similar to that of the localic Tychonoff's embedding theorem.[5]

**Proposition 5.4** *A formal topology is isomorphic to an enumerably completely regular formal topology if and only if it can be embedded into* $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$.

**Proof** ($\Rightarrow$) It suffices to show that any enumerably completely regular formal topology can be embedded into $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$. Let $\mathcal{S}$ be an enumerably completely regular formal topology equipped with a function $\mathsf{wc}\colon S \to \mathrm{Pow}(S)$ which satisfies the three conditions in Definition 5.2. Let $(b_n, a_n)_{n \in \mathbb{N}}$ be an enumeration of the set $\overline{\mathsf{wc}} = \{(b,a) \in S \times S \mid b \in \mathsf{wc}(a)\}$, and let $\mathsf{sc}$ be a choice of scale for $\mathsf{wc}$. Then for each $n \in \mathbb{N}$, the scale $\mathsf{sc}\big((b_n, a_n)\big)$ from $\{b_n\}$ to $\{a_n\}$ determines a formal topology map $r_n\colon \mathcal{S} \to \mathcal{I}[0,1]$ such that

(1) $r_n^{-}(0,\infty) \downarrow b_n \lhd \emptyset$,

(2) $r_n^{-}(-\infty,1) \lhd a_n$.

Let $r\colon \mathcal{S} \to \prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$ be the canonical formal topology map determined by the sequence $(r_n\colon \mathcal{S} \to \mathcal{I}[0,1])_{n \in \mathbb{N}}$. We show that $r$ is an embedding, that is $a \lhd r^{-} r^{-*} \mathcal{A}\{a\}$ for each $a \in S$. Let $a \in S$ and $b \in \mathsf{wc}(a)$, and let $n \in \mathbb{N}$ be the index of the pair $(b,a) \in \overline{\mathsf{wc}}$. Then

$$\begin{aligned} b &\lhd \big(r_n^{-}(-\infty,1) \cup r_n^{-}(0,\infty)\big) \downarrow b \\ &\lhd \big(r_n^{-}(-\infty,1) \downarrow b\big) \cup \big(r_n^{-}(0,\infty) \downarrow b\big) \\ &\lhd \emptyset \cup r_n^{-}(-\infty,1) \\ &=_{\mathcal{S}} r^{-}\,\{\{(n,(p,q))\} \mid (p,q) \in (-\infty,1)\} \lhd a. \end{aligned}$$

Thus $b \lhd r^{-} r^{-*} \mathcal{A}\{a\}$, and hence $a \lhd \mathsf{wc}(a) \lhd r^{-} r^{-*} \mathcal{A}\{a\}$.

($\Leftarrow$) Immediate from Lemma 5.3 and Proposition 2.25 (1). $\square$

## 6 Point-free one-point compactification

We prove a point-free analogue of the fact that every Bishop locally compact metric space has a one-point compactification. Our proof is analogous to the proof given by Bishop and Bridges [4, Chapter 4, Theorem 6.8] for the corresponding fact for Bishop locally compact metric spaces.

[5] As far as we know, the proof of Tychonoff's embedding theorem in terms of formal topology has not appeared explicitly before.

**Definition 6.1** Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. A *wb-scale* from $U$ to $V$ is a family $\left(U_q\right)_{q \in \mathbb{I}}$ of subsets of $S$ such that

(1) $U \lhd U_0$ and $U_1 \lhd V$,

(2) $\left(\forall p, q \in \mathbb{I}\right) p < q \implies U_p \ll U_q$.

**Definition 6.2** A formal topology $\mathcal{S}$ is *enumerably locally compact* if it is equipped with a function $\mathsf{wb} \colon S \to \mathrm{Pow}(S)$ such that

(1) $a \lhd \mathsf{wb}(a)$ for each $a \in S$,

(2) the relation $\overline{\mathsf{wb}} = \{(b, a) \in S \times S \mid b \in \mathsf{wb}(a)\}$ is countable, i.e. there exists a surjection $f \colon \mathbb{N} \to \overline{\mathsf{wb}}$,

(3) there exists a function $\mathsf{sc} \in \prod_{(b,a) \in \overline{\mathsf{wb}}} \mathrm{Sc}_{\ll}\left(\{b\}, \{a\}\right)$, called a *choice of wb-scale* for $\mathsf{wb}$,

where $\mathrm{Sc}_{\ll}\left(\{b\}, \{a\}\right)$ is the class of wb-scales from $\{b\}$ to $\{a\}$.

In a regular formal topology, any wb-scale is a scale by Lemma 2.33. Hence, we have the following.

**Lemma 6.3** *Any enumerably locally compact regular formal topology is enumerably completely regular.*

**Definition 6.4** Let $\mathcal{S}$ be an overt enumerably locally compact regular formal topology. A *one-point compactification* of $\mathcal{S}$ is a triple $(\mathcal{T}, \omega, r)$ consisting of a compact overt enumerably completely regular formal topology $\mathcal{T}$, a formal point $\omega \in \mathrm{Pt}(\mathcal{T})$, and an embedding $r \colon \mathcal{S} \to \mathcal{T}$ such that the image of $\mathcal{S}$ under $r$ is isomorphic to the open complement $\mathcal{T}_{\neg\omega}$ of the located subtopology determined by $\omega$.

Note that if $\mathcal{S}$ is a locally compact regular formal topology, any formal point of $\mathcal{S}$ is a located subset of $\mathcal{S}$ and thus determines a located subtopology of $\mathcal{S}$. This follows from Lemma 2.33.

**Theorem 6.5** *Any overt enumerably locally compact regular formal topology has a one-point compactification.*

The rest of this section is devoted to the proof of the theorem. In what follows, we fix an overt enumerably locally compact regular formal topology $\mathcal{S}$. Let Pos be the positivity of $\mathcal{S}$. Let $\mathsf{wb} \colon S \to \mathrm{Pow}(S)$ be a function which satisfies the three conditions in Definition 6.2. Let $(b_n, a_n)_{n \in \mathbb{N}}$ be an enumeration of the set

$\overline{\mathsf{wb}} = \{(b,a) \in S \times S \mid b \in \mathsf{wb}(a)\}$, and let $\mathsf{sc}$ be a choice of wb-scale for $\mathsf{wb}$. For each $n \in \mathbb{N}$, let $r_n \colon \mathcal{S} \to \mathcal{I}[0,1]$ be the formal topology map determined by the wb-scale $\mathsf{sc}\big((b_n, a_n)\big)$ from $\{b_n\}$ to $\{a_n\}$. Note that $r_n$ is defined by the condition (11) and satisfies

(1) $r_n^-(0,\infty) \downarrow b_n \lhd \emptyset$,

(2) $r_n^-(-\infty,1) \lhd a_n$.

Let $r \colon \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ be the embedding that is determined by the sequence $(r_n \colon \mathcal{S} \to \mathcal{I}[0,1])_{n\in\mathbb{N}}$, where $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1] = (S_\Pi, \lhd_\Pi, \leq)$ is the countable product of the formal unit interval $\mathcal{I}[0,1]$ described in Section 5. For each $n,k \in \mathbb{N}$ define

$$\begin{aligned} \mathcal{C}_k^n &\overset{\text{def}}{=} \left\{\{(n,(p,q))\} \in S_\Pi \mid q - p = 2^{-k}\right\}, \\ \mathcal{C}_k^{\leq n} &\overset{\text{def}}{=} \left\{\{(0,(p_0,q_0)),\dots,(n,(p_n,q_n))\} \in S_\Pi \mid (\forall i < n)\, q_i - p_i = 2^{-k}\right\}. \end{aligned}$$

By the axiom (R2') of $\mathcal{R}$ given in Example 2.35, we have $S_\Pi \lhd_\Pi \mathcal{C}_k^n$ for all $n,k \in \mathbb{N}$. Thus for any $n,k \in \mathbb{N}$, we have

$$S_\Pi \lhd_\Pi \mathcal{C}_k^0 \downarrow \cdots \downarrow \mathcal{C}_k^n \lhd_\Pi \mathcal{C}_k^{\leq n},$$

and hence $S \lhd r^- \mathcal{C}_k^{\leq n}$.

**Lemma 6.6** *For any $N \in \mathbb{N}$ such that $a_N \ll S$, there exists a compact overt subtopology $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{S}_{b_N} \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}_{a_N}$, where $\mathcal{S}_{b_N}$ and $\mathcal{S}_{a_N}$ are the open subtopologies of $\mathcal{S}$ determined by $\{b_N\}$ and $\{a_N\}$ respectively.*

**Proof** Let $N \in \mathbb{N}$, and suppose that $a_N \ll S$. For each $n \in \mathbb{N}$, there exists $\mathcal{E}_n \in \mathrm{Fin}\left(\mathcal{C}_{n+3}^{\leq n}\right)$ such that $a_N \lhd r^- \mathcal{E}_n$ and $\mathcal{E}_n \subseteq r\,\mathrm{Pos}$. By Countable Choice, there exists a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{E}_n \in \mathrm{Fin}\left(\mathcal{C}_{n+3}^{\leq n}\right), \qquad \mathcal{E}_n \subseteq r\,\mathrm{Pos}, \qquad a_N \lhd r^- \mathcal{E}_n$$

for all $n \in \mathbb{N}$. Write $\mathcal{E}_N = \{A_0, \dots, A_{n-1}\}$, and for each $i < n$ write $A_i = \{(0,(p_0^i,q_0^i)),\dots,(N,(p_N^i,q_N^i))\}$. Split $\mathcal{E}_N$ into finitely enumerable subsets $\mathcal{E}_N^+$ and $\mathcal{E}_N^-$ such that $\mathcal{E}_N = \mathcal{E}_N^+ \cup \mathcal{E}_N^-$ and that

(1) $A_i \in \mathcal{E}_N^+ \implies (p_N^i, q_N^i) \in (-\infty, 1/2)$,

(2) $A_i \in \mathcal{E}_N^- \implies (p_N^i, q_N^i) \in (1/4, \infty)$.

Let $S_\Pi^*$ be the set of finite lists of elements of $S_\Pi$. We let $\langle A_0, \ldots, A_{n-1}\rangle$ denote an element of $S_\Pi^*$ of length $n \in \mathbb{N}$. The concatenation of lists $l, l' \in S_\Pi^*$ is denoted by $l * l'$. Define a subset $T$ of $S_\Pi^*$ by

$$\begin{aligned} T_0 &\stackrel{\text{def}}{=} \left\{ \langle A\rangle \in S_\Pi^* \mid A \in \mathcal{E}_N^+ \right\}, \\ T_{n+1} &\stackrel{\text{def}}{=} \left\{ l * \langle A\rangle \in S_\Pi^* \mid l \in T_n \;\&\; l = l' * \langle A'\rangle \;\&\; A \in \mathcal{E}_{N+n+1} \;\&\; A' \asymp A \right\}, \\ T &\stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} T_n, \end{aligned}$$

where for each $A, B \in S_\Pi$, we define

$$A \asymp B \stackrel{\text{def}}{\iff} \big(\forall (i,(p,q)) \in A\big)\big(\forall (j,(s,t)) \in B\big)\; i = j \implies \max\{p,s\} < \min\{q,t\}.$$

Note that $T_n$ is finitely enumerably for each $n \in \mathbb{N}$. Define

$$\begin{aligned} U_T &\stackrel{\text{def}}{=} \bigcup \left\{ r^- A_l \mid l \in T \right\}, \\ K &\stackrel{\text{def}}{=} \{ a \in S \mid \mathrm{Pos} \between (U_T \downarrow a) \}, \end{aligned}$$

where $A_l$ denotes the last element of a list $l \in T$. We show that $K$ is a located subset of $\mathcal{S}$.

Note that $K$ is the positivity of the open subtopology $\mathcal{S}_{U_T}$ by Lemma 2.15 (2). Thus $K$ is a splitting subset of $\mathcal{S}$. Hence it remains to be shown that for each $L \in \mathbb{N}$, either $b_L \in \neg K$ or $a_L \in K$. Let $L \in \mathbb{N}$ and define

$$n_L \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } L \le N, \\ L - N & \text{if } L > N. \end{cases}$$

Then the following two cases arise:

(1) $\big(\exists l \in T_{n_L}\big)\big(\forall (i,(p,q)) \in A_l\big)\; i = L \implies (p,q) \in (-\infty, 3/4)$,

(2) $\big(\forall l \in T_{n_L}\big)\big(\forall (i,(p,q)) \in A_l\big)\; i = L \implies (p,q) \in (1/2, \infty)$.

In the first case, there exist $l \in T_{n_L}$ and $(L,(p,q)) \in A_l$ such that $(p,q) \in (-\infty, 3/4)$. Thus

$$r^- A_l \lhd r^- \{(L,(p,q))\} \lhd r_L^-(p,q) \lhd r_L^- \big(-\infty, 3/4\big) \lhd a_L.$$

Since $A_l \in r\,\mathrm{Pos}$, we have $\mathrm{Pos} \between (r^- A_l \downarrow a_L)$. Hence $a_L \in K$.

In the second case, suppose that $b_L \in K$. Then there exist $n \in \mathbb{N}$ and $l \in T_n$ such that $\mathrm{Pos} \between (r^- A_l \downarrow b_L)$. If $n > n_L$, then by letting $l = \langle A_0, \ldots, A_n\rangle$ where $A_l = A_n$, we have $A_{n_L} \asymp A_{n_L+1}, \ldots, A_{n-1} \asymp A_n$. Since $(p,q) \in (1/2, \infty)$ for an element $(L,(p,q)) \in A_{n_L}$, we have $(s,t) \in (0,\infty)$ for an element $(L,(s,t)) \in A_l$. Thus

$$r^- A_l \downarrow b_L \lhd r^- \{(L,(s,t))\} \downarrow b_L \lhd r_L^-(0,\infty) \downarrow b_L \lhd \emptyset,$$

and hence $\mathrm{Pos} \between \emptyset$, a contradiction. If $n \leq n_L$, then since $r^- A_l \lhd r_N^-(\infty, 3/4) \lhd a_N$, we have

$$r^- A_l \lhd r^-(\mathcal{E}_{N+n+1} \downarrow \cdots \downarrow \mathcal{E}_{N+n_L} \downarrow A_l).$$

Thus, there exist $A_{n+1} \in \mathcal{E}_{N+n+1}, \ldots, A_{n_L} \in \mathcal{E}_{N+n_L}$ such that

$$\mathrm{Pos} \between \left(r^-(A_l \downarrow A_{n+1} \downarrow \cdots \downarrow A_{n_L}) \downarrow b_L\right).$$

Then $l * \langle A_{n+1}, \ldots, A_{n_L}\rangle \in T_{n_L}$, and so

$$r^- A_{n_L} \downarrow b_L \lhd r_L^-(1/2, \infty) \downarrow b_L \lhd \emptyset.$$

Thus $\mathrm{Pos} \between \emptyset$, a contradiction. Hence $b_L \in \neg K$. Therefore $K$ is located.

Next, we show that $\mathcal{S}_{b_N} \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg K} \sqsubseteq \mathcal{S}_{a_N}$. Since $\mathcal{S}^{\mathcal{S}-\neg K}$ is the closure of $\mathcal{S}_{U_T}$ by Lemma 2.20, it suffices to show that $b_N \lhd U_T \lll a_N$. Since $b_N \lhd r^- \mathcal{E}_N$, we have $b_N \lhd \left(r^- \mathcal{E}_N \downarrow b_N\right) \cap \mathrm{Pos}$. Let $c \in r^- \mathcal{E}_N \downarrow b_N$ such that $\mathrm{Pos}(c)$. Then there exists $A \in \mathcal{E}_N$ such that $c \in r^- A \downarrow b_N$. If $A \in \mathcal{E}_N^-$, then

$$c \lhd r^- A \downarrow b_N \lhd r_N^-(1/4, \infty) \downarrow b_N \lhd \emptyset,$$

and thus $\mathrm{Pos} \between \emptyset$, a contradiction. Hence $A \in \mathcal{E}_N^+$, and so $c \lhd r^- \mathcal{E}_N^+$. Therefore

$$b_N \lhd r^- \mathcal{E}_N^+ \lhd U_T.$$

Let $n \in \mathbb{N}$ and $l \in T_n$, and write $l = \langle A_0, \ldots, A_n\rangle$. Since $A_i \asymp A_{i+1}$ for all $i < n$ and $(p, q) \in (-\infty, 1/2)$ for an element $(N, (p, q)) \in A_0$, we have

$$r^- A_l \lhd r_N^-(-\infty, 3/4) \lll a_N.$$

Hence $U_T \lhd r_N^-(-\infty, 3/4) \lll a_N$.

Lastly, since $\{a_N\}$ is bounded, $\mathcal{S}^{\mathcal{S}-\neg K}$ is compact by Proposition 2.30, and $\mathcal{S}^{\mathcal{S}-\neg K}$ is overt by Lemma 4.3. □

The following is a point-free version of Lemma 4.12.

**Proposition 6.7** *For any $U, V \subseteq S$ such that $U \ll V$, there exists a compact overt subtopology $\mathcal{S}' \sqsubseteq \mathcal{S}$ such that $\mathcal{S}_U \sqsubseteq \mathcal{S}' \sqsubseteq \mathcal{S}_V$.*

**Proof** Let $U, V \subseteq S$, and suppose that $U \ll V$. Since

$$V \lhd \bigcup \{\mathsf{wb}(u) \mid (\exists v \in V)\, u \in \mathsf{wb}(v)\},$$

there exists $\{(u_0,v_0),\dots,(u_{n-1},v_{n-1})\} \in \mathrm{Fin}(\overline{\mathsf{wb}})$ such that $U \lhd \{u_0,\dots,u_{n-1}\}$ and $\{v_0,\dots,v_{n-1}\} \ll V$. By Lemma 6.6, for each $i<n$ there exists a located subset $K_i$ of $\mathcal{S}$ such that

$$\mathcal{S}_{u_i} \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg K_i} \sqsubseteq \mathcal{S}_{v_i}.$$

Let $K \stackrel{\mathrm{def}}{=} \bigcup_{i<n} K_i$. Since a finite union of located subsets is located, $K$ is located. Moreover we have

$$\mathcal{S}_U \sqsubseteq \mathcal{S}_{U_0} \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg K} \sqsubseteq \mathcal{S}_{V_0} \sqsubseteq \mathcal{S}_V,$$

where $U_0 = \{u_0,\dots,u_{n-1}\}$ and $V_0 = \{v_0,\dots,v_{n-1}\}$. Since $V_0$ is bounded, $\mathcal{S}^{\mathcal{S}-\neg K}$ compact overt by Proposition 2.30. □

Let $\mathcal{S}_r$ be the image of $\mathcal{S}$ under the embedding $r\colon \mathcal{S} \to \prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Then $\mathcal{S}_r$ is overt with the positivity $r\,\mathrm{Pos}$ by Lemma 2.13. Define

$$\omega \stackrel{\mathrm{def}}{=} \left\{A \in S_\Pi \mid \left(\forall (n,(p,q)) \in A\right) p < 1 < q\right\}.$$

It is straightforward to show that $\omega$ is a formal point of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Moreover, $\omega$ is a decidable subset of $S_\Pi$. Let

$$\overline{\mathrm{Pos}} \stackrel{\mathrm{def}}{=} r\,\mathrm{Pos} \cup \omega.$$

Since $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ is compact regular by Proposition 2.26, it is locally compact by Proposition 2.34.

**Lemma 6.8** *$\overline{\mathrm{Pos}}$ is a located subset of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$.*

**Proof** Since $\overline{\mathrm{Pos}}$ is a union of splitting subsets $r\,\mathrm{Pos}$ and $\omega$, it is a splitting of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$. Let $\mathsf{wc}_\Pi$ be the function defined by (10) in Section 5. Let $A, A' \in S_\Pi$, and suppose that $A' \in \mathsf{wc}_\Pi(A)$. Then $A$ and $A'$ are of the forms

$$A = \{(m_0,(p_0,q_0)),\dots,(m_{n-1},(p_{n-1},q_{n-1}))\},$$
$$A' = \left\{(m_0,(p'_0,q'_0)),\dots,(m_{n-1},(p'_{n-1},q'_{n-1}))\right\}$$

such that $p_i < p'_i < q'_i < q_i$ for all $i<n$. By Proposition 2.34, it suffices to show that either $A' \in \neg\overline{\mathrm{Pos}}$ or $A \in \overline{\mathrm{Pos}}$.

Since $\omega$ is decidable, we have either $A \in \omega$ or $A \in \neg\omega$. In the former case, we have $A \in \overline{\mathrm{Pos}}$. In the latter case, there exists $i_* < n$ such that either $1 \le p_{i_*}$ or $q_{i_*} \le 1$. Suppose that $1 \le p_{i_*}$, and suppose further that $A' \in r\,\mathrm{Pos}$. Then there exists $a \in \mathrm{Pos}$ such that $a\, r\, A'$. Thus

$$a \lhd r^-_{m_{i_*}}\{(p_{i_*},q_{i_*})\} \lhd r^-_{m_{i_*}}\{(p_{i_*},q_{i_*}) \mid p_{i_*} < 1 \,\&\, 0 < q_{i_*}\} \lhd \emptyset$$

by the axiom (2) of $\mathcal{I}[0,1]$. Since $\mathrm{Pos}(a)$, we have $\mathrm{Pos} \between \emptyset$, a contradiction. Since $A \in \neg\omega$ implies $A' \in \neg\omega$, it follows that $A' \in \neg\overline{\mathrm{Pos}}$.

Now, suppose that $q_{i_*} \leq 1$. Then

$$r^-_{m_{i_*}}\{(p'_{i_*}, q'_{i_*})\} \ll a_{m_{i_*}},$$

where $a_{m_{i_*}}$ is the second component of the pair $(b_{m_{i_*}}, a_{m_{i_*}}) \in \overline{\mathsf{wb}}$ indexed by $m_{i_*} \in \mathbb{N}$. Let

$$U_* \overset{\text{def}}{=} r^-_{m_{i_*}}\{(p'_{i_*}, q'_{i_*})\}.$$

By Proposition 6.7 and Lemma 4.3, there exists a located subset $K$ of $\mathcal{S}$ such that $\mathcal{S}_{U_*} \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg K}$ and $\mathcal{S}^{\mathcal{S}-\neg K}$ is compact. Choose $k \in \mathbb{N}$ and $\theta \in \mathbb{Q}^{>0}$ such that $2^{-k} < \theta$ and that $p_i < p'_i - 2\theta < q'_i + 2\theta < q_i$ for each $i < n$. Since $S_\Pi \lhd_\Pi \mathcal{C}^n_k$ for all $n, k \in \mathbb{N}$, we have

$$\begin{aligned} S &\lhd r^- \left(\mathcal{C}^{m_0}_k \downarrow \cdots \downarrow \mathcal{C}^{m_{n-1}}_k\right) \\ &\lhd r^- \left\{\left\{\left(m_0, (s_0, t_0)\right), \ldots, \left(m_{n-1}, (s_{n-1}, t_{n-1})\right)\right\} \in S_\Pi \mid (\forall i < n)\ t_i - s_i = 2^{-k}\right\}. \end{aligned}$$

Let $\mathcal{C}_A \overset{\text{def}}{=} \left\{\left\{\left(m_0, (s_0, t_0)\right), \ldots, \left(m_{n-1}, (s_{n-1}, t_{n-1})\right)\right\} \in S_\Pi \mid \left(\forall i < n\right)\ t_i - s_i = 2^{-k}\right\}$. Since $\mathcal{S}^{\mathcal{S}-\neg K}$ is compact overt with the positivity $K$, there exist $B_0, \ldots, B_{N-1} \in \mathcal{C}_A$ such that $B_j \in rK$ for each $j < N$ and that $S \lhd^{\mathcal{S}-\neg K} r^- \{B_0, \ldots, B_{N-1}\}$. For each $j < N$, write

$$B_j = \left\{\left(m_0, (s_{j,0}, t_{j,0})\right), \ldots, \left(m_{n-1}, (s_{j,n-1}, t_{j,n-1})\right)\right\}.$$

Then either $(s_{j,i}, t_{j,i}) \leq_{\mathcal{R}} (p'_i - 2\theta, q'_i + 2\theta)$ for all $i < n$ or $(s_{j,i}, t_{j,i}) \in (-\infty, p'_i) \cup (q'_i, \infty)$ for some $i < n$. Thus the following two cases arise:

(1) $\left(\exists j < N\right)\left(\forall i < n\right)\ (s_{j,i}, t_{j,i}) \leq_{\mathcal{R}} (p'_i - 2\theta, q'_i + 2\theta)$,

(2) $\left(\forall j < N\right)\left(\exists i < n\right)\ (s_{j,i}, t_{j,i}) \in (-\infty, p'_i) \cup (q'_i, \infty)$.

In the first case, there exists $j < N$ such that $B_j \leq A$, and hence $r^- B_j \lhd r^- A$. Since $B_j \in rK$ and $K$ is a splitting subset of $\mathcal{S}$, we have $A \in rK \subseteq r\,\mathrm{Pos} \subseteq \overline{\mathrm{Pos}}$.

In the second case, suppose that $A' \in r\,\mathrm{Pos}$. Then there exists $a \in \mathrm{Pos}$ such that $a\ r\ A'$. Let $\mathrm{Pos}_{\neg K}$ be the positivity of $\mathcal{S}_{\neg K}$. Since $\mathrm{Pos} = \mathrm{Pos}_{\neg K} \cup K$, we have either $a \in \mathrm{Pos}_{\neg K}$ or $a \in K$. If $a \in \mathrm{Pos}_{\neg K}$ then $\mathrm{Pos} \between (\neg K \downarrow a)$. Since $\mathcal{S}_{U_*} \sqsubseteq \mathcal{S}^{\mathcal{S}-\neg K}$, we have

$$\neg K \downarrow a \lhd \neg K \downarrow r^- A' \lhd \neg K \downarrow U_* \lhd \emptyset,$$

and thus $\mathrm{Pos} \between \emptyset$, a contradiction. If $a \in K$, then since

$$a \lhd^{\mathcal{S}-\neg K} \left(r^- \{B_0, \ldots, B_{N-1}\}\right) \downarrow a,$$

there exists $j < N$ such that $K \between \left(r^- B_j \downarrow a\right)$. Thus there exists $i < n$ such that $(s_{j,i}, t_{j,i}) \in (-\infty, p'_i) \cup (q'_i, \infty)$. If $(s_{j,i}, t_{j,i}) \in (-\infty, p'_i)$, then

$$\begin{aligned} r^- B_j \downarrow a &\lhd r^-_{m_i}(-\infty, p'_i) \downarrow r^-_{m_i}(p'_i, q'_i) \\ &\lhd r^-_{m_i}\left((-\infty, p'_i) \downarrow (p'_i, q'_i)\right) \lhd \emptyset, \end{aligned}$$

and thus $K \between \emptyset$, a contradiction. If $(s_{j,i}, t_{j,i}) \in (q'_i, \infty)$, we similarly obtain a contradiction. Hence $A' \in \neg(r\,\mathrm{Pos})$, and so $A' \in \neg\overline{\mathrm{Pos}}$. Therefore $\overline{\mathrm{Pos}}$ is located. □

Thus, $\overline{\mathrm{Pos}}$ determines a located subtopology of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$, which is compact overt regular by Lemma 4.3 and Proposition 2.25 (2). Write $\mathcal{T} = (S_\Pi, \lhd^{\mathcal{T}}, \leq)$ for this subtopology. Since $\omega$ is a formal point of $\mathcal{T}$, it is a located subset of $\mathcal{T}$. Let $\mathcal{T}_{\neg\omega}$ be the open complement of the located subtopology determined by $\omega$ in $\mathcal{T}$. The cover $\lhd^{\mathcal{T}}_{\neg\omega}$ of $\mathcal{T}_{\neg\omega}$ is given by

$$A \lhd^{\mathcal{T}}_{\neg\omega} \mathcal{U} \overset{\text{def}}{\iff} A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathrm{Pos}} \cup \mathcal{U}$$

for all $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$.

**Lemma 6.9** *The embedding $r \colon \mathcal{S} \to \prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$ satisfies $S \lhd r^- \neg\omega$.*

**Proof** Let $a \in S$ and $b \in \mathsf{wb}(a)$, and let $n \in \mathbb{N}$ be the index of the pair $(b, a) \in \overline{\mathsf{wb}}$. Then

$$\begin{aligned} b &\lhd r^-_n\left((-\infty, 1) \cup (0, \infty)\right) \downarrow b \\ &\lhd \left(r^-_n(-\infty, 1) \downarrow b\right) \cup \left(r^-_n(0, \infty) \downarrow b\right) \\ &\lhd r^-_n(-\infty, 1) \lhd r^- \neg\omega. \end{aligned}$$

Hence $a \lhd \mathsf{wb}(a) \lhd r^- \neg\omega$, and therefore $S \lhd r^- \neg\omega$. □

**Lemma 6.10** *For any $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$,*

$$r^- A \lhd r^- \mathcal{U} \iff A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathrm{Pos}} \cup \mathcal{U}.$$

*That is $\mathcal{S}_r = \mathcal{T}_{\neg\omega}$.*

**Proof** Let $A \in S_\Pi$ and $\mathcal{U} \subseteq S_\Pi$. First, suppose that $A \downarrow \neg\omega \lhd_\Pi \neg\overline{\mathrm{Pos}} \downarrow \mathcal{U}$. By Lemma 6.9 we have

$$\begin{aligned} r^- A &\lhd r^- A \downarrow r^- \neg\omega \\ &\lhd r^-\,(A \downarrow \neg\omega) \\ &\lhd r^-\left(\neg\overline{\mathrm{Pos}} \cup \mathcal{U}\right) \\ &\lhd \left(r^-\,(\neg r\,\mathrm{Pos} \downarrow \neg\omega) \cup r^- \mathcal{U}\right) \cap \mathrm{Pos} \\ &\lhd \left(\left(r^- \neg r\,\mathrm{Pos}\right) \cap \mathrm{Pos}\right) \cup \left(r^- \mathcal{U} \cap \mathrm{Pos}\right) \\ &\lhd r^- \mathcal{U} \cap \mathrm{Pos} \lhd r^- \mathcal{U}. \end{aligned}$$

Conversely, suppose that $r^{-}A \lhd r^{-}\mathcal{U}$, and let $B \in A \downarrow \neg\omega$. We must show that $B \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \cup \mathcal{U}$. Write $B = \{(m_0,(p_0,q_0)),\dots,(m_{n_B-1},(p_{n_B-1},q_{n_B-1}))\}$. Since $B \in \neg\omega$, there exists $i_* < n_B$ such that either $1 \leq p_{i_*}$ or $q_{i_*} \leq 1$. If $1 \leq p_{i_*}$ we have

$$B \lhd_{\Pi} \{(m_{i_*},(p_{i_*},q_{i_*}))\} \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \cup \mathcal{U}.$$

Now, suppose that $q_{i_*} \leq 1$. Let $B' \in \mathsf{wc}_{\Pi}(B)$, so that $B'$ is of the form

$$B' = \left\{(m_0,(p'_0,q'_0)),\dots,(m_{n_B-1},(p'_{n_B-1},q'_{n_B-1}))\right\}$$

such that $p_i < p'_i < q'_i < q_i$ for each $i < n_B$. Since $q'_{i_*} < 1$, we have

$$r^{-}B' \lhd r^{-}_{m_{i_*}}(p'_{i_*},q'_{i_*}) \ll a_{m_{i_*}},$$

and since $B' \lll A$ in $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$, we have $r^{-}B' \lll r^{-}A$ in $\mathcal{S}$. Hence $r^{-}B' \ll r^{-}A$ by Lemma 2.31. Moreover, since $\mathcal{U} \lhd_{\Pi} \mathcal{U}_{<}$ where

$$\mathcal{U}_{<} \stackrel{\mathrm{def}}{=} \left\{C' \in S_{\Pi} \mid (\exists C \in \mathcal{U})\, C' \in \mathsf{wc}_{\Pi}(C)\right\},$$

we have $r^{-}A \lhd r^{-}\mathcal{U}_{<}$. Thus there exist $C_0,\dots,C_{n_{\mathcal{U}}-1} \in \mathcal{U}$ and $C'_0,\dots,C'_{n_{\mathcal{U}}-1} \in S_{\Pi}$ such that $r^{-}B' \lhd r^{-}\left\{C'_0,\dots,C'_{n_{\mathcal{U}}-1}\right\}$ and that for each $j < n_{\mathcal{U}}$, the sets $C_j$ and $C'_j$ are of the forms

$$\begin{aligned} C_j &= \left\{\left(l_{j,0},(s_{j,0},t_{j,0})\right),\dots,\left(l_{j,n_j-1},(s_{j,n_j-1},t_{j,n_j-1})\right)\right\},\\ C'_j &= \left\{\left(l_{j,0},(s'_{j,0},t'_{j,0})\right),\dots,\left(l_{j,n_j-1},(s'_{j,n_j-1},t'_{j,n_j-1})\right)\right\} \end{aligned}$$

such that $s_{j,i} < s'_{j,i} < t'_{j,i} < t_{j,i}$ for each $i < n_j$. Let

$$M \stackrel{\mathrm{def}}{=} \max\{l_{j,i} \mid j < n_{\mathcal{U}} \;\&\; i < n_j\},$$

and choose $k \in \mathbb{N}$ and $\theta \in \mathbb{Q}^{>0}$ such that $2^{-k} < \theta$ and

$$(\forall j < n_{\mathcal{U}})(\forall i < n_j)\; s_{j,i} < s'_{j,i} - \theta \;\&\; t'_{j,i} + \theta < t_{j,i}.$$

Then $B' \lhd^{\mathcal{T}} \left(B' \downarrow \mathcal{C}_k^{\leq M}\right) \cap \overline{\mathrm{Pos}}$. Let $B'' \in \left(B' \downarrow \mathcal{C}_k^{\leq M}\right) \cap \overline{\mathrm{Pos}}$. Then either $B'' \in r\,\mathrm{Pos}$ or $B'' \in \omega$. Since $B' \in \neg\omega$ we have $B'' \in \neg\omega$, so the latter case yields a contradiction. If $B'' \in r\,\mathrm{Pos}$, then since

$$r^{-}B'' \lhd r^{-}\left\{C'_0,\dots,C'_{n_{\mathcal{U}}-1}\right\} \downarrow r^{-}B'' \lhd r^{-}\left(\{C'_0,\dots,C'_{n_{\mathcal{U}}-1}\} \downarrow B''\right),$$

there exists $j < n_{\mathcal{U}}$ such that $r\,\mathrm{Pos} \between \left(C'_j \downarrow B''\right)$. Hence $C'_j \asymp B''$, so that $B'' \leq C_j \lhd^{\mathcal{T}} \mathcal{U}$ by the choice of $\theta$. Thus $B' \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \cup \mathcal{U}$, and so $B \lhd_{\Pi} \mathsf{wc}_{\Pi}(B) \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \cup \mathcal{U}$. Therefore $A \downarrow \neg\omega \lhd_{\Pi} \neg\overline{\mathrm{Pos}} \cup \mathcal{U}$. □

Finally, since $\prod_{n\in\mathbb{N}}\mathcal{I}[0,1]$ is enumerably completely regular and $\mathcal{T}$ is its subtopology, $\mathcal{T}$ is a compact overt enumerably completely regular formal topology.

# 7 Point-free characterisation

We show that the notion of inhabited enumerably locally compact regular formal topology characterises that of Bishop locally compact metric space up to isomorphism.

First, we recall the main result of our previous work [14, Theorem 4.3.2].

**Lemma 7.1** *Let $\mathcal{S}$ be a formal topology. Then the following are equivalent.*

(1) *$\mathcal{S}$ is isomorphic to a compact overt enumerably completely regular formal topology.*

(2) *$\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$.*

(3) *$\mathcal{S}$ is isomorphic to the localic completion of some compact metric space.*

By Proposition 4.14 and Theorem 6.5, we have the following proposition.

**Proposition 7.2** *For any overt enumerably locally compact regular formal topology $\mathcal{S}$, there exists a locally compact metric space $X$ such that $\mathcal{M}(X) \cong \mathcal{S}$.*

Note that the image of any inhabited formal topology under a formal topology map is inhabited. Hence Corollary 4.15 yields the following.

**Corollary 7.3** *For any inhabited enumerably locally compact regular formal topology $\mathcal{S}$, there exists a Bishop locally compact metric space $X$ such that $\mathcal{M}(X) \cong \mathcal{S}$.*

**Lemma 7.4** *The localic completion of a Bishop locally compact metric space is isomorphic to an inhabited enumerably locally compact regular formal topology.*

**Proof** Let $X$ be a Bishop locally compact metric space. Since $X$ is separable, we may assume that $X$ is countable by Theorem 3.2. Since the base $M_X$ of $\mathcal{M}(X)$ is a countable union of countable sets, $M_X$ is countable. Since $a <_X b$ implies $a \ll b$ and we have

$$\mathsf{b}(x,\varepsilon) \lhd_X \left\{\mathsf{b}(x,\delta) \in M_X \mid \delta \in \mathbb{Q}^{>0} \,\&\, \delta < \varepsilon\right\}$$

for each $\mathsf{b}(x,\varepsilon) \in M_X$, the function $\mathsf{wb}\colon M_X \to \mathrm{Pow}(M_X)$ define by

$$\mathsf{wb}(\mathsf{b}(x,\varepsilon)) \stackrel{\text{def}}{=} \left\{\mathsf{b}(x,\delta) \in M_X \mid \delta \in \mathbb{Q}^{>0} \,\&\, \delta < \varepsilon\right\}$$

makes $\mathcal{M}(X)$ locally compact. For each $\mathsf{b}(x,\varepsilon) \in M_X$, the subset $\mathsf{wb}(\mathsf{b}(x,\varepsilon))$ is countable by the standard enumeration of the rational interval $(0,\varepsilon)$. Thus the set $\overline{\mathsf{wb}} = \{(b,a) \in M_X \times M_X \mid b \in \mathsf{wb}(a)\}$ is countable.

Moreover, for each $\mathsf{b}(x,\delta) \in \mathsf{wb}(\mathsf{b}(x,\varepsilon))$ we can define an order preserving bijection $\varphi\colon \mathbb{I} \to [\delta,\varepsilon] \cap \mathbb{Q}$. Then the family $\left(\{\mathsf{b}(x,\varphi(q))\}\right)_{q\in\mathbb{I}}$ is a wb-scale from $\{\mathsf{b}(x,\delta)\}$ to $\{\mathsf{b}(x,\varepsilon)\}$. Thus we can define a function $\mathsf{sc} \in \prod_{(b,a)\in\overline{\mathsf{wb}}} \mathrm{Sc}_{\ll}\left(\{b\},\{a\}\right)$ which assigns to each $(b,a) \in \overline{\mathsf{wb}}$ the wb-scale from $\{b\}$ to $\{a\}$ as described above.

Since $X$ is inhabited, $\mathcal{M}(X)$ is an inhabited formal topology. Therefore $\mathcal{M}(X)$ is an inhabited enumerably locally compact regular formal topology with the function $\mathsf{wb}$ and the choice $\mathsf{sc}$ of wb-scale for $\mathsf{wb}$. □

By Corollary 7.3 and Lemma 7.4, we obtain the following.

**Theorem 7.5** *Let $\mathcal{S}$ be a formal topology. Then $\mathcal{S}$ is isomorphic to an inhabited enumerably locally compact regular formal topology if and only if $\mathcal{S}$ is isomorphic to the localic completion of some Bishop locally compact metric space.*

Let **BLCM** be the full subcategory of **LCM** consisting of Bishop locally compact metric spaces, and let **IELKReg** be the full subcategory of **FTop** consisting of formal topologies which are isomorphic to some inhabited enumerably locally compact regular formal topology.

Then, the localic completion functor $\mathcal{M}\colon \mathbf{LCM} \to \mathbf{FTop}$ restricts to a functor $\mathcal{M}\colon \mathbf{BLCM} \to \mathbf{IELKReg}$ by Lemma 7.4, and the restricted functor $\mathcal{M}$ is essentially surjective by Theorem 7.5. Since the restriction is still full and faithful, we have the following.

**Theorem 7.6** *The categories* **BLCM** *and* **IELKReg** *are equivalent.*[6]

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[6] Constructively, this is the equivalence in a weaker sense that there exists a full, faithful and essentially surjective functor from one category to the other. Under the Axiom of Choice, this notion is equivalent to the stronger notion of equivalence, i.e. the existence of adjoint functors $F \dashv G$ such that $FG$ and $GF$ are naturally isomorphic to the identity functors (see Mac Lane [15]).

# References


[1] **P Aczel**, *Aspects of general topology in constructive set theory*, Ann. Pure Appl. Logic 137 (2006) 3–29; doi: 10.1016/j.apal.2005.05.016

[2] **P Aczel**, **M Rathjen**, *Notes on Constructive Set Theory*, Technical Report 40, Institut Mittag-Leffler (2000/2001)

[3] **E Bishop**, *Foundations of Constructive Analysis*, McGraw-Hill, New York (1967)

[4] **E Bishop**, **D Bridges**, *Constructive Analysis*, Springer, Berlin (1985)

[5] **J Cederquist**, **S Negri**, *A constructive proof of the Heine–Borel covering theorem for formal reals*, from: "Types for Proofs and Programs", (S Berardi, M Coppo, editors), Lecture Notes in Computer Science 1158, Springer Berlin Heidelberg (1996) 62–75; doi: 10.1007/3-540-61780-9_62

[6] **T Coquand**, **E Palmgren**, **B Spitters**, *Metric complements of overt closed sets*, MLQ Math. Log. Q. 57 (2011) 373–378; doi: 10.1002/malq.201010011

[7] **T Coquand**, **G Sambin**, **J Smith**, **S Valentini**, *Inductively generated formal topologies*, Ann. Pure Appl. Logic 124 (2003) 71–106; doi: 10.1016/S0168-0072(03)00052-6

[8] **G Curi**, *Constructive metrisability in point-free topology*, Theoret. Comput. Sci. 305 (2003) 85–109; doi: 10.1016/S0304-3975(02)00699-0

[9] **G Curi**, *Exact approximations to Stone-Čech compactification*, Ann. Pure Appl. Logic 146 (2007) 103–123; doi: 10.1016/j.apal.2006.12.004

[10] **M H Escardó**, *The regular-locally compact coreflection of a stably locally compact locale*, J. Pure Appl. Algebra 157 (2001) 41–55; doi: 10.1016/S0022-4049(99)00172-3

[11] **M P Fourman**, **R J Grayson**, *Formal spaces*, from: "The L.E.J. Brouwer Centenary Symposium", (A S Troelstra, D van Dalen, editors), North-Holland, Amsterdam (1982) 107–122

[12] **C Fox**, *Point-set and Point-free Topology in Constructive Set Theory*, PhD thesis, University of Manchester (2005)

[13] **P T Johnstone**, *Stone Spaces*, Cambridge University Press (1982)

[14] **T Kawai**, *Bishop Compactness in Formal Topology*, PhD thesis, Japan Advanced Institute of Science and Technology (2015); url: http://hdl.handle.net/10119/12746

[15] **S Mac Lane**, *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*, second edition, Springer-Verlag, New York (1998)

[16] **S Negri**, **D Soravia**, *The continuum as a formal space*, Arch. Math. Logic 38 (1999) 423–447; doi: 10.1007/s001530050149

[17] **E Palmgren**, *A constructive and functorial embedding of locally compact metric spaces into locales*, Topology Appl. 154 (2007) 1854–1880; doi: 10.1016/j.topol.2007.01.018

[18] **E Palmgren**, *Open sublocales of localic completions*, Journal of Logic and Analysis 2 (2010) 1–22; doi: 10.4115/jla.2010.2.1

[19] **G Sambin**, *Intuitionistic formal spaces — a first communication*, from: "Mathematical Logic and its Applications", (D Skordev, editor), volume 305, Plenum Press, New York (1987) 187–204; doi: 10.1007/978-1-4613-0897-3_12

[20] **G Sambin**, *Some points in formal topology*, Theoret. Comput. Sci. 305 (2003) 347–408; doi: 10.1016/S0304-3975(02)00704-1

[21] **S Vickers**, *Localic completion of generalized metric spaces I*, Theory Appl. Categ. 14 (2005) 328–356

[22] **S Vickers**, *Some constructive roads to Tychonoff*, from: "From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics", (L Crosilla, P Schuster, editors), Oxford Logic Guides 48, Oxford University Press (2005) 223–238; doi: 10.1093/acprof:oso/9780198566519.003.0014

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