Topometric spaces and perturbations of metric structures

Itaï Ben Yaacov

Abstract We develop the general theory of *topometric spaces*, i.e., topological spaces equipped with a well-behaved lower semi-continuous metric. Spaces of global and local types in continuous logic are the motivating examples for the study of such spaces. In particular, we develop Cantor-Bendixson analysis of topometric spaces, which can serve as a basis for the study of local stability (extending the ad hoc development in Ben Yaacov I and Usvyatsov A, Continuous first order logic and local stability. Trans Am Math Soc, in press), as well as of global $\aleph_0$-stability. We conclude with a study of perturbation systems (see Ben Yaacov I, On perturbations of continuous structures, submitted) in the formalism of topometric spaces. In particular, we show how the abstract development applies to $\aleph_0$-stability up to perturbation.

0 Introduction

Topometric spaces, namely spaces equipped both with a topology and with a metric, are omnipresent in continuous logic and in fact predate it.

Global type spaces, in the sense of continuous logic, as well as in the sense of predecessors such as Henson's logic or metric compact abstract theories, are equipped with a logic topology as well as with a natural metric $d(p, q) = d\big(p(\bar{M}), q(\bar{M})\big)$ (where

I. Ben Yaacov (✉)
Université de Lyon, Université Lyon 1, Institut Camille Jordan, CNRS UMR 5208,
43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France
URL: http://math.univ-lyon1.fr/~begnac/

$\bar{M}$ is the monster model). Iovino's notion of a uniform structure on the type spaces [8] is an early attempt to put this metric structure in a more general setting, and as such may be viewed as a precursor to the formalism we propose here. The metric nature of global types spaces was used by Iovino and later by the author to define useful notions of Morley ranks. These ranks play a crucial role in the proof of Morley's Theorem for metric structures in [3].

Continuous logic was proposed and developed in [7] as a model-theoretic formalism for metric structures. Unlike its predecessors it provides a good notion of a local type space $S_\varphi(M)$, namely the space of φ-types for a fixed formula φ. Again, in addition to the logic topology, this space is equipped with a useful metric $d_\varphi(p, q) = \sup_{\bar{b}} |\varphi(\bar{x}, \bar{b})^p - \varphi(\bar{x}, \bar{b})^q|$. More examples comes from the study of perturbations of metric structures in [5], where perturbation metrics turn out to be alternative topometric structures on the type spaces.

In addition, topometric analogues of the classical (one should say "discrete") Cantor-Bendixson analysis in these spaces play important roles in various contexts. In global type spaces they can be used to characterise $\aleph_0$-stability and define the Morley ranks which were constructed (in a far more complicated manner) in [3]. In local type spaces they can be use to characterise local stability and independence, as in [7]. Finally, at the end of the present paper, we use them to for a rudimentary study of the notion of a theory being $\aleph_0$-*stable up to perturbation*, which occurs more and more in recently studied examples. In particular we show that this property is characterised by the existence of corresponding Morley ranks.

In the present paper we unite these examples under the single definition of a *topometric space*. We then proceed to study topometric spaces as such, much like general topology studies topological spaces, with a particular emphasis on Cantor-Bendixson analysis. Alongside this abstract study we provide many motivating examples from continuous logic as well as applications of our abstract results to the study of metric structures.

In Sect. 1 we define topometric spaces and the category of topometric spaces. While we are quite certain about the category of *compact* topometric spaces, we propose to extend our definitions to locally compact and even more general spaces, with some lower degree of certitude. In particular, we study questions such the existence of quotients which preserve part of the structure, which seem to be a little more complicated than for classical topological spaces.

In Sect. 2 we study various notions analogous to isolation in classical topological spaces. In the case of topometric type spaces, *d-isolated* types are indeed the correct analogues of isolated types in classical logic (such types were referred to in [3] as *principal*, following Henson's earlier terminology).

In Sect. 3 we study several natural notions of Cantor-Bendixson ranks, showing that they all give rise the same notion of Cantor-Bendixson analysability. We give several results characterising Cantor-Bendixson analysability and comparing the Cantor-Bendixson ranks of two spaces.

Finally, in Sect. 4 we study the special case of topometric spaces arising as type spaces equipped with perturbation metrics. We study notions such as λ-stability, and in particular $\aleph_0$-stability, up to perturbation.

For the purpose of examples we shall assume familiarity with the basics of continuous first order logic, as developed in [7]. For a general survey of continuous logic and the model theory of metric structures we refer the reader to [2].

1 Topometric spaces

1.1 Basic properties

While a metric is usually defined to take values in $[0, \infty)$, we allow infinite distances. If (X, d) is a metric space then distances between sets are defined as usual $d(A, B) = \inf\{d(x, y) \colon x \in A, y \in B\}$ and $d(x, A) = d(\{x\}, A)$. We follow the convention that $d(x, \varnothing) = d(A, \varnothing) = \inf \varnothing = \infty$.

Notation 1.1 Let (X, d) be a metric space, $A \subseteq X, r \in \mathbb{R}^+$. We define:

$$B(A, r) = \{x \in X \colon d(x, A) < r\}$$
$$\overline{B}(A, r) = \{x \in X \colon d(x, A) \leq r\}$$

When A is a singleton $\{a\}$ we may write $B(a, r)$ and $\overline{B}(a, r)$ instead.

Definition 1.2 A *(Hausdorff) topometric space* is a triplet $(X, \mathscr{T}, d) = (X, \mathscr{T}_X, d_X)$ where X is a set of points, $\mathscr{T}$ is a topology on X and d is a $[0, \infty]$-valued metric on X:

(i) The metric refines the topology. In other words, for every open $U \subseteq X$ and every $x \in U$ there is $r > 0$ such that $B(x, r) \subseteq U$.
(ii) The metric function $d \colon X^2 \to [0, \infty]$ is lower semi-continuous, i.e., for all $r \in \mathbb{R}^+$ the set $\{(a, b) \in X^2 \colon d(a, b) \leq r\}$ is closed in X^2.

Convention 1.3 We shall follow the convention that unless explicitly qualified otherwise, terms and notations from the vocabulary of general topology (e.g., compactness, continuity, etc.) refer to the topological space $(X, \mathscr{T})$, while terms from the vocabulary of metric spaces which are not applicable in general topology (e.g., uniform continuity, completeness) refer to the metric space (X, d).

The topological closure of a subset $Y \subseteq X$ is denoted by $\overline{Y}$. Note that the closed set $\overline{B(a, r)}$ should not be confused with $\overline{B}(a, r)$, which is defined in pure metric terms. Lower semi-continuity of d implies that $\overline{B}(a, r)$ is closed (more generally, we show in Lemma 1.8 below that $\overline{B}(F, r)$ is closed for every compact F), so $\overline{B(a, r)} \subseteq \overline{B}(a, r)$. A discrete 0/1 metric provides us with an extreme example of proper inclusion: $\overline{B(a, 1)} = \{a\} \neq X = \overline{B}(a, 1)$.

Recall that the *weight* of a topological space X, denoted $\mathrm{wt}(X)$, is the minimal cardinality of a base of open sets for X. Similarly, if X is a metric space, we use $\|X\|$ to denote its *density character*, i.e., the minimal size of a dense subset. In case X is a topometric space, $\mathrm{wt}(X)$ refers to its topological part while $\|X\|$ to its metric part. If X is a finite space then $\|X\| = \mathrm{wt}(X) = |X|$; otherwise both are infinite.

Lemma 1.4 *A topological space X equipped with a lower semi-continuous metric is Hausdorff. If X is in addition compact then the metric must refine the topology and X is a topometric space (in other words, for compact spaces the second item in Defininition 1.2 implies the first item).*

Proof Assume X is equipped with a lower semi-continuous metric d. Then the diagonal $\Delta X = d^{-1}(0) \subseteq X \times X$ is closed and X is Hausdorff.

We observed earlier that $\overline{B}(a, r)$ is closed for all $a \in X$ and $r \in \mathbb{R}^+$. If U is a neighbourhood of a then $\bigcap_{r>0} \overline{B}(a, r) = \{a\} \subseteq U$. If X is compact then $\overline{B}(a, r) \subseteq U$ for some $r > 0$, so the metric refines the topology, as desired.   □

Example 1.5 The motivating examples come from continuous first order logic [7]. Type spaces, which are naturally equipped with an intrinsic "logic topology", also admit one or several metric structures rendering them topometric spaces:

   (i)   The type spaces $S_n(T)$ of a theory T, equipped with what we call the *standard metric*:

$$d(p, q) = \inf\{d(a, b) \colon a \vDash p \text{ and } b \vDash q \text{ in a saturated model } M \vDash T\}.$$

            If T is incomplete and p, q belong to distinct completions then $d(p, q) = \infty$.

  (ii) Type spaces in unbounded continuous logic were defined in [5], with the distance between types defined as above. Unlike type spaces in standard continuous logic, these are merely locally compact.

 (iii) Local type spaces $S_\varphi(M)$ over a model:

$$d(p, q) = \sup\{|\varphi(x, b)^p - \varphi(x, b)^q| \colon b \in M\}.$$

 (iv) Perturbation systems are presented in [5] via an alternative system of topometric structures on the type spaces $S_n(T)$, where the metric is the "perturbation distance" $d_{\mathfrak{p}}$. Here $d(p, q) = \infty$ means that a realisation of p cannot be perturbed into a realisation of q.

  (v) The metric $\tilde{d}_{\mathfrak{p}}$, defined in [5] as a combination of d and $d_{\mathfrak{p}}$, also renders the topological space $S_n(T)$ a topometric space.

There are two extreme kinds of topometric spaces which arise naturally from standard topological and metric spaces:

Definition 1.6   (i) A *maximal* topometric space is one in which the metric is discrete.

  (ii) A *minimal* topometric space is one in which the metric coincides with the topology.

Example 1.7 Every Hausdorff topological space can be naturally viewed as a maximal topometric space. Similarly, every metric space can be naturally viewed as a minimal topometric space.

Clearly a topometric space X is minimal if and only if the metric function $d \colon X^2 \to \mathbb{R}^+$ is continuous (rather than merely lower semi-continuous). If X is compact, then this is further equivalent to the metric topology on X being compact.

A topometric space is both minimal and maximal if and only if it is topologically discrete: thus the minimal topometric spaces should be viewed as the topometric generalisation of classical discrete topological spaces.

Lemma 1.8 *Let X be a topometric space, $F \subseteq X$ compact. Then $\overline{B}(F, r)$ is closed for every $r \in \mathbb{R}^+$ (we say that F has closed metric neighbourhoods in X).*

Proof For $r \in \mathbb{R}^+$ let $G_r = F \times X \cap d^{-1}([0, r])$. Then $G_r \subseteq F \times X$ is closed. Since F is compact, the projection $\pi \colon F \times X \to X$ is closed, whereby $\pi(G_r)$ is closed. We conclude that $\overline{B}(F, r) = \bigcap_{r' > r} \pi(G_{r'})$ is closed. $\qquad\square$

In particular, if X is compact then every closed subset of X has closed metric neighbourhoods. In fact we can say something slightly stronger:

Lemma 1.9 *Let X be a Hausdorff topological space, d a metric on X. If X is compact and $d \colon X^2 \to [0, \infty]$ is lower semi-continuous then every closed set in X has closed metric neighbourhoods. Conversely, if X is regular (in particular, if X is locally compact) and every closed set has closed metric neighbourhoods then d is lower semi-continuous.*

Proof The first assertion is a consequence of Lemma 1.8. For the converse assume $\overline{B}(F, r)$ is closed for every closed F and $r > 0$. Assume that $d(x, y) > r$, and choose some intermediate values $d(x, y) > r_1 > r_2 > r$. First, we have $x \notin \overline{B}(y, r_1)$ and the latter is closed. We can therefore find an open set V such that $x \in V \subseteq \overline{V} \subseteq X \smallsetminus \overline{B}(y, r_1)$, whereby $d(\overline{V}, y) \geq r_1 > r_2$, and thus $y \notin \overline{B}(\overline{V}, r_2)$. Following the same reasoning we find U open such that $y \in U \subseteq \overline{U} \subseteq X \smallsetminus \overline{B}(\overline{V}, r_2)$. We conclude that $d(V, U) \geq d(\overline{V}, \overline{U}) \geq r_2 > r$, and $(x, y) \in V \times U \subseteq X^2 \smallsetminus d^{-1}([0, r])$, as desired. $\qquad\square$

Along with Lemma 1.4, this means that our definition of a *compact* topometric space coincides with that given in [7], based on closed sets having closed metric neighbourhoods. If we drop the compactness assumption then still in every "reasonable" space lower semi-continuity of the metric is weaker than the closed metric neighbourhoods assumption. It is nonetheless sufficient for our purposes in compact and locally compact spaces, and as all natural examples are at least locally compact we have no choice but to base our intuition on those. Finally, lower semi-continuity passes to product spaces (equipped with the supremum distance) whereas the closed metric neighbourhoods property does not seem to do so.

Lemma 1.10 *Let X be a topometric space, $K, F \subseteq X$ compact. Then $d(K, F) = \min\{d(x, y) \colon x \in K, y \in F\}$, i.e., the minimum is attained by some $x_0 \in K$ and $y_0 \in Y$.*

Proof Let $r = d(K, F) = \inf\{d(x, y) \colon x \in K, y \in F\}$. Then for every $r' > r \colon (K \times F) \cap d^{-1}([0, r']) \neq \varnothing$. As $d^{-1}([0, r]) = \bigcap_{r' > r} d^{-1}([0, r'])$ and this is a decreasing intersection of compact sets we get $(K \times F) \cap d^{-1}([0, r]) \neq \varnothing$, as requried. $\qquad\square$

Proposition 1.11 *Every compact topometric space is complete.*

Proof Let $\{x_n\}_{n<\omega}$ be a Cauchy sequence in X. For each n let $r_n = \sup\{d(x_n, x_m): m > n\}$, so $r_n \searrow 0$. Then $\overline{B}(x_n, r_n)$ is closed for all n (by Lemma 1.8) and contains x_m for all $m \geq n$. By compactness $F = \bigcap_{n<\omega} \overline{B}(x_n, r_n) \neq \varnothing$. It follows that $F = \{x\}$ where x is the metric limit of $\{x_n\}_{n<\omega}$. □

On the other hand, for non-compact spaces completeness is not isomorphism-invariant: indeed, the minimal spaces $(0, 1)$ and $\mathbb{R}$ are isomorphic as (minimal) topometric spaces, yet only one of them is complete. A more interesting example is that of type spaces of a theory T in unbounded continuous logic (see [5]): Each type space $S_n(T)$ is locally compact and complete. The compactification procedure described there consists of embedding it $S_n(T) \hookrightarrow S_n(T^\infty)$, where $S_n(T^\infty)$ is a compact type space of a standard (i.e., bounded) theory T^∞. This embedding is a morphism (continuous and locally uniformly continuous) and the image is incomplete—indeed, it is metrically dense in $S_n(T^\infty)$.

Question 1.12 Can a "single point" (or at least "few points") compactification be constructed for locally compact topometric spaces? In the case of an unbounded continuous theory T, we should like $S_n(T^\infty)$ to be a "few points compactification" of $S_n(T)$.

1.2 The category of topometric spaces

In this paper we deal mostly with compact or locally compact topometric spaces (by our convention this means the topology is compact or locally compact). The correct notion of a morphism of compact topometric spaces seems clear.

Definition 1.13 Let X and Y be topometric spaces, X compact. A *morphism* $f: X \to Y$ is a mapping which is both continuous (topologically) and uniformly continuous (metrically).

We seek a candidate for the definition of a morphism between general topometric spaces. In the non compact case uniform continuity seems too strong a requirement.

Definition 1.14 Let $f: X \to Y$ be a mapping between topometric spaces. We define some properties of f which depend both on the topological and the metric structures of X:

  (i)   We say that f is *locally uniformly continuous* if for every $x \in X$ has a neighbourhood on which f is uniformly continuous.
  (ii) We say that f is *weakly locally uniformly continuous* if for every $x \in X$ and $\varepsilon > 0$ there are a neighbourhood U of x and $\delta > 0$ such that if $y, y' \in U$ and $d_X(y, y') < \delta$ then $d_Y(f(y), f(y')) \leq \varepsilon$.
  (iii) We say that f is *uniformly continuous on every compact* if every restriction of f to a compact subset of X is uniformly continuous.

Lemma 1.15 *The properties defined in Definition 1.14 imply one another from top to bottom.*

Proof We only need to prove that if X is compact and $f: X \to Y$ is weakly locally uniformly continuous then f is uniformly continuous.

Indeed, let $\varepsilon > 0$. Then X admits an open cover $X = \bigcup_{i \in I} U_i$ such that for each U_i there is $\delta_i > 0$ such that if $x, x' \in U_i$ and $d_X(x, x') < \delta_i$ then $d_Y(f(x), f(x')) \leq \varepsilon$. The set $\bigcup_{i \in I} U_i \times U_i$ is a neighbourhood of the diagonal of X, which is equal to $\bigcap_{\delta > 0} d^{-1}([0, \delta])$. By definition of a topometric space each $d^{-1}([0, \delta])$ is closed. Thus by compactness we may assume that $\bigcup_{i < n} U_i \times U_i \supseteq d^{-1}([0, \delta'])$. Now let $\delta = \min\{\delta_i : i < n\} \cup \{\delta'\}$. If $x, x' \in X$ and $d(x, x') < \delta$ then $x, x' \in U_i$ for some $i < n$ and therefore $d(f(x), f(x')) \leq \varepsilon$. □

Thus for locally compact X all these properties agree. For non locally compact X even local uniform continuity seems too strong (e.g., for Proposition 1.17 below). On the other hand, uniform continuity on every compact is too weak (let X be the minimal space based on $\mathbb{R} \setminus \mathbb{Q}$ and Y the maximal one: then id: $X \to Y$ is uniformly continuous on every compact while not being even metrically continuous.) We find ourselves led to suggesting the intermediary property as the definition.

Definition 1.16 (Tentative) Let X and Y be topometric spaces. A *morphism* $f: X \to Y$ is a mapping which is both continuous and weakly locally uniformly continuous.

We leave it to the reader to check that the composition of two morphisms is indeed a morphism. By Lemma 1.15 this definition agrees with Definition 1.13.

Proposition 1.17 *Let* $f: X \to Y$ *be a continuous mapping between topometric spaces, and assume that either* X *is maximal or* Y *is minimal. Then* f *is a morphism.*

Proof The case where X is maximal is immediate, so we prove the case where Y is minimal. We need to show that f is weakly locally uniformly continuous. So let $\varepsilon > 0$ and $x \in X$. Let $V = B(f(x), \varepsilon/2) \subseteq Y$. As Y is minimal, V is open, so $U = f^{-1}(V)$ is open in X. Then $x \in U$, and for every $y, y' \in U$ we have $d(f(y), f(y')) \leq \varepsilon$ (no need for δ here). □

This justifies the terminology: a maximal topometric structure is the strongest possible such structure on a topological space, while a minimal structure is the weakest possible on a (metrisable) topological space.

We get another reassurance about our definition of a morphism of non-compact topometric spaces from the following result, telling us that for all intents and purposes we may identify classical Hausdorff topological spaces with maximal topometric spaces and classical metric spaces with minimal topometric spaces:

Proposition 1.18 *The construction of a maximal topometric space from a Hausdorff topological space is a functor* $\mathfrak{T} \to \mathfrak{TM}$, *from the category of Hausdorff spaces and continuous mappings to that of topometric spaces. As such it is the left-adjoint of the forgetful functor* $\mathfrak{TM} \to \mathfrak{T}$. *This functor is an equivalence of categories between Hausdorff topological spaces and maximal topometric spaces.*

Similarly the construction of a minimal topometric space from a metric one is a functor $\mathfrak{M} \to \mathfrak{TM}$, *from the category of metric spaces and (metrically) continuous mappings to topometric spaces. It is the right-adjoint of the forgetful functor* $\mathfrak{TM} \to \mathfrak{M}$, *and is an equivalence of categories between metric spaces and minimal topometric spaces.*

Proof Let $\psi_{\mathfrak{T}}\colon \mathfrak{T} \to \mathfrak{TM}$ be the maximal topometric space construction and $\psi_{\mathfrak{M}}\colon \mathfrak{M} \to \mathfrak{TM}$ be the minimal topometric space construction. Both are functors by Proposition 1.17. Let also $\varphi_{\mathfrak{T}}\colon \mathfrak{T} \to \mathfrak{TM}$ and $\varphi_{\mathfrak{M}}\colon \mathfrak{M} \to \mathfrak{TM}$ be the forgetful functors. It is immediate from the definition that $\varphi_{\mathfrak{T}}$ is indeed a functor; it is also not difficult to check that $\varphi_{\mathfrak{M}}$ is.

Let $X \in \mathfrak{T}$, $Y \in \mathfrak{TM}$, and $f\colon X \to Y$ a mapping between their underlying sets. Clearly if $f\colon \psi_{\mathfrak{T}}(X) \to Y$ is a morphism then $f\colon X \to \varphi_{\mathfrak{T}}(Y)$ is continuous, and the converse is by Proposition 1.17. Thus $\psi_{\mathfrak{T}}$ is the left-adjoint of $\varphi_{\mathfrak{T}}$. If $X, Y \in \mathfrak{T}$ we get $\mathrm{Hom}_{\mathfrak{T}}(X, Y) = \mathrm{Hom}_{\mathfrak{T}}(X, \varphi_{\mathfrak{T}} \circ \psi_{\mathfrak{T}}(Y)) = \mathrm{Hom}_{\mathfrak{TM}}(\psi_{\mathfrak{T}}(X), \psi_{\mathfrak{T}}(Y))$ whence the equivalence of categories.

The argument for minimal topometric spaces and metric spaces is similar.   □

Definition 1.19 Let X be a topometric space. A *(topometric) subspace* of X is a subset $Y \subseteq X$ equipped with the induced structure. One easily verifies that this is indeed a topometric space (this was in fact used implicitly in the proof of Lemma 1.15) and that the inclusion mapping $Y \hookrightarrow X$ is a morphisms.

A mapping $f\colon Y \to X$ is a *monomorphism* if it is an isomorphism with a subspace of X (its image).

When dealing with quotients we feel more secure restricting to compact spaces. Recall that a continuous surjective mapping from a compact space to a Hausdorff space is automatically a topological quotient mapping, so we only need to worry about the metric structure.

Lemma 1.20 *Let (X, d_X) be a compact topometric space and $\pi\colon X \to Y$ a Hausdorff topological quotient of X. Let $d_Y(y, y') = d_X\big(\pi^{-1}(y), \pi^{-1}(y')\big)$. Then (Y, d_Y) is a topometric space and $\pi\colon (X, d_X) \to (Y, d_Y)$ is a contractive homomorphism of topometric spaces.*

Proof For $r \geq 0$ let $\Delta_X^r = \{(x, x') \in X^2\colon d_X(x, x') \leq r\}$ and define $\Delta_Y^r \subseteq Y^2$ similarly. The set Δ_X^r is closed in X^2 by the lower semi-continuity of d_X and by Lemma 1.10 we have $\Delta_Y^r = (\pi, \pi)(\Delta_X^r)$ so Δ_Y^r is closed as well. Therefore d_Y is lower semi-continuous. By Lemma 1.9 d_Y refines the topology on Y and (Y, d_Y) is a topometric space. The rest follows directly from the construction.   □

For many purposes we shall require a somewhat stronger notion of quotient.

Definition 1.21 Let X and Y be topometric spaces, X compact. An *epimorphism* $f\colon X \to Y$ is a surjective morphism satisfying that for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ and $y \in Y$:

$$d_Y(f(x), y) < \delta \implies d_X(x, f^{-1}(y)) \leq \varepsilon.$$

We then say that Y is a *quotient space* of X, and that f is a *quotient mapping*.

Note that this property is in some sense a converse of uniform continuity, which may be stated as:

$$(\forall \varepsilon)(\exists \delta)(\forall xy)\big(d_X(x, f^{-1}(y)) < \delta \implies d_Y(f(x), y) \leq \varepsilon\big).$$

Also, a mapping is a surjective monomorphism if and only if it is an injective epimorphism if and only if it is an isomorphism.

Definition 1.22 Let us call a mapping between metric spaces $f: X \to Y$ is *precise* if for every $x \in X$ and $y \in Y$: $d_Y(f(x), y) = d_X(x, f^{-1}(y))$. We follow the convention that $d(x, \varnothing) = \infty$.

Lemma 1.23 *Let* $f: X \to Y$ *be a continuous precise mapping between topometric spaces. Then*:

(i) *It is uniformly continuous, and in particular a morphism.*
(ii) *If f is surjective and X is compact then f is an epimorphism. (Any generalisation of the definition of epimorphisms to non-compact spaces should preserve this property.)*
(iii) *The distance from* $f(X)$ *to its complement* $Y \smallsetminus f(X)$ *is infinite. Thus, if we exclude in Y the infinite distance then a precise mapping is necessarily surjective.*
(iv) *If f is injective then it is isometric.*

Proof Immediate. □

1.3 Existence of precise quotients

We shall need a result saying that interesting quotients of compact topometric spaces exist. By interesting we mean ones which preserve some prescribed piece of information without preserving too much else. Throughout this subsection X is a compact topometric space (much of what we say can be extended to locally compact spaces in a straightforward manner).

First, let us describe precise topometric quotients of X.

Lemma 1.24 *Let* $\sim$ *be an equivalence relation on X. Let* $[x] = \{x' \in X: x \sim x'\}$ *and* $Y = X/\sim = \{[x]: x \in X\}$ *be the quotient set,* $\pi: X \twoheadrightarrow Y$ *the projection map. Then the following are equivalent*:

(i) *The relation* $\sim$ *is closed and for all* $x, y \in X$: $d_X(x, [y]) = d_X([x], y)$.
(ii) *There exists a topometric structure on Y such that* π *is a precise quotient map.*

Moreover, if such a topometric structure on Y exists then it is unique, given by the quotient topology in Y and the metric:

$$d_Y([x], [y]) = d_X([x], [y]) = \inf\{d_X(x', y'): x' \sim x, y' \sim y\}.$$

Proof Assume first $\sim$ is closed and $d_X(x, [y]) = d_X([x], y)$. Then Y is compact and Hausdorff, and that each equivalence class is compact. By Lemma 1.10, for all $x, y \in X$ there are $x' \in [x]$ and $y' \in [y]$ such that $d_X([x], [y]) = d_X(x', y')$. We obtain that $[x] \neq [y] \implies d_X([x], [y]) > 0$ and $d_X([x], [y]) = d_X(x', [y]) = d_X([x], y) = d_X(x, [y])$, whereby $d_X([x], [z]) \leq d_X([x], y) + d_X(y, [z]) = d_X([x], [y]) + d_X([y], [z])$. Thus $d_Y([x], [y]) = d_X([x], [y])$ is a metric.

For all $r \in \mathbb{R}^+$ the set $\{(x, y) \colon d_X(x, [y]) \leq r\} \subseteq X^2$ is closed as the projection on the first and last coordinate of $\{(x, y, z) \colon d_X(x, y) \leq r$ and $y \sim z\} \subseteq X^3$. Thus $\{([x], [y]) \colon d_Y([x], [y]) \leq r\} \subseteq Y^2$ is compact and therefore closed, and $(Y, \mathcal{T}_Y, d_Y)$ is a topometric structure. The property $d_Y([x], [y]) = d_X(x, [y])$ also implies that π is precise.

Conversely, assume that π is precise. Then it is immediate that the topometric structure on Y is as prescribed in the moreover part, and that for all $x, y \in X$:

$$d_X(x, [y]) = d_Y([x], [y]) = d_X([x], y).$$

$\square$

Lemma 1.25 *Let $\sim$ be a closed equivalence relation on X. Then the following are equivalent*:

(i)   *For all $x, y \in X$: $d(x, [y]) = d([x], y)$.*
(ii) *For every class $[x] \subseteq X$ and $r \in \mathbb{R}^+$: $\overline{B}([x], r)$ is closed under $\sim$.*
(iii) *For every closed set $F \subseteq X$ and $r \in \mathbb{R}^+$, if F is closed under $\sim$ then so is $\overline{B}(F, r)$.*
(iv) *There exists a family B of closed sets in X such that*:
    (a) *B is closed under finite intersections.*
    (b) *For all $x \not\sim y$ there is $F \in B$ such that $x \in F$, $y \notin F$.*
    (c) *For all $F \in B$ and $r \in \mathbb{Q}^+$: $\overline{B}(F, r)$ is closed under $\sim$.*

Proof (i) $\Longrightarrow$ (ii). For all $y \sim y'$ we have $d(y, [x]) = d([y], x) = d(y', [x])$, so one belongs to $\overline{B}([x], r)$ if and only if the other does.

(ii) $\Longrightarrow$ (iii). As F and each $[x]$ are compact and closed under $\sim$, we have from Lemma 1.10 that $\overline{B}(F, r) = \bigcup_{x \in F} B(x, r) = \bigcup_{x \in F} B([x], r)$.

(iii) $\Longrightarrow$ (iv). Immediate.

(iv) $\Longrightarrow$ (i). It is enough to show that $d(x, [y]) \geq d([x], y)$. Let B be a family as in the assumption, and assume that $d(y, [x]) > r \in \mathbb{Q}^+$. Then we can find a family $B_0 \subseteq B$ closed under finite intersections such that $\bigcap B_0 = [x]$. As $\overline{B}(y, r) \cap [x] = \varnothing$, we obtain by compactness that $\overline{B}(y, r) \cap F = \varnothing$ for some $F \in B_0$. Then $y \notin \overline{B}(F, r)$, and since the latter is closed under $\sim$: $[y] \cap \overline{B}(F, r) = \varnothing$. In particular $d([y], x) > r$.

$\square$

Let $\mathfrak{A} = C(X, [0, 1])$. It is known that the closed equivalence relations on X are precisely those of the form $\sim_{\mathfrak{B}}$ where $\mathfrak{B} \subseteq \mathfrak{A}$ and $x \sim_{\mathfrak{B}} y \iff (\forall f \in \mathfrak{B})(f(x) = f(y))$. In this case for each $f \in \mathfrak{B}$ there is a unique $f' \in C(Y, [0, 1])$ such that $f = f' \circ \pi$. If $\mathfrak{B}$ is closed under $\neg$, $\frac{1}{2}$ and $\dot{-}$, then the set $\mathfrak{B}' = \{f' \colon f \in \mathfrak{B}\}$ dense in uniform convergence in $C(Y, [0, 1])$.

Lemma 1.26 *Let $\mathfrak{B} \subseteq \mathfrak{A}$, and assume $\mathfrak{B}$ is closed under $\neg$, $\frac{1}{2}$ and $\dot{-}$. Then the following are equivalent*:

(i) *For all $x, y \in X$: $d(x, [y]) = d([x], y)$.*
(ii) *For all $r \in \mathbb{Q}^+$, $f \in \mathfrak{B}$: $\overline{B}(f^{-1}(\{0\}, r))$ is closed under $\sim_{\mathfrak{B}}$.*

Proof (i) $\Longrightarrow$ (ii). Follows from Lemma 1.25, as every set of the form $f^{-1}(\{0\})$ is closed in X and closed under $\sim_{\mathfrak{B}}$.

(ii) $\Longrightarrow$ (i). Let $B = \{f^{-1}(\{0\}): f \in \mathfrak{B}\}$. For every $f, g \in \mathfrak{B}$ we also have $f \vee g \in \mathfrak{B}$, so B is closed under finite intersections. If $x \not\sim_{\mathfrak{B}} y$, there is $f \in \mathfrak{B}$ such that $f(x) \neq f(y)$. Possibly replacing f with $\neg f \in \mathfrak{B}$ we have $f(x) < f(y)$. If $f(x) < s < f(y)$ and $s \in [0, 1]$ is dyadic, then we can replace f with $f \dot{-} s \in \mathfrak{B}$, and have $f(x) = 0 < f(y)$. Thus B satisfies the hypotheses of Lemma 1.25(iv) . $\qquad\square$

As we use continuous functions to prescribe equivalence relations, we need to measure how many such functions are required to capture a piece of information.

Definition 1.27 (i) The *definition complexity* $\chi_{\mathrm{def}}(K)$ of a closed subset $K \subseteq X$ is the smallest infinite cardinal κ such that there exists a family of continuous functions $\{f_i : i < \kappa\} \subseteq \mathfrak{A}$ which define K in the sense that $K = \bigcap_i f_i^{-1}(\{0\})$.

(ii) The definition complexity of the metric d is defined as:

$$\chi_{\mathrm{def}}(d) = \sup\{\chi_{\mathrm{def}}(\overline{B}(K, r)): r > 0, \chi_{\mathrm{def}}(K) = \aleph_0\}.$$

Note that if $\chi_{\mathrm{def}}(K) = \aleph_0$ then K is the zero set of a single continuous function. In case $\chi_{\mathrm{def}}(K) = \kappa > \aleph_0$ we can find $\{K_i : i < \kappa\}$ such that $\chi_{\mathrm{def}}(K_i) = \aleph_0$, $K = \bigcap_i K_i$, and the family $\{K_i : i < \kappa\}$ is closed under finite intersections. It follows by compactness that $\overline{B}(K, r) = \bigcap_i \overline{B}(K_i, r)$. We conclude that $\chi_{\mathrm{def}}(\overline{B}(K, r)) \leq \chi_{\mathrm{def}}(K) + \chi_{\mathrm{def}}(d)$.

Theorem 1.28 *Let X be a compact topometric space, $\mathfrak{B} \subseteq \mathfrak{A} = C(X, [0, 1])$. Then there exists $\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{A}$ such that:*

(i) *$|\mathfrak{B}'| \leq |\mathfrak{B}| + \chi_{def}(d)$.*
(ii) *$Y = X/\sim_{\mathfrak{B}'}$ admits a topometric structure for which the projection $\pi : X \to Y$ is a precise topometric quotient map.*

Moreover, this quotient structure is unique given by the quotient topology and the metric:

$$d_Y([x], [y]) = d_X([x], [y]) = \inf\{d_X(x', y'): x' \sim_{\mathfrak{B}'} x, y' \sim_{\mathfrak{B}'} y\}.$$

Proof For each $f \in \mathfrak{A}$ and $r \in \mathbb{Q}^+$ choose a family $\mathfrak{B}_{f,r} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}_{f,r}| \leq \chi_{\mathrm{def}}(d)$ and $\overline{B}(f^{-1}(\{0\}), r) = \bigcap\{g^{-1}(\{0\}): g \in \mathfrak{B}_{f,r}\}$.

Let $\mathfrak{B}_0$ be the closure of $\mathfrak{B}$ under $\neg$, $\frac{1}{2}$ and $\dot{-}$, so $|\mathfrak{B}_0| \leq |\mathfrak{B}| + \aleph_0$. Given $\mathfrak{B}_n \subseteq \mathfrak{A}$ closed under these connectives, let $\mathfrak{B}_{n+1}$ be the closure under the connectives of $\mathfrak{B}_n \cup \bigcup_{f \in \mathfrak{B}_n, r \in \mathbb{Q}^+} \mathfrak{B}_{f,r}$.

Let $\mathfrak{B}' = \bigcup_n \mathfrak{B}_n$. Then $\mathfrak{B} \subseteq \mathfrak{B}' \subseteq \mathfrak{A}$, $|\mathfrak{B}'| \leq |\mathfrak{B}| + \chi_{\mathrm{def}}(d)$, and $\mathfrak{B}'$ satisfies the hypotheses of Lemma 1.26. We conclude using Lemma 1.24. $\qquad\square$

Corollary 1.29 *Let X be a compact topometric space, $\mathfrak{B} \subseteq \mathfrak{A} = C(X, [0, 1])$. Then X admits a precise topometric quotient $\pi: X \to Y$ such that each member of $\mathfrak{B}$ factors through π and $\mathrm{wt}(Y) \leq |\mathfrak{B}| + \chi_{def}(d)$.*

Proof Immediate from Theorem 1.28, since $\mathrm{wt}(X/\sim_{\mathfrak{B}'}) \leq |\mathfrak{B}'| + \aleph_0 \leq |\mathfrak{B}| + \chi_{\mathrm{def}}(d)$. $\qquad\square$

Definition 1.30 Let X be a compact topometric space.

(i) A family $\mathscr{Q}$ of (isomorphism classes of) quotients of X is *sufficient* if for every subset $\mathfrak{B} \subseteq C(X, [0, 1])$ there is a quotient $(Y, \pi) \in \mathscr{Q}$ (where $\pi \colon X \to Y$ is the quotient map) such that $\mathrm{wt}(Y) \leq |\mathfrak{B}| + \aleph_0$ and every member of $\mathfrak{B}$ factors via π.

(ii) We say that X has *enough quotients* if X admits a sufficient family of quotients.

(iii) We say that X has *enough precise quotients* if X admits a sufficient family of precise quotients.

Theorem 1.31 *Let X be a topometric space. Then the following are equivalent:*

(i) $\chi_{def}(d_X) = \aleph_0$.

(ii) *The family of all precise quotients of X is sufficient.*

(iii) *X has enough precise quotients.*

Proof (i) $\Longrightarrow$ (ii). By Corollary 1.29.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). Let $K \subseteq X$ be a zero set, $K' = \overline{B}(K, r)$, and we need to show that $\chi_{\mathrm{def}}(K') = \aleph_0$ as well. Say that $K = f^{-1}(0)$, and let $\pi \colon X \to Y$ be a precise quotient such that $\mathrm{wt}(Y) = \aleph_0$ and $f = f' \circ \pi$. Let $\bar{K} = f'^{-1}(0)$, $\bar{K}' = \overline{B}(\bar{K}, r)$. Then $K = \pi^{-1}(\bar{K})$, so $K' = \pi^{-1}(\bar{K}')$ by preciseness. On the other hand, as $\mathrm{wt}(Y) = \aleph_0$ every closed set is a zero set, so say $\bar{K}' = g^{-1}(0)$. Then $K' = (g \circ \pi)^{-1}(0)$, as desired. $\qquad\square$

The topometric spaces we are interested in are type spaces, with either the standard metric or some other (e.g., perturbation) metric. Such spaces almost always have enough precise quotients.

Proposition 1.32 (i) *Let T be a theory in a language of arbitrary size. Let d' be a metric on $\mathrm{S}_n(T)$, and assume that for every r the set of $2n$-tuples $\{(\bar{a}, \bar{b}) \colon d'(\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{b})) \leq r\}$ is type-definable using only countably many symbols from the language. Then $(\mathrm{S}_n(T), d')$ has enough precise quotients.*

(ii) *Let T be a theory in a language of arbitrary size. Then $(\mathrm{S}_n(T), d)$ has enough precise quotients where T is the standard metric. (Since we may name parameters in the language, this also applies to $\mathrm{S}_n(A)$ for any set of parameters A.)*

(iii) *Let M be an $\aleph_1$-saturated and strongly $\aleph_1$-homogeneous structure in a countable language, and let d' be a lower semi-continuous metric on $\mathrm{S}_n(M)$ invariant under the action of $\mathrm{Aut}(M)$. Then $(\mathrm{S}_n(M), d')$ has enough precise quotients.*

(iv) *Let T be a theory in a language of arbitrary size, $\varphi(\bar{x}, \bar{y})$ a formula, M a model. Then $(\mathrm{S}_\varphi(M), d_\varphi)$ has enough precise quotients where d_φ is the standard metric on $\mathrm{S}_\varphi(M)$.*

Proof For the first item, let $K \subseteq \mathrm{S}_n(T)$ be a zero set. Then K can be defined using countably many symbols from the language, so $\overline{B}(K, r)$ can also be defined using countably many symbols and is therefore a zero set as well. It follows that $\chi_{\mathrm{def}}(d') =$

$\aleph_0$, and conclude using Theorem 1.31. The second and third items are special cases of the first one.

For the last item, even though we allow a language of arbitrary size we may replace it with a countable sub-language containing all symbols appearing in φ. By downward Löwenheim-Skolem, the family of all spaces $S_\varphi(M')$ where $M' \preceq M$ is a sufficient family of precise quotients. □

2 *d*-isolation

Usually we do not expect to find topologically isolated points in a non-maximal topometric space, since by definition the topology cannot be stronger than the metric. Instead, we define a notion of isolation relative to the metric by requiring the topology to be as strong as possible around a point, i.e., to coincide with the metric:

Definition 2.1 (i) A point $x \in X$ is *d-isolated* if the topology and the metric agree near x, i.e., if $x \in B(x, r)^\circ$ for all $r > 0$.

(ii) It is *weakly d-isolated* if we only have $B(x, r)^\circ \neq \varnothing$ for all $r > 0$.

The need for two notions of isolation may be bothering. Indeed, in the case of the standard metric on type spaces d-isolation and weak d-isolation are equivalent (see [6, Fact 1.8]). On the other hand, the distinction is unavoidable in some cases, e.g., that of perturbation metrics, and each notion plays its own role.

For example, we have:

Lemma 2.2 *The set of weakly d-isolated points in a topometric space X is metrically closed.*

Proof Let $x \in X$, and assume that $B(x, r)$ contains a weakly d-isolated point for all $r > 0$. This means that $B(x, r/2)$ contains a weakly d-isolated point x_r, and $B(x_r, r/2)^\circ \neq \varnothing$. Thus $B(x, r)^\circ \neq \varnothing$ for all $r > 0$, and x is weakly d-isolated as well. □

In particular, a metric limit of d-isolated points is weakly d-isolated, and we do not know in general that it is d-isolated.

One can push the notion of weak d-isolation a bit further, allowing us to improve the previous observation a little. Since the usefulness of this exercise is not clear we do it briefly. Define the *weak d-isolation rank* of a point as follows: the rank of every point is at least zero, and the rank of x is at least $\alpha + 1$ if $B(x, r)^\circ$ contains a point of rank α for every $r > 0$. Thus a point is weakly d-isolated if and only if it has weak d-isolation rank one or more, and every truly d-isolated point has rank ∞.

Lemma 2.3 (i) *The set of all points of weak d-isolation rank $\geq \alpha$ is metrically closed.*

(ii) *In a complete topometric space the set of point of weak d-isolation rank ∞ is precisely the metric closure of the set of d-isolated points.*

Proof The first item is proved like Lemma 2.2, and yields that every limit of d-isolated points has weak d-isolation rank ∞. Conversely, assume x has weak

d-isolation rank ∞. Fix $r > 0$, and let $x_0 = x$, $r_0 = r$. Given x_n of weak d-isolation rank ∞ and $r_n > 0$ the set $B(x_n, r_n)^\circ$ contains at least one point x_{n+1} of weak d-isolation rank ∞ (as else the set of all ranks of points there is bounded by some ordinal). As the metric refines the topology $B(x_n, r_n)^\circ$ is metrically open, is there is $r_{n+1} > 0$ such that $\overline{B}(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n)^\circ$, and we may further assume that $r_{n+1} < r_n/2$. The sequence $(x_n : n > \omega)$ is Cauchy and thus converges to some point y. We observe that by construction $x_m \in \overline{B}(x_n, r_n)$ for all $m < n$ whereby $y \in \overline{B}(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n)^\circ$. In particular $d(x, y) < r$. More generally, for every $\varepsilon < 0$ there is n such that $r_n < \varepsilon/2$ in which case $y \in B(x_n, r_n)^\circ \subseteq B(y, \varepsilon)^\circ$ and y is d-isolated. $\square$

On the other hand, weak d-isolation does not seem to pass to sub-spaces while full d-isolation does.

Lemma 2.4 *Let $X \subseteq Y$ be topometric spaces, where X carries the induced structure from Y, and let $x \in X$ be d-isolated in Y. Then it is d-isolated in X.*

Proof Let $r > 0$ and $U = B_Y(x, r)^\circ$. Then $x \in U \cap X$ by assumption, $U \cap X$ is open in X and $U \cap X \subseteq B_X(x, r)$. $\square$

Recall that a mapping of topological spaces $f : X \to Y$ is *open* if the image of every open set is open. Say it is *weakly open* if the image of every non-empty open set has non-empty interior. Then we have the following:

Theorem 2.5 *Let $\pi : X \to Y$ be a morphism between locally compact topometric spaces. Let $x \in X$, $y = \pi(x) \in Y$, and $Z = \pi^{-1}(y)$ the fibre over y with the induced topometric structure. Then:*

(i) *If x is d-isolated in X then it is d-isolated in Z.*
(ii) *If x is (weakly) d-isolated in X and π is (weakly) open then y is (weakly) d-isolated in Y.*
(iii) *If π is an epimorphism, x is (weakly) d-isolated in Z and y (weakly) d-isolated in Y then x is (weakly) d-isolated in X.*

Proof There is no harm in replacing X with a compact neighbourhood of x and replacing Y with its image, so we may assume all spaces are compact, and thus that π is uniformly continuous. Let $\Delta : (0, \infty) \to (0, \infty)$ be a uniform continuity modulus for π, meaning that for all $\varepsilon > 0$, if $d_X(x', x'') < \Delta(\varepsilon)$ then $d_Y(\pi(x'), \pi(x'')) \le \varepsilon$.

For (i) just use Lemma 2.4.

For (ii) assume first that x is weakly d-isolated and π weakly open. Fix $r > 0$. Let $U = B_X(x, \Delta(r))^\circ$ in X. Then $U \ne \varnothing$, whereby $\varnothing \ne \pi(U)^\circ \subseteq B_Y(y, r)^\circ$ in Y. If x is d-isolated and π open we have furthermore that $x \in U$ and $y \in \pi(U) = \pi(U)^\circ$.

We now prove (iii). As π is assumed to be an epimorphism we may further assume that if $d_Y(y', \pi(x')) < \Delta(\varepsilon)$ then $d_X(\pi^{-1}(y'), x') \le \varepsilon$. Assume that x is weakly d-isolated in Z and y is weakly d-isolated in Y. Fix $r > 0$. Let $U = B_Z(x, r/2)^\circ$ in Z, so $U \ne \varnothing$ and we can fix some $z \in U$. Then there is an open set $V \subseteq X$ such that $U = Z \cap V$. We can find an open subset $V' \subseteq X$ such that $z \in V' \subseteq \bar{V}' \subseteq V$. Then $\bar{V}' = \bigcap_{s>0} \overline{B}(V', s)$, and as $\bar{V}'$ has closed metric neighbourhoods and X is compact

there is $s > 0$ such that $\overline{B}_X(\overline{V}', s) \subseteq V$. Decreasing s further we may assume that $s \leq r/2$ and $\overline{B}_X(z, s) \subseteq V'$.

Let $W = B_Y(y, \Delta(s))^\circ$, and $R = \pi^{-1}(W) \cap V'$, so R is open. Then $W \neq \varnothing$, and given $y' \in W$ we know that $d_X(z, \pi^{-1}(y')) \leq s$. As $\pi^{-1}(y')$ is compact, we obtain by Lemma 1.10 that $\varnothing \neq \pi^{-1}(y') \cap \overline{B}_X(z, s) \subseteq R$. Finally we show that $R \subseteq B_X(x, r)$. Indeed, let $x' \in R$. Then $\pi(x') \in W$, so $d_Y(\pi(x'), y) < \Delta(s) \Longrightarrow d(x', Z) \leq s$, and by Lemma 1.10 there is $x'' \in Z$ such that $d_X(x', x'') \leq s$. Then $x' \in V' \Longrightarrow x'' \in \overline{B}(V', s) \subseteq V$, so $x'' \in Z \cap V = U \subseteq B_Z(x, r/2)$. Thus $x' \in B_X(x, r)$, as desired. We conclude that $B_X(x, r)^\circ \supseteq R \neq \varnothing$, so x is weakly d-isolated.

In case x is d-isolated in Z and y in Y then we may choose $z = x \in V'$ and we know that $y \in W$ whereby $x \in R$, so the same argument shows that x is d-isolated in X. $\square$

Corollary 2.6 *Let $\pi \colon X \to Y$ be an open epimorphism of locally compact topometric spaces. Let $x \in X$, $y = \pi(x) \in Y$, and $Z = \pi^{-1}(y)$ the fibre over y with the induced topometric structure.*

Then x is d-isolated in X if and only if x is d-isolated in Z and y is d-isolated in Y.

Corollary 2.7 *Let $\pi \colon X \to Y$ be an epimorphism of locally compact topometric spaces. Let $x \in X$, $y = \pi(x) \in Y$, and $Z = \pi^{-1}(y)$ the fibre over y with the induced topometric structure. Let (Z, d_Z) be another topometric structure with the same underlying topological space Z, where d_Z is finer than d_X, so $\mathrm{id} \colon (Z, d_Z) \to (Z, d_X)$ is a morphism.*

If x is d_Z-isolated in Z and y is d_Y-isolated in Y then x is d_X-isolated in X.

Proof Since $\mathrm{id} \colon (Z, d_Z) \to (Z, d_X)$ is open we have that x is d_X-isolated in Z by Theorem 2.5(ii). Now apply Theorem 2.5(iii). $\square$

3 Cantor-Bendixson ranks

In classical topological spaces the Cantor-Bendixson derivative consists of removing isolated points. One crucial property of the derivative is that it is a closed subspace. In the topometric setting the situation is more complicated. If we simply tried to take out the (weakly) d-isolated points the derivative would no longer be closed, and the machinery would break down. We resolve this difficulty by viewing the classical Cantor-Bendixson derivative as consisting of removing open sets which are "small" (singletons, or finite sets). In a topometric space the metric gives rise to notions of smallness which allow to recover much of the classical theory concerning Cantor-Bendixson analysis.

Similar extensions of the classical Cantor-Bendixson analysis were also defined and used by Newelski [11].

3.1 General definitions

Fix a topometric space $(X, \mathcal{T}, d)$. We consider several natural notions of smallness which depend on a parameter $\varepsilon > 0$, none of which *a priori* better than another:

Definition 3.1 Let $A \subseteq X$, $\varepsilon > 0$, $\alpha \leq \omega$. We say that A is ε-α-*finite* if there is no subset $\{a_i : i \leq \alpha\} \subseteq A$ satisfying $d(a_i, a_j) > \varepsilon$ for all $i < j \leq \alpha$.

  (i)   We observe that A is ε-1-finite if and only if $\mathrm{diam}(A) \leq \varepsilon$.
  (ii) We say that A is ε-*finite* if it is ε-n-finite for some $n < \omega$.
(iii) We say that A is ε-*bounded* if it is ε-ω-finite.

These notions of smallness relate as follows:

Lemma 3.2 *Let $U \subseteq X$ be open, $\varepsilon > 0$.*

  (i)   *If $\mathrm{diam}(U) \leq \varepsilon$ then U is ε-finite.*
  (ii) *If U is ε-finite then it is ε-bounded.*
(iii) *If U is ε-bounded then it is the union of open sets of diameter $\leq 2\varepsilon$.*

Proof All but the last property are obvious. So assume U is ε-bounded. Let $a_0 \in U$ and construct by induction a maximal set $\{a_i : i < \alpha\} \subseteq U$ satisfying $d(a_i, a_j) > \varepsilon$ for all $i < j < \alpha$. Since U is ε-bounded α must by finite. Let $V = U \smallsetminus \bigcup_{0 < i < \alpha} \overline{B}(a_i, \varepsilon)$. Then V is open, since each $\overline{B}(a_i, \varepsilon)$ is closed, $a_0 \in V \subseteq U$, and $\mathrm{diam}(V) \leq 2\varepsilon$ by the maximality of the set $\{a_i : i < \alpha\}$.   □

To each notion of smallness we associate a Cantor-Bendixson derivative and rank. We can also define Cantor-Bendixson derivative based on (weakly) d-isolation, but showing these have the desired properties is trickier.

Definition 3.3 Let $\varepsilon > 0$ and $* \in \{d, f, b, i, wi\}$

  (i)   We define the $(*, \varepsilon)$-*Cantor-Bendixson derivative* of X as:

$$
\begin{aligned}
X'_{d,\varepsilon} &= X \smallsetminus \bigcup\{U \subseteq X : U \text{ open, } \mathrm{diam}(U) \leq \varepsilon\} \\
X'_{f,\varepsilon} &= X \smallsetminus \bigcup\{U \subseteq X : U \text{ open and } \varepsilon\text{-finite}\} \\
X'_{b,\varepsilon} &= X \smallsetminus \bigcup\{U \subseteq X : U \text{ open and } \varepsilon\text{-bounded}\} \\
X'_{i,\varepsilon} &= X \smallsetminus \bigcup\{\overline{B}(a, \varepsilon)^\circ : a \in X \ d\text{-isolated}\} \\
X'_{wi,\varepsilon} &= X \smallsetminus \bigcup\{\overline{B}(a, \varepsilon)^\circ : a \in X \text{ weakly } d\text{-isolated}\}.
\end{aligned}
$$

  (ii) We define the $(*, \varepsilon)$-*Cantor-Bendixson derivative sequence*:

$$
\begin{aligned}
X^{(0)}_{*,\varepsilon} &= X, \\
X^{(\alpha+1)}_{*,\varepsilon} &= \left(X^{(\alpha)}_{*,\varepsilon}\right)'_{*,\varepsilon}, \\
X^{(\alpha)}_{*,\varepsilon} &= \bigcap_{\beta < \alpha} X^{(\beta)}_{*,\varepsilon}, \qquad (\alpha \text{ limit}) \\
X^{(\infty)}_{*,\varepsilon} &= \bigcap_\alpha X^{(\alpha)}_{*,\varepsilon} = X^{|X|^+}_{*,\varepsilon}.
\end{aligned}
$$

(iii) For $A \subseteq X$ we define its $(*, \varepsilon)$-*Cantor-Bendixson rank*:

$$
\mathrm{CB}_{*,\varepsilon}(A) = \sup\{\alpha : X^{(\alpha)}_{*,\varepsilon} \cap A \neq \varnothing\}.
$$

Note that if A is compact then supremum is attained as a maximum.

(iv) We say that X is $(*, \varepsilon)$-*CB-analysable* if $CB_{*,\varepsilon}(X) < \infty$, i.e., if $X_{*,\varepsilon}^{(\infty)} = \varnothing$. It is $*$-*CB-analysable* if it is $(*, \varepsilon)$-CB-analysable for all $\varepsilon > 0$.

Definition 3.4 In case $K \subseteq X$ is compact and has a Cantor-Bendixson rank we may define its *Cantor-Bendixson degree*, in a manner depending on the kind of rank in question:

(i)   If $CB_{d,\varepsilon}^X(K) = \alpha < \infty$ then $X_{d,\varepsilon}^{(\alpha)} \cap K \subseteq \bigcup_{i < n} U_i$ where each $U_i \subseteq X_{d,\varepsilon}^{(\alpha)}$ is open of diameter $\leq \varepsilon$, and we define $CBd_{d,\varepsilon}^X(K)$ to be the minimal such n.

(ii) If $CB_{f,\varepsilon}^X(K) = \alpha < \infty$ then $X_{f,\varepsilon}^{(\alpha)} \cap K \subseteq U$ where $U \subseteq X_{f,\varepsilon}^{(\alpha)}$ is open and ε-n-finite for some n, and we define $CBd_{f,\varepsilon}^X(K)$ to be the minimal such n.

(iii) If $CB_{b,\varepsilon}^X(K) = \alpha < \infty$ then $X_{b,\varepsilon}^{(\alpha)} \cap K \subseteq U$ where $U \subseteq X_{b,\varepsilon}^{(\alpha)}$ is open and ε-bounded, and thus 2ε-n-finite for some n, and we define $CBd_{b,\varepsilon}^X(K)$ to be the minimal such n.

(iv) The definition in the previous item is based on the one we have already given in [3] (see Remark 3.5 below). Alternatively, we observe that since U is ε-bounded there exists a maximal finite subset $\{a_i : i < n\} \subseteq U$ verifying $d(a_i, a_j) > \varepsilon$ for all $i < j < n$ and we may define $CBd_{b',\varepsilon}^X(K)$ to be the smallest n for which this is possible.

Remark 3.5 (i) In the case of a maximal topometric space, which is just the topometric representation of a classical topological space, these notions all coincide (for ε small enough) with the classical Cantor-Bendixson ranks and derivatives.

(ii) The Cantor-Bendixson rank which was defined in [7] is $CB_{d,\varepsilon}$.

(iii) Also, one can verify that the ε-Morley rank of a closed or open set A as defined in [3] coincides with $CB_{b,\varepsilon}(A)$ where we view A as a subset of the space of types over a sufficiently saturated model, equipped with the standard metric. The ε-Morley degree defined there coincides with $CBd_{b,\varepsilon}(A)$.

The three notions of Cantor-Bendixson rank based on notions of smallness are tightly related, and in particular define a unique notion of CB-analysability.

Proposition 3.6 *For every topometric space X, ordinal α and $\varepsilon > 0$:*

$$X_{d,2\varepsilon}^{(\alpha)} \subseteq X_{b,\varepsilon}^{(\alpha)} \subseteq X_{f,\varepsilon}^{(\alpha)} \subseteq X_{d,\varepsilon}^{(\alpha)}.$$

If follows for all $A \subseteq X$:

$$CB_{d,2\varepsilon}(A) \leq CB_{b,\varepsilon}(A) \leq CB_{f,\varepsilon}(A) \leq CB_{d,\varepsilon}(A).$$

In particular, being $*$-CB-analysable for $* \in \{d, f, b\}$ are all equivalent properties, and from now on we shall refer to them as being CB-analysable.

Proof It follows from Lemma 3.2 that if $X \subseteq Y \subseteq Z \subseteq W$ then $X_{d,2\varepsilon}' \subseteq Y_{b,\varepsilon}' \subseteq Z_{f,\varepsilon}' \subseteq W_{d,\varepsilon}'$, and from there proceed by induction. $\qquad\square$

As we shall see below, the ranks $CB_{f,\varepsilon}$ and $CB_{b,\varepsilon}$ are somewhat easier to study than the $CB_{d,\varepsilon}$. At the same time, the degrees associated to $CB_{d,\varepsilon}$ and $CB_{f,\varepsilon}$ are more elegant than those associated with $CB_{b,\varepsilon}$. This "comparative study" suggests that among these three, the most convenient rank to use is $CB_{f,\varepsilon}$.

Proposition 3.7 *Assume that X is locally compact and CB-analysable. Then the d-isolated points are dense.*

Proof Let $U \subseteq X$ be open, non-empty. As X is locally compact, we may replace U with a non-empty open subset such that $\bar{U}$ is compact. We construct a decreasing sequence of non-empty open sets $(U_i : i < \omega)$ and numbers $(\varepsilon_i > 0 : i < \omega)$ such that $U_0 = U$, $\mathrm{diam}(U_{i+1}) \leq 2^{-i}$ and $\bar{U}_{i+1} \subseteq U_i$.

Start with $U_0 = U$. Given U_i open, non-empty, let V_i be open and non-empty such that $\bar{V}_i \subseteq U_i$.

Let $x \in V_i$ be such that $CB_{d,2^{-i}}(x) = \alpha$ is minimal. This means that $V_i \subseteq X^{(\alpha)}_{d,2^{-i}}$ and that there is an open subset W_{i+1} of $X^{(\alpha)}_{d,2^{-i}}$ such that $x \in W_{i+1}$ and $\mathrm{diam}(W_{i+1}) \leq 2^{-i}$. Let $U_{i+1} = V_i \cap W_{i+1}$. Since $V_i \subseteq X^{(\alpha)}_{d,2^{-i}}$, U_{i+1} is open in X, $\mathrm{diam}(U_{i+1}) \leq 2^{-i}$ and $\bar{U}_{i+1} \subseteq \bar{V}_i \subseteq U_i$.

In the end $\bigcap U_i = \bigcap \bar{U}_i$ is non-empty as a decreasing intersection on non-empty compact sets, and in fact consists of a single point $\{a\}$. It follows from the construction that for all $\varepsilon > 0$ the set $B(a, \varepsilon)$ contains U_i for some i, so $a \in U$ is d-isolated. $\square$

Corollary 3.8 *Let X be a locally compact topometric space. Then the following are equivalent*:

(i)   *X is CB-analysable.*
(ii) *For all locally compact $\varnothing \neq Y \subseteq X$ the d-isolated points of the topometric space Y (with the induced structure) are dense.*
(iii) *For all closed $\varnothing \neq Y \subseteq X$ the topometric space Y (with the induced structure) contains a weakly d-isolated point.*

Proof (i) $\Longrightarrow$ (ii). By Proposition 3.7.
(ii) $\Longrightarrow$ (iii). Immediate.
(iii) $\Longrightarrow$ (i). Let $\varepsilon > 0$, and assume that α is such that $X^{(\alpha)}_{d,\varepsilon} \neq \varnothing$. Then by assumption there is a point $a \in X^{(\alpha)}_{d,\varepsilon}$ which is weakly d-isolated there. Then the set $B(a, \varepsilon/2)^\circ$ (as calculated inside $X^{(\alpha)}_{d,\varepsilon}$) is non-empty and of diameter $\leq \varepsilon$, so $X^{(\alpha+1)}_{d,\varepsilon} \subsetneq X^{(\alpha)}_{d,\varepsilon}$. Therefore $X^{(\infty)}_{d,\varepsilon} = \varnothing$. $\square$

We also obtain a converse for Lemma 2.2:

Corollary 3.9 *Let X be a locally compact CB-analysable topometric space. Then the set of weakly d-isolated points is the metric closure of the set of d-isolated points.*

Proof One inclusion follows from Lemma 2.2. For the other, let x be a weakly d-isolated point. Then for each $r > 0$ the set $B(x, r)^\circ$ is non-empty and by Proposition 3.7 contains a d-isolated point. $\square$

We can now show the relation with the Cantor-Bendixson ranks based on (weakly) d-isolated points:

Theorem 3.10 *Let X be a locally compact topometric space. Then all notions of CB-analysability defined so far are equivalent. Moreover, if X is CB-analysable then for every ordinal α and $\varepsilon > 0$:*

$$X_{d,2\varepsilon}^{(\alpha)} \subseteq X_{wi,\varepsilon}^{(\alpha)} \subseteq X_{i,\varepsilon}^{(\alpha)} \subseteq X_{d,\varepsilon}^{(\alpha)}$$

Whereby for all $A \subseteq X$:

$$\mathrm{CB}_{d,2\varepsilon}(A) \leq \mathrm{CB}_{wi,\varepsilon}(A) \leq \mathrm{CB}_{i,\varepsilon}(A) \leq \mathrm{CB}_{d,\varepsilon}(A).$$

Proof Clearly if $X \subseteq Y \subseteq Z$ then $X'_{d,2\varepsilon} \subseteq Y'_{wi,\varepsilon} \subseteq Z'_{i,\varepsilon}$, whereby $X_{d,2\varepsilon}^{(\alpha)} \subseteq X_{wi,\varepsilon}^{(\alpha)} \subseteq X_{i,\varepsilon}^{(\alpha)}$ for all ε. Thus i-CB-analysable implies wi-CB-analysable implies CB-analysable. To close the circle assume that X is CB-analysable. By Proposition 3.7, if $X_{i,\varepsilon}^{(\alpha)}$ is non-empty then it contains a d-isolated point, so $X_{i,\varepsilon}^{(\alpha+1)} \subsetneq X_{i,\varepsilon}^{(\alpha)}$. Thus $X_{i,\varepsilon}^{(\infty)} = \varnothing$.

For the moreover part assume X is CB-analysable. Assume $U \subseteq X$ is open and $\mathrm{diam}(U) \leq \varepsilon$. By Proposition 3.7 U contains a d-isolated point a, and clearly $U \subseteq \overline{B}(a, \varepsilon)^\circ$. It follows that $X'_{i,\varepsilon} \subseteq X'_{d,\varepsilon}$, and the rest follows. □

Remark 3.11 Clearly, if $\varepsilon > \delta$ then $X'_{*,\varepsilon} \subseteq X'_{*,\delta}$. One may therefore define $X'_{*,\varepsilon^-} = \bigcap_{\delta < \varepsilon} X'_{*,\delta}$ and then proceed to define $X_{*,\varepsilon^-}^{(\alpha)}$ and $\mathrm{CB}_{*,\varepsilon^-}^X$ accordingly. We observe that for all $\varepsilon > \varepsilon'$:

$$X_{*,\varepsilon}^{(\alpha)} \subseteq X_{*,\varepsilon^-}^{(\alpha)} \subseteq X_{*,\varepsilon'}^{(\alpha)}, \qquad \mathrm{CB}_{*,\varepsilon}^X(A) \leq \mathrm{CB}_{*,\varepsilon^-}^X(A) \leq \mathrm{CB}_{*,\varepsilon'}^X(A).$$

In particular, no new notion of CB-analysability arises with these ranks. For some specific model-theoretic considerations (e.g., local Shelah stability ranks) the $\mathrm{CB}_{*,\varepsilon^-}$ are more convenient to use than the $\mathrm{CB}_{*,\varepsilon}$ ranks we defined earlier. In the present paper we shall restrict our attention to ranks of the form $\mathrm{CB}_{*,\varepsilon}$.

3.2 Cantor-Bendixson analysis of subspaces and quotients

Lemma 3.12 *Let $X \subseteq Y$ be topometric spaces. Then for every $* \in \{d, b, f, i\}$, $x \in X$, $\varepsilon > 0$: $\mathrm{CB}_{*,\varepsilon}^X(x) \leq \mathrm{CB}_{*,\varepsilon}^Y(x)$. In particular if Y is CB-analysable then so is X.*

Proof It is straightforward to verify that $X \subseteq Y$ implies $X'_{*,\varepsilon} \subseteq Y'_{*,\varepsilon}$ for $* \in \{d, b, f, i\}$ (though not for $* = wi$, since a weakly d-isolated point is not necessarily so in a subspace). The statement follows. □

Lemma 3.13 *Let $\pi : X \to Y$ be a surjective morphism of compact topometric spaces, and let Δ be such that $d(x, y) \leq \Delta(\varepsilon) \implies d(\pi(x), \pi(y)) \leq \varepsilon$. Then for all $b \in Y$ and $\varepsilon > 0$: $\mathrm{CB}_{f,\varepsilon}^Y(y) \leq \mathrm{CB}_{f,\Delta(\varepsilon)}^X(\pi^{-1}(y))$, i.e., $Y_{f,\varepsilon}^{(\alpha)} \subseteq \pi\left(X_{f,\Delta(\varepsilon)}^{(\alpha)}\right)$ for all α. In particular, if X is CB-analysable so is Y.*

Proof We show that $Y_{f,\varepsilon}^{(\alpha)} \subseteq \pi\left(X_{f,\Delta(\varepsilon)}^{(\alpha)}\right)$ by induction on α. For $\alpha = 0$ and limit this is clear from the induction hypothesis. For $\alpha + 1$, let us assume that $y \in Y_{f,\varepsilon}^{(\alpha+1)} \setminus \pi\left(X_{f,\Delta(\varepsilon)}^{(\alpha+1)}\right)$. Then each $x \in \pi^{-1}(y)$ has an open neighbourhood V_x such that $V_x \cap X_{f,\Delta(\varepsilon)}^{(\alpha)}$ is $\Delta(\varepsilon)$-finite. Since $\pi^{-1}(y)$ is compact it can be covered by a finite sub-family: $\pi^{-1}(y) \subseteq V_{x_0} \cup \cdots \cup V_{x_{k-1}} = V$. Then $V \cap X_{f,\Delta(\varepsilon)}^{(\alpha)}$ is $\Delta(\varepsilon)$-n-finite for some $n < \omega$.

Let $N(y)$ denote the set of all open neighbourhoods of y. Then for all $U \in N(y)$ there are points $y_{U,i} \in U \cap Y_{f,\varepsilon}^{(\alpha)}$ for $i \leq n$ such that $i < j \implies d(y_{U,i}, y_{U,j}) > \varepsilon$. By the induction hypothesis there are $x_{U,i} \in \pi^{-1}(y_{U,i}) \cap X_{f,\Delta(\varepsilon)}^{(\alpha)}$. By assumption on $\pi, \Delta \colon i < j \implies d(x_{U,i}, x_{U,j}) > \Delta(\varepsilon)$.

For each $i \leq n$, the net $(y_{U,i} : U \in N(y))$ converges to y. As X is compact we can find a directed partially ordered set $(S, \leq)$ and a decreasing function $\sigma \colon S \to N(y)$ sending $s \mapsto U_s$ such that for each $i \leq n$ the sub-net $(x_{U_s,i} : s \in S)$ converges in X, say to z_i. Then necessarily $\pi(z_i) = y$, i.e., $z_i \in \pi^{-1}(y)$. Thus for some $s \in S$ and $U = U_s \in N(y)$ we have $x_{U,i} \in V$ for all $i \leq n$, in contradiction with $\Delta(\varepsilon)$-n-finiteness of $V \cap X_{f,\Delta(\varepsilon)}^{(\alpha)}$. This contradiction concludes the proof. $\square$

Definition 3.14 Let X be a topometric space and $\varepsilon > 0$. An *ε-perfect tree* in X is a tree of compact non-empty sets $\{F_\sigma : \sigma \in 2^{<\omega}\}$ where:

(i)   If $\sigma, \tau \in 2^{<\omega}$ and $\sigma < \tau$ (i.e., τ extends σ) then $F_\tau \subseteq F_\sigma$.
(ii) For all $\sigma \in 2^{<\omega}$: $d(F_{\sigma 0}, F_{\sigma 1}) > \varepsilon$.

Lemma 3.15 *Let X be a locally compact topometric space, $\varepsilon > 0$, and assume an ε-perfect tree $\{F_\sigma : \sigma \in 2^{<\omega}\}$ exists in X. Let $\mathscr{B}$ be a base of closed sets for the topology on X, closed under finite intersections, and such that for every compact set $F \subseteq X$ there is a compact $F' \in \mathscr{B}$ such that $F \subseteq F'$. Then there exists in X an ε-perfect tree $\{F_\sigma' : \sigma < 2^{<\omega}\} \subseteq \mathscr{B}$, such that moreover $F_\sigma' \supseteq F_\sigma$ for all $\sigma \in 2^{<\omega}$.*

Proof We let $F_\varnothing'$ be any compact member of $\mathscr{B}$ containing $F_\varnothing$.

We now proceed by induction on $|\sigma|$. Assume $F_\sigma \subseteq F_\sigma' \in \mathscr{B}$ has been chosen. By assumption $\overline{B}(F_{\sigma 0}, \varepsilon) \cap F_\sigma' \cap F_{\sigma 1} = \varnothing$. Since $F_{\sigma 0}$ is compact, $\overline{B}(F_{\sigma 0}, \varepsilon)$ is closed and thus $\overline{B}(F_{\sigma 0}, \varepsilon) \cap F_\sigma'$ is compact. Since $F_{\sigma 1}$ is an intersection of members of $\mathscr{B}$, there is a finite sub-intersection $F_{\sigma 1}'$ satisfying $\overline{B}(F_{\sigma 0}, \varepsilon) \cap F_\sigma' \cap F_{\sigma 1}' = \varnothing$. Since $F_\sigma' \in \mathscr{B}$ we may assume that $F_\sigma' \supseteq F_{\sigma 1}'$, so $F_{\sigma 1}'$ is compact and $\overline{B}(F_{\sigma 0}, \varepsilon) \cap F_{\sigma 1}' = \varnothing$. Since $F_{\sigma 0}$ is compact as well we get $d(F_{\sigma 0}, F_{\sigma 1}') > \varepsilon$, and since $\mathscr{B}$ is closed under finite intersections we have $F_{\sigma 1}' \in \mathscr{B}$. We can now do the same thing to find $F_{\sigma 0}' \in \mathscr{B}$ compact such that $F_{\sigma 0} \subseteq F_{\sigma 0}' \subseteq F_\sigma'$ and $d(F_{\sigma 0}', F_{\sigma 1}') > \varepsilon$. This completes the induction step. $\square$

Remark 3.16 Notice that the second requirement on $\mathscr{B}$ is not superfluous. Indeed, the set of all complements of bounded open sets in $\mathbb{R}$ is a base for the closed sets and is closed under finite intersections, but equipping $\mathbb{R}$ with the maximal (i.e., discrete) metric we obtain a locally compact topometric space in which Lemma 3.15 fails.

On the other hand, let X be a locally compact space and let $\mathscr{B}$ be the family of all zero sets of functions in $C(X, [0, 1])$. Then it is easy to verify that $\mathscr{B}$ satisfies all the assumptions of Lemma 3.15.

Proposition 3.17 *Let X be locally compact. Then X is CB-analysable if and only if for no $\varepsilon > 0$ is there an ε-perfect tree in X.*

Proof Assume X is not CB-analysable, say it is not (d, ε)-CB-analysable. We may replace X with $X_{d,\varepsilon}^{(\infty)}$, so every non-empty open set has diameter greater than ε. Given an open set $\varnothing \neq U \subseteq X$ there are $x, y \in F$ be such that $d(x, y) > \varepsilon$. Then we can find compact neighbourhoods F_0 and F_1 of x and y, respectively, such that $d(F_0, F_1) > \varepsilon$. In particular F_0° and F_1° are non-empty open sets. We can thus proceed by induction to construct the tree.

Conversely, assume an ε-perfect tree $(F_\sigma : \sigma \in 2^{<\omega})$ exists. Let $F_n = \bigcup_{\sigma \in 2^n} F_\sigma$ and $F = \bigcap_n F_n$. Then $F \subseteq X$ is compact, and there is a natural surjective mapping $\pi: F \to 2^\omega$ sending $F_\tau = \bigcap_n F_{\tau \upharpoonright n}$ to τ for all $\tau \in 2^\omega$. Viewing 2^ω with the natural topology and the discrete metric, π is a surjective morphism. Since 2^ω is not CB-analysable, F is not CB-analysable by Lemma 3.13, so X is not CB-analysable by Lemma 3.12. □

Theorem 3.18 *Let X be a compact topometric space with enough precise quotients, and let $\mathscr{Q}$ be a sufficient family of precise quotients of X (see Definition 1.30). Then the following are equivalent:*

   (i)   *X is CB-analysable.*
  (ii)   *All homomorphic images of X are CB-analysable.*
 (iii)   *All precise quotients of X are CB-analysable.*
 (iv)   *All $Y \in \mathscr{Q}$ admitting a countable base are CB-analysable.*

Proof The first implication follows from Lemma 3.13. The second and third are immediate.

For the last, assume X is not CB-analysable. By Proposition 3.17, for some $\varepsilon > 0$ there exists an ε-perfect tree $\{F_\sigma : \sigma \in 2^{<\omega}\}$ in X. Let $\mathscr{B}$ be the collection of zero sets of continuous functions $f \in C(X, [0, 1])$. Then $\mathscr{B}$ satisfies the assumptions of Lemma 3.15, so we may assume that each F_σ is the zero set of some $f_\sigma \in C(X, [0, 1])$. Let $\mathfrak{B} = \{f_\sigma : \sigma \in 2^{<\omega}\}$. By definition there is a quotient $(Y, \pi) \in \mathscr{Q}$ such that $\mathrm{wt}(Y) = \aleph_0$ and each f_σ factors as $f'_\sigma \circ \pi$ with $f'_\sigma \in C(Y, [0, 1])$. Let $F'_\sigma = \pi(F_\sigma)$ for each $\sigma \in 2^{<\omega}$. Then as f_σ factors through π we have $F_\sigma = \pi^{-1}(F'_\sigma)$, and since π is precise we see that $d_Y(F'_{\sigma 0}, F'_{\sigma 1}) > \varepsilon$ for all σ, so $\{F'_\sigma : \sigma \in 2^{<\omega}\}$ is an ε-perfect tree in Y, and Y is not CB-analysable. □

We now seek to relate CB-analysability of a topometric space, its metric density character and its topological weight.

Proposition 3.19 *Let X be a CB-analysable topometric space. Then $\|X\| \leq \mathrm{wt}(X)$.*

Proof We may assume $\mathrm{wt}(X) \geq \aleph_0$.

Fix a base $\mathscr{B}$ of open sets for X, $|\mathscr{B}| = \mathrm{wt}(X)$. For each $U \in \mathscr{B}$ and $n < \omega$, if U contains a point of maximal $\mathrm{CB}_{d,2^{-n}}$-rank we let $x_{U,n}$ be such a point, otherwise $x_{U,n} \in U$

is an arbitrary point. Let $A = \{x_{U,n}\colon U \in \mathscr{B}, n < \omega\}$. Clearly $|A| \leq \aleph_0 \cdot |\mathscr{B}| = \mathrm{wt}(X)$, and we claim A is metrically dense.

Indeed, let $x \in X$ and $\varepsilon > 0$. Then for some n: $\varepsilon > 2^{-n}$. Let $\alpha = \mathrm{CB}_{d,2^{-n}}(x)$, and let $U_0 \subseteq X_{d,2^{-n}}^{(\alpha)}$ be relatively open of diameter $\leq 2^{-n}$. Let $U \subseteq X$ be open so that $U_0 = U \cap X_{d,2^{-n}}^{(\alpha)}$. Then $\mathrm{CB}_{d,2^{-n}}(U) = \alpha$, so $\mathrm{CB}_{d,2^{-n}}(x_{U,n}) = \alpha$, whereby $x_{U,n} \in U_0$. Thus $x_{U,n} \in B(x, \varepsilon) \cap A$. □

The converse does not hold in general (the disjoint union of a small non-CB-analysable space with a large CB-analysable one would be a counterexample). The converse does hold when $\mathrm{wt}(X)$ is countable:

Proposition 3.20 *Let X be a locally compact topometric space with a countable base. Then X is CB-analysable if and only if $\|X\| \leq \aleph_0$ if and only if $\|X\| < 2^{\aleph_0}$.*

Proof If X is CB-analysable then $\|X\| \leq \mathrm{wt}(X) = \aleph_0 < 2^{\aleph_0}$. Conversely, assume X is not CB-analysable. Then for some $\varepsilon > 0$ there is an ε-perfect tree $\{F_\sigma\colon \sigma < 2^{<\omega}\}$ in X. By compactness, for each $\tau \in 2^\omega$ the intersection $F_\tau = \bigcap_{n<\omega} F_{\tau\restriction_n}$ is non-empty, and we may choose $x_0 \in F_\tau$. Then $\tau \neq \tau' \implies d(x_\tau, x_{\tau'}) > \varepsilon$, whereby $\|X\| \geq 2^{\aleph_0}$. □

In conjunction with Theorem 3.18 we obtain:

Corollary 3.21 *Let X be a compact topometric space with enough precise quotients, and let $\mathscr{Q}$ be a sufficient family of precise quotients of X.*

(i) *X is CB-analysable.*
(ii) *Every homomorphic image Y of X satisfies $\|Y\| \leq \mathrm{wt}(Y)$.*
(iii) *Every precise quotient Y of X satisfies $\|Y\| \leq \mathrm{wt}(Y)$.*
(iv) *Every $Y \in \mathscr{Q}$ with a countable base is metrically separable.*

3.3 Comparing Cantor-Bendixson ranks of two spaces

Earlier we compared the Cantor-Bendixson ranks of two topometric spaces admitting a special relation, such as an inclusion or a surjective homomorphism from one to the other. Such (and other) homomorphisms can be identified with their graphs, which are a special kind of closed relations between two spaces. We shall now explore inequalities of Cantor-Bendixson ranks between spaces admitting an arbitrary closed relation.

Notation 3.22 Let X, Y be two compact spaces, $R \subseteq X \times Y$ a closed relation. For $x \in X$ and $A \subseteq Y$ we define:

$$R_x = \{y \in Y\colon (x, y) \in R\},$$
$$R^{\forall A} = \{x \in X\colon R_x \subseteq A\},$$
$$R^{\exists A} = \{x \in X\colon R_x \cap A \neq \varnothing\}.$$

Note that:

(i) For all $A \subseteq Y$: $R^{\forall(Y \smallsetminus A)} = X \smallsetminus R^{\exists A}$.
(ii) If $A \subseteq Y$ is closed then $R^{\exists A}$ is closed.
(iii) If $A \subseteq Y$ is open then $R^{\forall A}$ is open.

For example, if $Y \subseteq X$ then $R = \Delta_Y \subseteq Y \times X$ is a closed relation, $R_y = \{y\}$, $R^{\forall A} = R^{\exists A} = A \cap Y$. If $\pi\colon X \to Y$ is a projection, then $R = \{(\pi(x), x)\colon x \in X\} \subseteq Y \times X$ is again a closed relation, $R_y = \pi^{-1}(y)$, $R^{\exists A} = \pi(A)$.

The formalism of closed relations allows us to compare Cantor-Bendixson ranks of spaces (or of subsets thereof) in very general situations. In particular, the following result, albeit more technical, is a proper generalisation of Lemma 3.13 and a partial generalisation of Lemma 3.12.

Let us fix:

- A pair of locally compact topometric spaces X, Y.
- A closed relation $R \subseteq X \times Y$ such that R_x is compact for every $x \in X$.
- $\varepsilon, \delta > 0$ such that for all $(x, y), (x', y') \in R$: if $d_Y(y, y') \leq \delta$ then $d_X(x, x') \leq \varepsilon$.

Lemma 3.23 *If $U \subseteq Y$ is open and δ-finite (δ-bounded) then $R^{\forall U} \cap (R^{\exists Y})^\circ$ is open and ε-finite (ε-bounded).*

Proof Since both $R^{\forall U}$ and $(R^{\exists Y})^\circ$ are open so is their intersection. Assume now that for some ordinal $\alpha \leq \omega$ we have $x_i \in R^{\forall U} \cap (R^{\exists Y})^\circ$ for $i \leq \alpha$ satisfying $i < j \leq \alpha \implies d(x_i, x_j) > \varepsilon$. Then we can find $y_i \in R_{x_i} \subseteq U$ for all $i \leq \alpha$ which necessarily satisfy $i < j \leq \alpha \implies d(y_i, y_j) > \delta$. $\qquad\square$

This remains true if we replace δ-finite with diameter $\leq \delta$ (i.e., δ-1-finite). However, it will not be of much use since the family of open sets of diameter $\leq \delta$ is not closed under finite unions.

Lemma 3.24 *Under the assumptions above, let $* \in \{f, b\}$, and set $X^{(\alpha)} = X^{(\alpha)}_{*,\varepsilon}$, $Y^{(\alpha)} = Y^{(\alpha)}_{*,\delta}$. Then for every ordinal α: $(R^{\exists Y})^\circ \cap X^{(\alpha)} \subseteq R^{\exists Y^{(\alpha)}}$.*

Proof Indeed for $\alpha = 0$ this holds by assumption, and for α limit a compactness argument shows that $R^{\exists \bigcap_{\beta<\alpha} Y^{(\beta)}} = R^{\exists \bigcap_{\beta<\alpha} Y^{(\beta)}} = \bigcap_{\beta<\alpha} R^{\exists Y^{(\beta)}}$.

Assume now that $(R^{\exists Y})^\circ \cap X^{(\alpha)} \subseteq R^{\exists Y^{(\alpha)}}$, and we shall prove this for $\alpha + 1$. Let $x \in ((R^{\exists Y})^\circ \cap X^{(\alpha)}) \smallsetminus R^{\exists Y^{(\alpha+1)}}$: we need to show that $x \notin X^{(\alpha+1)}$. Let $K = R_x \cap Y^{(\alpha)} = (R^{(\alpha)})_x$, where $R^{(\alpha)} = R \cap (X^{(\alpha)} \times Y^{(\alpha)})$.

Then K is compact, and by assumption on x: $K \neq \varnothing$, $K \cap Y^{(\alpha+1)} = \varnothing$. The latter means that K admits a covering $K \subseteq \bigcup_{i<\lambda} U_i$ where each $U_i \subseteq Y^{(\alpha)}$ is relatively open and δ-finite in case $* = f$ (δ-bounded in case $* = b$). By compactness of K we may take this union to be finite. Since a finite union of δ-finite (δ-bounded) sets is such, we find that $K \subseteq U$ where $U \subseteq Y^{(\alpha)}$ is open and δ-finite (δ-bounded). Let $U' = (R^{(\alpha)})^{\forall U} \cap (R^{\exists Y})^\circ \subseteq X^{(\alpha)}$. Then $x \in U'$, and by Lemma 3.23 applied to $R^{(\alpha)}$, U' is open in $X^{(\alpha)}$ and ε-finite (ε-bounded). Thus $U' \cap X^{(\alpha+1)} = \varnothing$ and $x \notin X^{(\alpha+1)}$, as desired. $\qquad\square$

Theorem 3.25 *Let X, Y be two locally compact topometric spaces. Let $R \subseteq X \times Y$ be a closed relation such that R_x is compact for all $x \in X$. Let $\varepsilon, \delta > 0$ be such that for all $(x, y), (x', y') \in R$: if $d_Y(y, y') \leq \delta$ then $d_X(x, x') \leq \varepsilon$. Let $K \subseteq X$ be compact and $F \subseteq Y$ any set such that $K \subseteq (R^{\exists Y})^\circ \cap R^{\forall F}$. Then $\mathrm{CB}^X_{*,\varepsilon}(K) \leq \mathrm{CB}^Y_{*,\delta}(F)$ for $* \in \{f, b\}$.*

Proof Assume that $\mathrm{CB}^X_{*,\varepsilon}(K) \geq \alpha$, i.e., there exists $x \in K \cap X^{(\alpha)}_{*,\varepsilon} \subseteq (R^{\exists Y})^\circ \cap X^{(\alpha)}_{*,\varepsilon} \subseteq R^{\exists Y^{(\alpha)}_{*,\delta}}$, so there is $y \in R_x \cap Y^{(\alpha)}_{*,\delta}$. Since $K \subseteq R^{\forall F}$ we have $y \in R_x \subseteq F$. Thus $F \cap Y^{(\alpha)}_{*,\delta} \neq \varnothing$ and $\mathrm{CB}^Y_{*,\delta}(F) \geq \alpha$ as well. $\square$

Corollary 3.26 *Lemma 3.13 can be obtained as a special case of Theorem 3.25. In fact it is enough to assume that X and Y are locally compact and $\pi \colon X \to Y$ is surjective with compact fibres.*

Proof We follow the notations and assumptions of Lemma 3.13. Let $R \subseteq Y \times X$ be the transposed graph of π, i.e., $R = \{(\pi(x), x) \colon x \in X\}$. Then $R_y = \pi^{-1}(y)$ is compact for all $y \in Y$ by assumption. As π is surjective, $R^{\exists X} = Y$. Let $\varepsilon > 0$ and $y \in Y$, and set $\delta = \Delta(\varepsilon)$, $K = \pi^{-1}(y) = R_y \subseteq X$. Then $y \in (R^{\exists X})^\circ \cap R^{\forall K}$, and by Theorem 3.25: $\mathrm{CB}^Y_{f,\varepsilon}(y) \leq \mathrm{CB}^X_{f,\delta}(\pi^{-1}(y))$. $\square$

The same argument shows that $\mathrm{CB}^Y_{b,\varepsilon}(y) \leq \mathrm{CB}^X_{b,\delta}(\pi^{-1}(y))$ as well, improving Lemma 3.13.

In particular, if $\pi \colon X \to Y$ is precise, then $\mathrm{CB}_{f,\varepsilon}$ ranks go down, in the sense that $\mathrm{CB}^Y_{f,\varepsilon}(K) \leq \mathrm{CB}^X_{f,\varepsilon}(\pi^{-1}(K))$ for $K \subseteq Y$. We now seek sufficient conditions for equality.

Definition 3.27 Let $\pi \colon X \to Y$ be a precise surjective mapping of compact spaces. We say that X is *homogeneous over Y* (or more precisely, *over π*) if for every $K \subseteq U \subseteq Y$, where K is compact and U open, and every countable set $X_0 \subseteq \pi^{-1}(K)$, there is an isometric automorphism f of X such that $\pi \circ f \restriction_{X_0}$ is isometric with image in U.

(All the results below hold if we replace "compact" with "locally compact" and require in addition that if $K \subseteq Y$ is compact then so is $\pi^{-1}(K)$.)

Proposition 3.28 *Let $M \preceq N$ be two structures, M approximately $\aleph_0$-saturated and N strongly $\aleph_1$-homogeneous. Then $\mathrm{S}_n(N)$ is homogeneous over $\mathrm{S}_n(M)$ and $\mathrm{S}_\varphi(N)$ is homogeneous over $\mathrm{S}_\varphi(M)$, each with the respective standard metric.*

Proof Let $K \subseteq U \subseteq \mathrm{S}_n(M)$ be closed and open, respectively, and let $X_0 \subseteq \pi^{-1}(K) \subseteq \mathrm{S}_n(N)$ be countable, say $X_0 = \{p_i \colon i < \omega\}$. Then one can find a formula $\varphi(\bar{x}, \bar{b})$ with parameters $\bar{b} \in M$ such that $K \subseteq [\varphi(\bar{x}, \bar{b}) = 0] \subseteq [\varphi(\bar{x}, \bar{b}) \leq 1/2] \subseteq U$. Then there is some $\varepsilon > 0$ such that for all $\bar{b}' \in M$, if $d(\bar{b}, \bar{b}') < \varepsilon$ then $[\varphi(\bar{x}, \bar{b}') = 0] \subseteq U$ as well.

For each $i < j < \omega$ one can find a countable set $A_{ij} \subseteq N$ such that $d(p_i \restriction_{A_{ij}}, p_j \restriction_{A_{ij}}) = d(p_i, p_j)$. Then $A = \bigcup_{i < j < \omega} A_{ij}$ is countable. By approximate $\aleph_0$-saturation of M (see [6, Fact 1.4]) we can find $\bar{b}'A' \subseteq M$ such that $A\bar{b} \equiv A'\bar{b}'$ and

$d(\bar{b}, \bar{b}') < \varepsilon$. By homogeneity of N there exists $f \in \mathrm{Aut}(N)$ sending $A\bar{b} \mapsto A'\bar{b}'$. Then f induces an isometric automorphism $\tilde{f}$ of $\mathrm{S}_n(N)$, $\pi \circ \tilde{f}(X_0) \subseteq [\varphi(\bar{x}, \bar{b}') = 0] \subseteq U$, and since $A' \subseteq M$ we get that $\pi \circ \tilde{f}$ is isometric on X_0.

The proof for $\pi \colon \mathrm{S}_\varphi(N) \to \mathrm{S}_\varphi(M)$ is essentially the same. □

Lemma 3.29 *Let $\pi \colon X \to Y$ be precise, and assume X is homogeneous over Y. Let $x \in X$ and $y = \pi(x) \in Y$. Then $\mathrm{CB}^Y_{*,\varepsilon}(y) \geq \mathrm{CB}^X_{*,\varepsilon}(x)$ for all $\varepsilon > 0$, $* \in \{d, f, b\}$.*

Proof We only prove the case $* = f$, the others being similar. It is enough to show by induction on α that $\pi(X^{(\alpha)}_{f,\varepsilon}) \subseteq Y^{(\alpha)}_{f,\varepsilon}$.

For $\alpha = 0$ or limit this is clear. For $\alpha + 1$, assume that $U \subseteq Y$ is open such that $U \cap Y^{(\alpha)}_{f,\varepsilon}$ is ε-finite, say ε-n-finite. Let $y \in U \cap Y^{(\alpha)}_{f,\varepsilon}$. Then it will suffice to find $\pi^{-1}(y) \subseteq V$ open such that $V \cap X^{(\alpha)}_{f,\varepsilon}$ is ε-n-finite as well.

Indeed, find $U' \subseteq Y$ open such that $y \in U' \subseteq \bar{U}' \subseteq U$, and let $V = \pi^{-1}(U')$. We claim that $V \cap X^{(\alpha)}_{f,\varepsilon}$ is ε-n-finite. If not, then there are $x_0, \ldots, x_n \in V \cap X^{(\alpha)}_{f,\varepsilon}$ such that $i < j \leq n \implies d(x_i, x_j) > \varepsilon$. By the homogeneity assumption there exists a precise automorphism f of X such that $\pi \circ f \restriction_{\{x_0, \ldots, x_n\}}$ is isometric with image in U. Since f is a precise automorphism it leaves $X^{(\alpha)}_{f,\varepsilon}$ invariant, so $f(x_0), \ldots, f(x_n) \in X^{(\alpha)}_{f,\varepsilon}$ as well. By the induction hypothesis $\pi \circ f(x_0), \ldots, \pi \circ f(x_n) \in Y^{(\alpha)}_{f,\varepsilon}$, contradicting the assumption that $U \cap Y^{(\alpha)}_{f,\varepsilon}$ is ε-n-finite. □

Theorem 3.30 *Let $\pi \colon X \to Y$ be precise, and assume X is homogeneous over Y. Then for all $K \subseteq Y$, $* \in \{f, b\}$ and $\varepsilon > 0$: $\mathrm{CB}^Y_{*,\varepsilon}(K) = \mathrm{CB}^X_{*,\varepsilon}(\pi^{-1}(K))$.*

Moreover, if this common rank is ordinal then: $\mathrm{CBd}^Y_{*,\varepsilon}(K) = \mathrm{CBd}^X_{*,\varepsilon}(\pi^{-1}(K))$.

Proof Only the moreover part is left to be proved. Indeed, assume first that $\mathrm{CB}^Y_{f,\varepsilon}(K) = \mathrm{CB}^X_{f,\varepsilon}(\pi^{-1}(K)) = \alpha < \infty$. Let $X^{(\alpha)} = X^{(\alpha)}_{f,\varepsilon}$ and $Y^{(\alpha)} = Y^{(\alpha)}_{f,\varepsilon}$. By the first part $\pi(X^{(\alpha)}) = Y^{(\alpha)}$.

Let $n = \mathrm{CBd}^Y_{f,\varepsilon}(K)$. Then there is an open set $K \subseteq U \subseteq Y$ such that $U \cap Y^{(\alpha)}$ is ε-n-finite. By the same argument as in the proof of Lemma 3.29 we find $V \subseteq X$ open such that $\pi^{-1}(K) \subseteq V$ and $V \cap X^{(\alpha)}$ is ε-n-finite so $n \geq \mathrm{CBd}^X_{f,\varepsilon}(\pi^{-1}(K))$. On the other hand, let $m = \mathrm{CBd}^X_{f,\varepsilon}(\pi^{-1}(K))$, so there is an open set $\pi^{-1}(K) \subseteq V \subseteq X$ such that $V \cap X^{(\alpha)}$ is ε-m-finite. Let $U = Y \setminus \pi(X \setminus V)$. Then U is open, $K \subseteq U$, and $\pi^{-1}(U) \subseteq V$. It follows that $\pi^{-1}(U \cap Y^{(\alpha)}) \subseteq V \cap X^{(\alpha)}$, and since π is precise $U \cap Y^{(\alpha)}$ is ε-m-finite, so $m \geq \mathrm{CBd}^Y_{f,\varepsilon}(K)$.

The case $* = b$ is similar. □

Together with Proposition 3.28 this means we can define the ε-*Morley rank* of a type-definable set X as $\mathrm{RM}_\varepsilon(X) = \mathrm{CB}^{\mathrm{S}_n(M)}_{f,\varepsilon}([X])$ where M is any approximately $\aleph_0$-saturated model containing the parameters for X. Indeed, if $\bar{M}$ is the monster model then $\mathrm{S}_n(\bar{M})$ is homogeneous over $\mathrm{S}_n(M)$ whereby $\mathrm{CB}^{\mathrm{S}_n(M)}_{f,\varepsilon}([X]^{\mathrm{S}_n(M)}) = \mathrm{CB}^{\mathrm{S}_n(\bar{M})}_{f,\varepsilon}([X]^{\mathrm{S}_n(\bar{M})})$. Similarly we define the ε-*Morley degree* of X as $\mathrm{dM}_\varepsilon(X) = \mathrm{CBd}^{\mathrm{S}_n(M)}_{f,\varepsilon}([X])$.

The same is true for $\mathrm{CB}_{b,\varepsilon}$ ranks and degrees (which coincide with the Morley ranks and degrees defined in [3]). However, $\mathrm{CB}_{f,\varepsilon}$ ranks seem to have the advantage of a more natural notion of degree.

3.4 Measures

We conclude this section with a few results concerning Borel probability measures on CB-analysable spaces. Some of these results come from joint work with Anand Pillay, whom we wish to thank for allowing their inclusion here.

Recall that a measure on a topological space X is *regular* if for every measurable S:

$$\mu(S) = \sup\{\mu(K)\colon S \supseteq K \text{ compact}\} = \inf\{\mu(U)\colon S \subseteq U \text{ open}\}.$$

Regular Borel measures on X are in bijection with positive integration functionals on $C(X, \mathbb{R})$ (or $C(X, [0, 1])$). Following our convention concerning terminology, the *Borel σ-algebra* of a topometric space is the σ-algebra generated by the topology.

Theorem 3.31 *Let X be a locally compact, CB-analysable topometric space, μ a regular Borel probability measure on X. Then for every $\varepsilon > 0$, $1 - \varepsilon$ of the mass of μ is supported by a metrically compact set.*

Proof Fix $r, \delta > 0$, and let $U_\alpha = X \smallsetminus X_{f,r}^{(\alpha)}$. Thus $U_{\alpha+1} \cap X_{f,r}^{(\alpha)}$ is a union of open r-finite subsets of $X_{f,r}^{(\alpha)}$.

Assume first that $\mu(U_1) > \delta$. By regularity there is a compact subset $F \subseteq U_1$ such that $\mu(F) > \mu(U_1) - \delta$. Since $F \subseteq U_1$ is compact it is covered by finitely many r-finite sets, and is therefore r-finite. It follows that for arbitrary α, if $\mu(U_{\alpha+1} \smallsetminus U_\alpha) > \delta$ then there is a compact set $F \subseteq U_{\alpha+1} \smallsetminus U_\alpha$ such that $\mu(F) > \mu(U_{\alpha+1} \smallsetminus U_\alpha) - \delta$.

We now claim that $\mu(U_\alpha) = \sum_{\beta < \alpha} \mu(U_{\beta+1} \smallsetminus U_\beta)$ for all α. In case α is countable this is just by σ-additivity, but for the general case a small inductive argument is required. $\alpha = 0$ (and indeed, α countable) is immediate, as is the successor case. For α limit, we have $U_\alpha = \bigcup_{\beta < \alpha} U_\beta$. Then every compact set $K \subseteq U_\alpha$ is contained is some U_β, so by regularity: $\mu(U_\alpha) = \sup_{\beta < \alpha} \mu(U_\beta)$.

By CB-analysability we have $\mu(X) = \sum_\alpha \mu(U_{\alpha+1} \smallsetminus U_\alpha)$. Only countably many summands can be non-zero, and we may enumerate their indexes by $\{\alpha_n : n < \omega\}$ (not necessarily in order). For each n find $F_n \subseteq U_{\alpha_n+1} \smallsetminus U_{\alpha_n}$ compact, r-finite and satisfying $\mu(F_n) > \mu(U_{\alpha_n+1} \smallsetminus U_{\alpha_n}) - 2^{-n-1}\delta$. Then for some m: $\mu(\bigcup_{n<m} F_n) > \mu(X) - \delta$, and $K = \bigcup_{n<m} F_n$ is r-finite and compact, and therefore complete.

Letting r and δ vary, we see that for all n we can find $K_n \subseteq X$ compact, 2^{-n}-finite, satisfying $\mu(K_n) > \mu(X) - 2^{-n-1}\varepsilon$. Let $K = \bigcap K_n$. Then K totally bounded and complete, i.e., it is metrically compact, and $\mu(K) > \mu(X) - \varepsilon$. ☐

Corollary 3.32 *Let X be a locally compact, CB-analysable topometric space, μ a regular Borel probability measure on X. Then there is a sequence $\{\mu_n : n < \omega\}$ of probability measures with finite support converging weakly to μ, i.e., $\int f \, d\mu_n \to \int f \, d\mu$ for every continuous function with compact support $f \in C_c(X, \mathbb{C})$. Moreover, if $\mathscr{F} \subseteq C_c(X, \mathbb{C})$ is a family of uniformly bounded, and equally uniformly continuous functions then $\int f \, d\mu_n \to \int f \, d\mu$ uniformly for all $f \in \mathscr{F}$.*

Proof For each $n < \omega$, choose $K_n \subseteq X$ metrically compact such that $\mu(K_n) > 1 - 2^{-n}$. By compactness we can find points $\{x_{n,i} : i < \ell_n\} \subseteq K_n$ such that $K_n \subseteq \bigcup \overline{B}(x_{n,i}, 2^{-n})$. For $i < \ell_n$ let $S_{n,i} = K_n \cap \overline{B}(x_{n,i}, 2^{-n-1}) \smallsetminus \bigcup_{j<i} S_{n,j}$. Then each $S_{n,j}$ is a Borel set, and they form a partition of K_n. Let $\mu_{n,0} = \sum_{i<\ell_n} \mu(S_{n,i})\delta_{x_{n,i}}$, where δ_x denotes the Dirac measure concentrated at x. Then $\mu_n = \mu(K_n)^{-1}\mu_{n,0}$ is a probability measure with finite support.

Now let $f \colon X \to \mathbb{C}$ be a continuous function with compact support. Then f is uniformly continuous and bounded. Let $M = \sup|f|$, and for each $\delta > 0$ let $\Delta^{-1}(\delta) = \sup\{|f(x) - f(y)| \colon d(x, y) < \delta\}$. Then $\Delta^{-1}(\delta) \to 0$ as $\delta \to 0$, so:

$$
\begin{aligned}
\left| \int f \, d\mu - \int f \, d\mu_n \right| &\leq \sum_{i<\ell_n} \left| \int_{S_{n,i}} f \, d\mu - \int_{S_{n,i}} f \, d\mu_{n,0} \right| \\
&\quad + \left| \int_{X \smallsetminus K_n} f \, d\mu \right| + \left| \int f \, d\mu_n - \int f \, d\mu_{n,0} \right| \\
&\leq \sum_{i<\ell_n} \Delta^{-1}(2^{-n})\mu(S_{n,i}) + M2^{-n} + M2^{-n}\left(1 - \frac{1}{\mu(K_n)}\right) \\
&\leq \Delta^{-1}(2^{-n}) + 3 \cdot 2^{-n}M.
\end{aligned}
$$

This goes to 0 as $n \to \infty$. The moreover part is implicit in the proof above. $\square$

Measures on type spaces were originally studied by Keisler [10].

Definition 3.33 Let T be a theory, A a set of parameters. An n-ary *Keisler measure* $\mu(\bar{x})$ over a set A is a Borel probability measure on $S_n(A)$.

Such measures generalise the notion of a type (the Dirac measures being in bijection with types).

Let $\mu(\bar{x})$ be a Keisler measure over A. Then a definable predicate $\varphi(\bar{x})$ with parameters in A is a continuous function $\varphi \colon S_n(A) \to [0, 1]$, and we can calculate its integral $I_\mu\varphi = \int \varphi^p \, d\mu(p)$. Conversely, the integration functional I_μ determines μ.

Let now $\mu(\bar{x})$ be a Keisler measure over a model M. Let $\varphi(\bar{x}, \bar{y})$ be a definable predicate, possibly with some parameters in M, and let us restrict our attention to instances $\varphi(\bar{x}, \bar{b})$ over M. This defines a predicate $\bar{b} \mapsto I_\mu\varphi(\bar{x}, \bar{b})$ on M, which we denote by $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$.

Let μ_φ be the image measure of μ on $S_\varphi(M)$. Then μ_φ is a Borel probability measure as well, so we say it is a *Keisler φ-measure* over M. Conversely, μ_φ determines $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$, so the family of all such μ_φ determines μ (in fact we only need the family of μ_φ where $\varphi(\bar{x}, \bar{y})$ are formulae without hidden parameters).

Corollary 3.34 *Assume $\varphi(\bar{x}, \bar{y})$ is a stable formula, M a model, and μ_φ a φ-Keisler measure over M. Then μ_φ is definable, meaning that the predicate $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$ is equal on M to some definable predicate $\psi(\bar{y})$ with parameters in M (it follows that ψ is unique).*

Moreover, ψ is a $\tilde{\varphi}$-predicate, namely a continuous function on $S_{\tilde{\varphi}}(M)$ where $\tilde{\varphi}(\bar{y}, \bar{x}) = \varphi(\bar{x}, \bar{y})$.

Proof By [7], $(S_\varphi(M), d_\varphi)$ is CB-analysable. By Corollary 3.32 μ is the weak limit of a sequence of finitely supported measures μ_n. The family of all functions of the form $\varphi(\bar{x}, \bar{b})$ is uniformly bounded (by 1) and equally uniformly continuous (all are 1-Lipschitz) on $S_\varphi(M)$. By the moreover part of Corollary 3.32 the predicates $I_{\mu_n(\bar{x})}\varphi(\bar{x}, \bar{y})$ converge uniformly to $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$ on M.

Let us write $\mu_n = \sum_{i < \ell_n} a_{n,i} \delta_{p_{n,i}}$. Each $p_{n,i}$ is definable, and let $\psi_{n,i}(\bar{y})$ denote its definition. Set

$$\psi_n(\bar{y}) = \sum_{i < \ell_n} a_{n,i} \psi_{n,i}(\bar{y}) = I_{\mu_n(\bar{x})}\varphi(\bar{x}, \bar{y}).$$

Then the definable predicates $\psi_n(\bar{y})$ converge uniformly (on a model M and therefore everywhere) and their limit is the desired definable predicate ψ.

As each $\psi_{n,i}$ is a $\tilde{\varphi}$-predicate so is each ψ_n and therefore ψ.   □

Let us now look more closely at Keisler measures over a set A which is not a model. Let $\varphi(\bar{x}, \bar{y})$ be a definable predicate, possibly with some parameters in A. Recall from [7, Section 6] that a definable φ-predicate over A is a definable predicate $\chi(\bar{x}, C)$ which is at the same time over A and (equivalent to) a uniform limit of continuous combinations of instances of φ which need not be over A. We may write it as an infinitary continuous combination $\chi(\bar{x}, C) = \theta \circ (\varphi(\bar{x}, \bar{c}_i))_{i < \omega}$. It may be more convenient to write $\chi(\bar{x}, Y) = \theta \circ (\varphi(\bar{x}, \bar{y}_i))_{i < \omega}$ and then replace it with $\chi(\bar{x}, z)$ where z is in the sort of canonical parameters for $\theta \circ (\varphi(\bar{x}, \bar{y}_i))_{i < \omega}$. Call such a $\chi(\bar{x}, z)$ a *φ-scheme*. Thus a φ-predicate over A can always be written as $\chi(\bar{x}, c)$ where $\chi(\bar{x}, z)$ is a φ-scheme and $c \in \mathrm{dcl}(A)$, and conversely every such $\chi(\bar{x}, c)$ is a φ-predicate over A.

We define $S_\varphi(A)$ as the quotient of $S_n(M)$ through which all the A-definable φ-predicates factor. It is thus a quotient both of $S_\varphi(M)$ and of $S_n(A)$. This construction does not depend on M and yields a compact Hausdorff space where the continuous mappings $S_\varphi(A) \to [0, 1]$ are precisely the A-definable φ-predicates. Applying Lemma 1.20 to the quotient mapping $S_\varphi(M) \to S_\varphi(A)$ we may equip $S_\varphi(A)$ with a topometric structure. If φ is stable then $S_\varphi(M)$ is CB-analysable and by Lemma 3.13 so is $S_\varphi(A)$.

A Keisler measure μ over A gives rise to predicates $\bar{b} \mapsto I_\mu \psi(\bar{x}, \bar{b})$ on $\mathrm{dcl}(A)$, which we denote by $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$. Let μ_φ be the image measure of μ on $S_\varphi(A)$. Then μ_φ is a Borel probability measure as well, so we say it is a *φ-Keisler measure* over A. The measure μ_φ determines, and is determined by, the family of predicates $I_{\mu(\bar{x})}\chi(\bar{x}, z)$ defined above where χ varies over all φ-schemes. We say that μ_φ is definable if all these predicates $I_{\mu(\bar{x})}\chi(\bar{x}, z)$ are definable over A. In case A is a model this agrees with our earlier notion of definable φ-measure in which we only considered instances of φ.

Recall that the *Galois group* $\mathrm{Gal}(A)$ of a set A is defined as $\mathrm{Aut}(\mathrm{acl}^{eq}(A)/A)$, namely the group of elementary permutations of $\mathrm{acl}^{eq}(A)$ fixing A point-wise. This is a compact group in the topology of point-wise convergence of its A-invariant action on $\mathrm{acl}^{eq}(A)$.

Lemma 3.35 *Let μ be a Keisler measure over A. Let $G = \mathrm{Gal}(A)$ and let H be its Haar measure. Then μ admits a unique extension to $\mathrm{acl}^{eq}(A)$ which is invariant under the action of G.*

More precisely, if $\varphi(\bar{x}, \bar{b})$ is a definable predicate over $\mathrm{acl}(A)$ then $\bar{x} \mapsto \int_G \varphi(\bar{x}, g\bar{b}) \, dH$ is a φ-predicate over A which we may denote by $\varphi^G(\bar{x}, \bar{b})$. The unique invariant extension is then given by $I_{\tilde{\mu}}\varphi(\bar{x}, \bar{b}) = I_\mu \varphi^G(\bar{x}, \bar{b})$.

Similarly, a Keisler φ-measure μ_φ over A admits a unique A-invariant extension to $\mathrm{acl}^{eq}(A)$ given by $I_{\tilde{\mu}}\chi(\bar{x}, c) = I_\mu \chi^G(\bar{x}, c)$ for every φ-predicate $\chi(\bar{x}, c)$ over $\mathrm{acl}^{eq}(A)$.

Proof If $\bar{b} \in \mathrm{acl}(A)$ then its orbit over A is metrically compact. We can therefore approximate $\int_G \varphi(\bar{x}, g\bar{b}) \, dH$ arbitrarily well by weighted sums $\sum_{i<k} a_i \varphi(\bar{x}, g_i\bar{b})$, so it is indeed a φ-predicate. It is clearly A-invariant and therefore over A. In particular if $\chi(\bar{x}, c)$ is a φ-predicate over $\mathrm{acl}(A)$ then $\chi^G(\bar{x}, c)$ is a χ-predicate, and therefore a φ-predicate, over A. Thus our definition of $\tilde{\mu}$ makes sense and it is not difficult to see that it does indeed define a regular Borel probability measure on $S_n(\mathrm{acl}^{eq}(A))$ (or on $S_\varphi(\mathrm{acl}^{eq}(A))$).

For uniqueness, assume that $\tilde{\mu}$ extends μ to $\mathrm{acl}^{eq}(A)$ and is A-invariant. Let $\varphi(\bar{x}, \bar{b})$ be over $\mathrm{acl}^{eq}(A)$ and let $\sum_{i<k} a_i \varphi(\bar{x}, g_i\bar{b})$ be an ε-approximation of $\varphi^G(\bar{x}, \bar{b})$. Then

$$I_{\tilde{\mu}}\varphi(\bar{x}, \bar{b}) = \sum_{i<k} I_{\tilde{\mu}}\varphi(\bar{x}, g_i\bar{b}) = I_{\tilde{\mu}} \sum_{i<k} \varphi(\bar{x}, g_i\bar{b})$$
$$\approx_\varepsilon I_{\tilde{\mu}}\varphi^G(\bar{x}, \bar{b}) = I_\mu \varphi^G(\bar{x}, \bar{b}).$$

Therefore $I_{\tilde{\mu}}\varphi(\bar{x}, \bar{b}) = I_\mu \varphi^G(\bar{x}, \bar{b})$, as desired. □

Corollary 3.36 *Let A be an algebraically closed set, M a strongly $|A|^+$-homogeneous and $|A|^+$-saturated model containing A. Let φ be a stable formula and let $X \subseteq S_\varphi(M)$ the collection of types which do not fork over A. Then every Keisler φ-measure μ_φ over M which is A-invariant is supported on X.*

Proof Assume not. Then by Theorem 3.31 there is a metrically compact set $Y \subseteq S_\varphi(M)$, $Y \cap X = \varnothing$, such that $\mu_\varphi(Y) > 0$. For each $p \in Y$ there exists $\varepsilon_p > 0$ and an infinite family $\{q_{p,i}\}$ of A-conjugates of p such that $d(q_{p,i}, q_{p,j}) \geq 2\varepsilon_p$ for all $i \neq j$. We may assume that ε_p is maximal for which such a family exists. Notice that $\varepsilon_p \geq \varepsilon_{p'} - d(p, p')$ for all $p, p' \in Y$. Since Y is metrically compact it admits a countable dense subset $\{p_i\}_{i<\omega}$. For $p \in Y$ there exists some $p_i \in B(p, \varepsilon_p/2)$. Thus $\varepsilon_{p_i} > \varepsilon_p/2$ and $p \in B(p_i, \varepsilon_{p_i})$. In other words, $Y = \bigcup_{i<\omega} B(p_i, \varepsilon_{p_i})$. Since $\mu(Y) > 0$ there is p such that $\mu_\varphi(B(p, \varepsilon_p)) = r > 0$. Then $\mu_\varphi(B(q_{p,i}, \varepsilon_p)) = r$ for each A-conjugate $q_{p,i}$ of p in the family we chose earlier. Since the sets $B(q_{p,i}, \varepsilon_p)$ are disjoint μ has infinite total measure, a contradiction. This contradiction concludes the proof. □

We can now improve Corollary 3.34.

Corollary 3.37 (*Compare with Keisler [9, Theorem 2.1].*)
Assume $\varphi(\bar{x}, \bar{y})$ is a stable formula, A a set, and μ_φ a φ-Keisler measure over A. Then

μ_φ is definable. Moreover, it admits a unique definition scheme over A which defines a Keisler φ-measure over any set containing A, and this definition scheme necessarily consists of $\tilde{\varphi}$-predicates.

Proof By Lemma 3.35 we may assume that $A = \mathrm{acl}^{eq}(A)$ is algebraically closed. Fix a very saturated and homogeneous model $M \supseteq A$ and let $X \subseteq \mathrm{S}_\varphi(M)$ be the collection of φ-types which do not fork over A, i.e., which are definable over A. The restriction mapping $\pi \colon X \to \mathrm{S}_\varphi(A)$ is a homeomorphism, so we may use it to pull μ_φ to a regular Borel probability measure $\tilde{\mu}_\varphi$ on $\mathrm{S}_\varphi(M)$ which is supported on X. By Corollary 3.34 $\tilde{\mu}_\varphi$ is definable. Since automorphisms which fix A necessarily fix every point in X they fix $\tilde{\mu}_\varphi$ and therefore fix its definition. Therefore $\tilde{\mu}_\varphi$ is definable over A.

For uniqueness, assume that $\tilde{\mu}'_\varphi$ is another A-invariant extension of μ_φ. By Corollary 3.36 it must be supported on X. Now fix $\bar{b} \in M$ and let $q(\bar{y}) \in \mathrm{S}_{\tilde{\varphi}}(M)$ be the unique non-forking extension of $\mathrm{tp}_{\tilde{\varphi}}(\bar{b}/A)$ to M, $\psi(\bar{x})$ is definition. Then ψ is a φ-predicate over A. By forking symmetry, if $p \in X$ then $\varphi(\bar{x}, \bar{b})^p = \psi(\bar{x})^p$. Therefore

$$
\begin{aligned}
I_{\tilde{\mu}'}\varphi(x, \bar{b}) &= \int \varphi(x, \bar{b})^p d\tilde{\mu}'_\varphi = \int_X \varphi(x, \bar{b})^p d\tilde{\mu}'_\varphi \\
&= \int_X \psi(x)^p d\tilde{\mu}'_\varphi = \int \psi(x)^p d\tilde{\mu}'_\varphi \\
&= I_{\tilde{\mu}'}\psi(x) = I_\mu \psi(x) = \ldots = I_{\tilde{\mu}}\varphi(x, \bar{b}).
\end{aligned}
$$

$\square$

Given the uniqueness of the good definitions we may always regard $I_\mu \varphi(\bar{x}, \bar{y})$ as $\tilde{\varphi}$-predicate over the parameter set of μ (or of μ_φ) without ambiguity.

Corollary 3.38 (*Compare with Keisler [10, Corollary 6.16].*)
Let A be a set and let $\varphi(\bar{x}, \bar{y})$ be any stable formula (or even definable predicate, possibly with hidden parameters in A). Let $\mu(\bar{x})$ and $\nu(\bar{y})$ be two Keisler measures over A. Then Fubini's Theorem holds for μ and ν:

$$
I_{\mu(\bar{x})} I_{\nu(\bar{y})} \varphi(\bar{x}, \bar{y}) = I_{\nu(\bar{y})} I_{\mu(\bar{x})} \varphi(\bar{x}, \bar{y}).
$$

Proof We know by Corollary 3.37 that $I_{\nu(\bar{y})}\varphi(\bar{x}, \bar{y})$ is a φ-predicate and $I_{\mu(\bar{x})}\varphi(\bar{x}, \bar{y})$ is a $\tilde{\varphi}$-predicate, so in fact the statement is only concerned with μ_φ and $\nu_{\tilde{\varphi}}$. In case these are Dirac measures, i.e., complete types, this is just standard forking symmetry:

$$
\begin{aligned}
I_{p(\bar{x})} I_{q(\bar{y})} \varphi(\bar{x}, \bar{y}) &= I_{p(\bar{x})} d_q \tilde{\varphi}(\bar{x}) = d_q \tilde{\varphi}(\bar{x})^p \\
&= d_p \varphi(\bar{y})^q = I_{q(\bar{y})} d_p \varphi(\bar{y}) = I_{q(\bar{y})} I_{p(\bar{x})} \varphi(\bar{x}, \bar{y}).
\end{aligned}
$$

In the general case proceed use Corollary 3.32 to approximate μ_φ and $\nu_{\tilde{\varphi}}$ by finite sums of complete types with weights, and apply the first case. $\square$

Note that if T is stable we can then define a Keisler measure $(\mu \times \nu)(\bar{x}, \bar{y})$ over A by: $I_{\mu \times \nu}\varphi(\bar{x}, \bar{y}) = I_{\mu(\bar{x})} I_{\nu(\bar{y})} \varphi(\bar{x}, \bar{y})$. This is the free product of μ and ν, generalising free product of types.

4 Perturbation metrics

In this section we apply earlier results to questions around perturbations of continuous structures, originally studied in [5]. We shall therefore leave the abstract setting and deal exclusively with topometric structures on type spaces $S_n(T)$ (or $S_n(A)$) in the context of continuous logic.

4.1 Definitions and characterisations

We recall from [5] that a *perturbation system* $\mathfrak{p}$ for a theory T can be given by a system of *perturbation metrics* $d_{\mathfrak{p},n}$ on $S_n(T)$ for each $n < \omega$ such that $n \mapsto (S_n(T), d_{\mathfrak{p}})$ is a precise topometric functor, namely:

(i)   Each $(S_n(T), d_{\mathfrak{p}})$ is a topometric space.
(ii) For every $n, m < \omega$ and mapping $\sigma: n \to m$, the corresponding mapping $\sigma^*: S_m(T) \to S_n(T)$ is a precise morphism of topometric spaces.

We shall follow the notation from [5] and denote $\overline{B}_{d_{\mathfrak{p}}}(X, r)$ by $X^{\mathfrak{p}(r)}$.

Definition 4.1 Let $\mathfrak{p}$ be a perturbation system for T, $M, N \vDash T$, and $r \geq 0$.

(i)   A *partial $\mathfrak{p}(r)$-perturbation* from M to N is a partial mapping $f: M \dashrightarrow N$ such that for all $\bar{a} \in \mathrm{dom}(f)$: $d_{\mathfrak{p}}(\mathrm{tp}_M(\bar{a}), \mathrm{tp}_N(f(\bar{a}))) \leq r$.
(ii) If f above is bijective then it is a *$\mathfrak{p}(r)$-perturbation* of M into N. We denote the set of all such mappings by $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$.

Remark 4.2 What we call a perturbation here was called a *bi-perturbation* in [5], the term perturbation being reserved there for the somewhat weaker notion of a total (but not necessarily surjective) partial perturbation. The distinction is more important when dealing with asymmetric perturbation radii and pre-radii with which much of that paper was concerned and which do not appear here at all. We apologise for the inconvenience.

The preciseness of σ^* has a concrete meaning for various special cases for σ:

- Preciseness of σ^* when $\sigma: 2 \to 1$ is the unique mapping is equivalent to the property that if $p(x, y) \in S_2(T)$ then either $p \in [x = y]$ or $d_{\mathfrak{p}}(p, [x = y]) = \infty$ (since $p \notin [x = y]$ implies that $d(q, (\sigma^*)^{-1}(p)) = d(q, \varnothing) = \infty$ for all $q \in S_1(T)$). By compactness it follows that for every $r > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that $[d(x, y) < \delta]^{\mathfrak{p}(r)} \subseteq [d(x, y) \leq \varepsilon]$, and by symmetry: $[d(x, y) > \varepsilon]^{\mathfrak{p}(r)} \subseteq [d(x, y) \geq \delta]$. In particular every partial $\mathfrak{p}(r)$-perturbation is uniformly continuous and injective, and its inverse is a partial $\mathfrak{p}(r)$-perturbation by symmetry of $d_{\mathfrak{p}}$.
- Preciseness of σ^* when $\sigma: n \to n$ is a permutation just means that a permutation of the variables is an isometry of $(S_n(T), d_{\mathfrak{p}})$, i.e., the notion of perturbation does not depend on the order of an enumeration.
- Preciseness of σ^* when $\sigma: n \to n + 1$ is the inclusion tells us that a partial $\mathfrak{p}(r)$-perturbation can be extended to one more element.

Along with a standard back-and-forth argument this yields:

Fact 4.3 *Let $\mathfrak{p}$ be a perturbation system for T, $r > 0$. Let $M, N \vDash T$, $\bar{a} \in M^n$, $\bar{b} \in N^n$. Then the following are equivalent*:

  (i) $d_{\mathfrak{p}}(\mathrm{tp}_M(\bar{a}), \mathrm{tp}_N(\bar{b})) \leq r$.
  (ii) *There are $M' \succeq M$, $N' \succeq N$ and $f \in \mathrm{Pert}_{\mathfrak{p}(r)}(M', N')$ such that $f(\bar{a}) = \bar{b}$.*

This means that the perturbation system is determined by the mapping $(M, N, r) \mapsto \mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$, and it will be useful to give a general characterisation of mappings of this form.

Theorem 4.4 *Let $\mathfrak{p}$ be a perturbation system for T. Then for each $r \in \mathbb{R}^+$ and $M, N \in \mathrm{Mod}(T)$, $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$ is a set of bijections of M with N satisfying the following properties*:

  (i)   *Monotonicity*: $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N) = \bigcap_{s > r} \mathrm{Pert}_{\mathfrak{p}(s)}(M, N)$.
  (ii)   *Strict reflexivity*: $\mathrm{Pert}_{\mathfrak{p}(0)}(M, N)$ *is the set of isomorphisms of M with N.*
  (iii)   *Symmetry*: $f \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$ *if and only* $f^{-1} \in \mathrm{Pert}_{\mathfrak{p}(r)}(N, M)$.
  (iv)   *Transitivity*: *if $f \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$ and $g \in \mathrm{Pert}_{\mathfrak{p}(s)}(N, L)$ then $g \circ f \in \mathrm{Pert}_{\mathfrak{p}(r+s)}(M, L)$.*
  (v)   *Uniform continuity*: *for each $r \in \mathbb{R}^+$, all members of $\mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$, where M, N vary over all models of T, satisfy a common modulus of uniform continuity.*
  (vi)   *Ultraproducts*: *If $f_i \in \mathrm{Pert}_{\mathfrak{p}(r)}(M_i, N_i)$ for $i \in I$, and $\mathscr{U}$ is an ultrafilter on I then $\prod_{\mathscr{U}} f_i \in \mathrm{Pert}_{\mathfrak{p}(r)}(\prod_{\mathscr{U}} M_i, \prod_{\mathscr{U}} N_i)$. (Note that $\prod_{\mathscr{U}} f_i$ exists by the uniform continuity assumption).*
  (vii) *Elementary substructures*: *If $f \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$, $M_0 \preceq M$, and $N_0 = f(M_0) \preceq N$ then $f\!\restriction_{M_0} \in \mathrm{Pert}_{\mathfrak{p}(r)}(M_0, N_0)$.*

Conversely, every mapping associating to every triplet $(r, M, N) \in \mathbb{R}^+ \times \mathrm{Mod}(T)^2$ a set of bijections $\mathrm{Pert}'_r(M, N)$ satisfying the properties above is of the form $(r, M, N) \mapsto \mathrm{Pert}_{\mathfrak{p}(r)}(M, N)$ for a unique perturbation system $\mathfrak{p}$.

Proof The first part is fairly immediate from facts we already know, so we only prove the converse. The uniqueness part follows from Fact 4.3 so we prove existence.

Say that "$d_{\mathfrak{p}}(p, q) \leq r$" (in quotes, since we have not yet given a value to $d_{\mathfrak{p}}(p, q)$) if there are models $M, N \vDash T$ and $f \in \mathrm{Pert}'_r(M, N)$ sending a realisation of p to one of q.

First we claim that "$d_{\mathfrak{p}}(p, q) \leq r$" if and only if "$d_{\mathfrak{p}}(p, q) \leq s$" for all $s > r$. Left to right is immediate from monotonicity. For right to left, let $s_n \searrow r$. For each n let $f_n \in \mathrm{Pert}'_{s_n}(M_n, N_n)$ witness that "$d_{\mathfrak{p}}(p, q) \leq s_n$" sending $\bar{a}_n \in M_n$ to $\bar{b}_n \in N_n$. Let $\mathscr{U}$ be a non-principal ultrafilter on ω, and let $(M, N, f) = \prod_{\mathscr{U}}(M_n, N_n, f_n)$. Then $f \in \mathrm{Pert}'_{s_n}(M, N)$ for all n by the ultraproduct property and $f([a_n]) = [b_n]$, whereby $f \in \mathrm{Pert}'_r(M, N)$ witnesses that "$d(p, q) \leq r$".

We may therefore define $d_{\mathfrak{p}}(p, q) = \inf\{r \colon\text{"}d_{\mathfrak{p}}(p, q) \leq r\text{"}\} \in [0, \infty]$ and drop the quotes. Strict reflexivity tells us that $d_{\mathfrak{p}}(p, q) = 0 \Longleftrightarrow p = q$, and symmetry and transitivity imply symmetry of $d_{\mathfrak{p}}$ and the triangle inequality, so $d_{\mathfrak{p}}$ is a metric. The ultraproduct property implies that $\{(p, q) \in S_n(T)^2 \colon d_{\mathfrak{p}}(p, q) \leq r\}$ is closed, and $(S_n(T), d_{\mathfrak{p}})$ is a topometric space for all n.

For preciseness, let now $\sigma : n \to m$, $p \in S_m(T)$, $q \in S_n(T)$, and we need to show that $d_{\mathfrak{p}}(p, (\sigma^*)^{-1}(q)) = d_{\mathfrak{p}}(\sigma^*(p), q)$. Assume first that $d_{\mathfrak{p}}(p, (\sigma^*)^{-1}(q)) = r < \infty$, so $d_{\mathfrak{p}}(p, q') = r$ for some $q' \in (\sigma^*)^{-1}(q)$. Let $f \in \text{Pert}'_r(M, N)$ witness this, sending $\bar{a} \vDash p$ (of length m) to $f(\bar{a}) \vDash q'$. Let $a'_i = a_{\sigma(i)}$ for $i < n$. Then $\bar{a}' \vDash \sigma^*(p)$ and $f(\bar{a}') \vDash q$, whereby $d_{\mathfrak{p}}(\sigma^*(p), q) \leq r$. Conversely, assume $d_{\mathfrak{p}}(\sigma^*(p), q) = r < \infty$. Let $f \in \text{Pert}'_r(M, N)$ witness this, sending $\bar{a} \vDash \sigma^*(p)$ (of length m) to $f(\bar{a}) \vDash q$. By the ultraproduct property we may replace M with an $\aleph_1$-saturated elementary extension, in which there is a tuple $c_{<m}$ such that $a_i = c_{\sigma(i)}$ for $i < n$. Then $f(\bar{c})$ realises a type in $(\sigma^*)^{-1}(q)$, showing that $d_{\mathfrak{p}}(p, (\sigma^*)^{-1}(q)) \leq r$. Equality follows.

We have shown that $d_{\mathfrak{p}}$ is indeed the perturbation metric associated to a perturbation system $\mathfrak{p}$. The inclusion $\text{Pert}'_r(M, N) \subseteq \text{Pert}_{\mathfrak{p}(r)}(M, N)$ is immediate from the construction. Assume now that $f \in \text{Pert}_{\mathfrak{p}(r)}(M, N)$. Let $A \subseteq M$ be a finite subset and enumerate it as a tuple $\bar{a}$. Let $p = \text{tp}(\bar{a})$, $q = \text{tp}(f(\bar{a}))$, so $d_{\mathfrak{p}}(p, q) \leq r$. This is witnessed by some $f_A \in \text{Pert}'_r(M_A, N_A)$ sending $\bar{a}' \vDash p$ to $f_A(\bar{a}') \vDash q$. In other words, there are partial elementary mappings $\theta_A : M \dashrightarrow M_A$ and $\theta'_A : N \dashrightarrow N_A$, with domains A and $f(A)$, respectively, such that $f_A \circ \theta_A = \theta'_A \circ f$ on A. Set $I = \{A \subseteq M : |A| < \infty\}$ and let $\mathcal{U}$ be an ultrafilter on I containing the set $\{A \in I : a \in A\}$ for each $a \in M$. Let $(\tilde{M}, \tilde{N}, \tilde{f}) = \prod_{\mathcal{U}} (M_A, N_A, f_A)$, so $\tilde{f} \in \text{Pert}'_r(\tilde{M}, \tilde{N})$ by the ultraproduct property. Define $\theta : M \to \tilde{M}$ by $\theta(a) = [a_A]_{A \in I}$ where $a_A = \theta_A(a)$ if $a \in A$, and anything otherwise. Define $\theta' : N \to \tilde{N}$ similarly. Then θ and θ' are (total) elementary embeddings, and $\tilde{f} \circ \theta = \theta' \circ f$. It follows that $f \in \text{Pert}'_r(M, N)$ by the elementary substructures property, as desired. $\square$

Remark 4.5 Let $\mathscr{L}_f^2$ be the language for triplets (M, N, f), consisting of two disjoint copies of $\mathscr{L}$ plus a new function symbol f going from one to the other. Then the ultraproduct and elementary substructures properties are *not* equivalent (modulo previous axioms) to the elementarity of the class

$$\{(M, N, f) : M, N \vDash T, f \in \text{Pert}_{\mathfrak{p}}(M, N)\}.$$

Indeed, The assumption that $(M, N, f) \subseteq (M', N', f')$ and $M \preceq M'$, $N \preceq N'$ does not imply that $(M, N, f) \preceq (M', N', f')$, so elementarity is not strong enough to imply the elementary substructures property.

The correct equivalent assumption is that the class above is elementary and that its theory is "universal over $\mathscr{L}$". In case T eliminates quantifiers (which we may always assume) this just means that it is given by:

$$\{M \vDash T\} \cup \{N \vDash T\} \cup \{\text{some universal axioms (involving both copies and } f)\}.$$

4.2 Extensions of perturbation systems

By definition, a perturbation system $\mathfrak{p}$ for a theory T compares complete types *without* parameters, telling us the by how much one needs to be perturbed in order to obtain the other. What about types *with* parameters? Adding a parameter in a set $A \subseteq M$ to the language consists of two steps: adding new constant symbols to the language for

the members of A, and replacing T with $T(A) = \mathrm{Th}_{\mathscr{L}(A)}(M)$. Given a perturbation system for T in $\mathscr{L}(A)$, the second step merely consists of the restriction to a smaller family of type, so let us concentrate on the first step. More generally, let us explore the extension of perturbation systems to a bigger language $\mathscr{L}' \supseteq \mathscr{L}$. Replacing a function symbol $f(\bar{x})$ with the predicate $G_f(\bar{x}, y) = d(f(\bar{x}), y)$, we may assume only predicate symbols are added.

Let us first consider the case where a single new symbol is added: $\mathscr{L}_P = \mathscr{L} \cup \{P\}$. Let T' denote the set of $\mathscr{L}_P$-consequence of T (i.e., T viewed as an $\mathscr{L}_P$-theory). Then it is natural (to the author, at least, which is what counts at the moment) to extend $\mathfrak{p}$ to a perturbation system $\mathfrak{p}_P$ for T' allowing small perturbations of P. Thus for every $(M, P^M), (N, P^N) \models T'$ we define:

$$\mathrm{Pert}_{\mathfrak{p}_P(r)}\big((M, P^M), (N, P^N)\big) = \left\{\theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N) : \begin{array}{l} \text{for all } \bar{b} \in M: \\ |P^M(\bar{b}) - P^N(\theta(\bar{b}))| \le r \end{array}\right\}.$$

It is fairly straightforward to verify that this definition satisfies the list of properties from Theorem 4.4, and thus indeed defines a perturbation system $\mathfrak{p}_P$.

In case we wish to add several new symbols $\bar{P} = \{P_i : i < k\}$, we merely iterate this construction.

$$\mathrm{Pert}_{\mathfrak{p}_{\bar{P}}(r)}\big((M, P_0^M, \ldots), (N, P_0^N, \ldots)\big)$$
$$= \left\{\theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N) : \begin{array}{l} \text{for all } i < k \text{ and } \bar{b} \in M: \\ |P_i^M(\bar{b}) - P_i^N(\theta(\bar{b}))| \le r \end{array}\right\}.$$

This is particularly elegant as it does not depend on the order in which we add the symbols. However, one could come up with several variants of this definition, such as:

$$\mathrm{Pert}_{\mathfrak{p}'_{\bar{P}}(r)}\big((M, P_0^M, \ldots), (N, P_0^N, \ldots)\big)$$
$$= \left\{\theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N) : \begin{array}{l} \text{for all } i < k \text{ and } \bar{b} \in M: \\ |P_i^M(\bar{b}) - P_i^N(\theta(\bar{b}))| \le 2^{-i}r \end{array}\right\}.$$

Or:

$$\mathrm{Pert}_{\mathfrak{p}''_{\bar{P}}(r)}\big((M, P_0^M, \ldots), (N, P_0^N, \ldots)\big)$$
$$= \left\{\theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N) : \begin{array}{l} \text{for all } i < k \text{ and } \bar{b} \in M: \\ |P_i^M(\bar{b}) - P_i^N(\theta(\bar{b}))| \le 2^i r \end{array}\right\}.$$

As long as we only add finitely many symbols, all three definitions are equivalent, in the sense that the metrics $d_{\mathfrak{p}_{\bar{P}}}$, $d_{\mathfrak{p}'_{\bar{P}}}$ and $d_{\mathfrak{p}''_{\bar{P}}}$ are all uniformly equivalent metrics. Of course, one can come up with many more variants of this kind, but as long as we allow to perturb each of the finitely many new symbols, we are always going to get something equivalent to $\mathfrak{p}_{\bar{P}}$, which, as we said, seems the most elegant of the lot.

Let us now consider the case where countably many new symbols are added. All three constructions suggested above admit an obvious generalisation to $\bar{P} = \{P_i : i < \omega\}$. However, an essential distinction now presents itself between $\mathfrak{p}_{\bar{P}}$, $\mathfrak{p}'_{\bar{P}}$ on the one hand, and $\mathfrak{p}''_{\bar{P}}$ on the other.

Indeed, $\mathfrak{p}''_{\bar{P}}$ is a very relaxed perturbation system, as a positive perturbation distance only takes into account finitely many of $\{P_i : i < \omega\}$. More precisely, the question whether or not $\theta \in \text{Pert}_{\mathfrak{p}''_{\bar{P}}(r)}(M, N)$ depends only on $\{P_i : 0 \leq i < -\log_2 r\}$. This is essentially the only way of getting a non trivial perturbation system for first order logic. For example, let T be a theory in a countable language $\mathscr{L}$. Let $\mathfrak{p}$ be the trivial perturbation system for the language of equality, and let $\mathfrak{p}''_{\mathscr{L}}$ be as above. Then T is $\mathfrak{p}''_{\mathscr{L}}$-$\aleph_0$-categorical if and only if every restriction of T to a finite sub-language is $\aleph_0$-categorical. In the case of an uncountable tuple $\bar{P}$, in order to obtain a "perturbation system" with similar properties we should have to replace metrics with non-metrisable uniform structures. At the moment we do not see the point in doing so, as the usefulness of $\mathfrak{p}''_{\bar{P}}$ for infinite $\bar{P}$ is not at all clear. In particular, type spaces over infinitely many parameters in classical logic do not involve non-trivial perturbation systems, so the "correct" way to extend $\mathfrak{p}$ to types over an infinite set A should go through another construction.

In contrast, $\mathfrak{p}_{\bar{P}}$ and $\mathfrak{p}'_{\bar{P}}$ can be arbitrarily strict when applied judiciously to infinitely many new symbols. Indeed, in the case of $\mathfrak{p}'_{\bar{P}}$, we may enumerate $\bar{P}$ with repetitions, repeating each symbols infinitely many times (or, if this bothers the reader, we could enumerate many copies of each new symbol and later add to T the axioms that all copies of a single symbol coincide). In that case, a $\mathfrak{p}'_{\bar{P}}(r)$-perturbation of $\mathscr{L}_{\bar{P}}$-structures would necessarily fix the interpretation of every new predicate symbol P_i. In case of the apparently more relaxed $\mathfrak{p}_{\bar{P}}$, the same can be achieved by replacing each new symbol P_i with a tree of symbols $\{P_i^\sigma : \sigma \in 2^{<\omega}\}$, viewing $P_i^\varnothing$ as P_i, and adding axioms that $P_i^{\sigma 0} = 2P_i^\sigma \wedge 1$ and $P_i^{\sigma 1} = (2P_i^\sigma - 1) \vee 0$. Then for $r < 1$ a $\mathfrak{p}_{\bar{P}}(r)$-perturbation of models of these axioms would necessarily fix all the new symbols.

The somewhat philosophical discussion in the previous paragraph is meant to convince the reader that when adding infinitely many new symbols to the language, the most reasonable (and canonical) way of extending $\mathfrak{p}$ is by fixing all the new symbols. In that case we might as well define directly the extension $\mathfrak{p}{\upharpoonright}_{\bar{P}}$ ("$\mathfrak{p}$ over $\bar{P}$") by:

$$\text{Pert}_{\mathfrak{p}{\upharpoonright}_{\bar{P}}(r)}\big((M, \bar{P}^M), (N, \bar{P}_0^N)\big) = \Big\{\theta \in \text{Pert}_{\mathfrak{p}(r)}(M, N) \colon P^M = P^N \circ \theta \text{ for all } P \in \bar{P}\Big\}.$$

In particular, when we extend $\mathfrak{p}$ to $S_n(A)$ where A is infinite we shall use $\mathfrak{p}{\upharpoonright}_A$.

Let us now re-examine the distance between two types $p, q \in S_n(T)$. If $\mathfrak{p}$ is a perturbation system then $d_\mathfrak{p}(p, q)$ measures by how much a realisation of p needs to be perturbed in order to get a realisation of q. But we can also identify p and q with completions of T in the language $\mathscr{L}(\bar{c})$, where $\bar{c}$ is an n-tuple of new constant symbols, which we denote by $p', q' \in S_0^{\mathscr{L}(\bar{c})}(T)$. Let id be the trivial perturbation system for T, and let $\text{id}_{\bar{c}}$ be constructed as above. Then $\text{Pert}_{\text{id}_{\bar{c}}}((M, \bar{a}), (N, \bar{b}))$ consists of all isomorphisms of $\theta \colon M \to N$ such that $d(\bar{b}, \theta(\bar{a})) \leq r$, and $d_{\text{id}_{\bar{c}}}(p', q')$ is simply the standard distance $d(p, q)$, measuring by how much a realisation of p needs to be

moved in order to obtain a realisation of q. Finally, we can combine both constructions defining $\tilde{d}_{\mathfrak{p}}(p, q) = d_{\mathfrak{p}_{\bar{c}}}(p', q')$. We obtain a notion of distance which allows both to perturb the underlying structure and to move the realisations. We could also define it directly (as was done in [5]) as:

$$\tilde{d}_{\mathfrak{p}}(p, q) = \inf \left\{ r \geq 0 : \begin{array}{c} \Big(\exists\, M \vDash p(\bar{a}),\ N \vDash q(\bar{b}),\ \theta \in \mathrm{Pert}_{\mathfrak{p}(r)}(M, N) \Big) \\[2mm] \Big(\forall\, c \in M,\ i < n \Big) \Big(|d^M(c, a_i) - d^N(\theta(c), b_i)| \leq r \Big) \end{array} \right\}.$$

In the terminology of Sect. 2 we can restate [5, Theorem 3.8 and Proposition 3.9] as:

A complete countable theory T is $\mathfrak{p}$-$\aleph_0$-categorical if and only if every finite tuple $\bar{a}$, every type $p \in \mathrm{S}_1(\bar{a})$ is weakly $\tilde{d}_{\mathfrak{p}_{\bar{a}}}$-isolated.

And:

If T is a complete countable theory and every type $p \in \mathrm{S}_n(T)$ is $\tilde{d}_{\mathfrak{p}}$-isolated, then T is $\mathfrak{p}$-$\aleph_0$-categorical.

Remark 4.6 C. Ward Henson pointed out that if (X, d) is a complete topometric space, then the following are equivalent:

(i)   Every point $x \in X$ is weakly d-isolated.
(ii) The set of d-isolated points in d-dense in X.

This is a special case of Lemma 2.3.

4.3 λ-stability up to perturbation

In the course of studying metric structures one encounters many which should, according to all moral standards, be $\aleph_0$-stable (or at least superstable), but are not. Examples for this are probability spaces with a generic automorphism [1] or Nakano spaces [4]. Reassuringly enough, both turn out to be $\aleph_0$-stable up to a natural perturbation system. Our earlier work allows us to conclude almost immediately that this notion of $\aleph_0$-stability, and more generally, of λ-stability, satisfies some expected properties. In particular, $\aleph_0$-stability coincides with the existence of appropriate Morley ranks. As pointed out in the introduction, definitions and results of Iovino [8] can be viewed as precursors to some presented here.

Convention 4.7 Henceforth, when $\mathfrak{p}$ is a perturbation system for T and A a set of parameters, we always interpret $d_{\mathfrak{p}}$ on $\mathrm{S}_n(A)$ as $d_{\mathfrak{p}\restriction_A}$, and accordingly, $\tilde{d}_{\mathfrak{p}}$ as $\tilde{d}_{\mathfrak{p}\restriction_A}$.

Definition 4.8 Let T be a theory, $\lambda \geq |\mathscr{L}|$, and $\mathfrak{p}$ a perturbation system for T. We say that T is $\mathfrak{p}$-λ-stable if $\|(\mathrm{S}_n(A), \tilde{d}_{\mathfrak{p}})\| \leq \lambda$ whenever $|A| \leq \lambda$.

First of all it should be pointed out that for any perturbation system $\mathfrak{p}$, $\mathfrak{p}$-λ-stability is weaker than λ-stability, since $\tilde{d}_{\mathfrak{p}}$ is always coarser than the standard metric d (which coincides with $\tilde{d}_{\mathrm{id}}$). We thus need to make sure that $\mathfrak{p}$-λ-stability is still strong enough to imply stability.

Fact 4.9 *Let T be a theory, $\lambda \geq |\mathscr{L}|$, and $\mathfrak{p}$ a perturbation system for T. Then T is $\mathfrak{p}$-λ-stable if and only if for any model $M \vDash T$: $\|M\| = \lambda \Longrightarrow \|(S_n(M), \tilde{d}_{\mathfrak{p}})\| = \lambda$.*

Proof By Löwenheim-Skolem and the fact that if $A \subseteq M$ is dense then $(S_n(A), \tilde{d}_{\mathfrak{p}}) \cong (S_n(M), \tilde{d}_{\mathfrak{p}})$. □

Lemma 4.10 *Let T be a theory, $\mathfrak{p}$ a perturbation system for T, and $M \vDash T$. Let $\varphi(\bar{x}, \bar{y})$ be any formula, $|\bar{x}| = n$. Then $\pi_\varphi \colon (S_n(M), \tilde{d}_{\mathfrak{p}}) \to (S_\varphi(M), d_\varphi)$ is uniformly continuous and thus a morphism of topometric spaces.*

Proof We need to show that for all $\varepsilon > 0$ there is $\delta > 0$ such that if $p, q \in S_n(M)$ and $\tilde{d}_{\mathfrak{p}}(p, q) < \delta$ then for all $\bar{b} \in M$: $|\varphi(\bar{x}, \bar{b})^p - \varphi(\bar{x}, \bar{b})^q| \leq \varepsilon$. Indeed, as $\mathfrak{p}$ is a perturbation system for T, one can find $\delta_1 > 0$ such that whenever $\theta \in \mathrm{Pert}_{\mathfrak{p}(\delta_1)}(M, N)$ then for all $\bar{a}, \bar{b} \in M$: $|\varphi(\bar{a}, \bar{b})^M - \varphi(\theta(\bar{a}), \theta(\bar{b}))^N| \leq \varepsilon/2$. By uniform continuity one can find $\delta_2 > 0$ such that if $\bar{a}, \bar{a}', \bar{b} \in M$ and $d(\bar{a}, \bar{a}') < \delta_2$ then $|\varphi(\bar{a}, \bar{b})^M - \varphi(\bar{a}', \bar{b})^M| \leq \varepsilon/2$. Now $\delta = \min\{\delta_1, \delta_2\}$ will do. □

Proposition 4.11 *Let $\mathfrak{p}$ be any perturbation system for T. Then T is stable if and only if T is $\mathfrak{p}$-λ-stable for some λ, if and only if T is $\mathfrak{p}$-λ-stable for all $\lambda = \lambda^{|\mathscr{L}|}$.*

Proof Assume that T is stable and let $\lambda = \lambda^{|\mathscr{L}|}$. Then $|S_n(M)| = \lambda$ whenever $\|M\| = \lambda$, so T is $\mathfrak{p}$-λ-stable independently of $\mathfrak{p}$. In particular, T is $\mathfrak{p}$-$2^{|\mathscr{L}|}$-stable.

Conversely, assume T is unstable, say due to an unstable formula φ, and let $\lambda \geq |\mathscr{L}|$. Then there exists $M \vDash T$ such that $\|M\| = \lambda$ and $\|S_\varphi(M)\| > \lambda$. Since the projection $(S_n(M), \tilde{d}_{\mathfrak{p}}) \to (S_\varphi(M), d_\varphi)$ is uniformly continuous, it follows that $\|(S_n(M), \tilde{d}_{\mathfrak{p}})\| > \lambda$, and T is not $\mathfrak{p}$-λ-stable. □

Remark 4.12 Recall the various alternatives we considered for the extension of a perturbation system $\mathfrak{p}$ to countably many new symbols (in this case, constant symbols): the strict variants $\mathfrak{p}_A$ and $\mathfrak{p}'_A$ allowed us essentially to fix A (as long as we enumerate it judiciously enough), while the relaxed variant $\mathfrak{p}''_A$ only considers finite parts of A. Had we chosen the latter as a basis for extending perturbation systems to new parameters, the previous Proposition would fail. Indeed, if T is any small theory and A any countable set, then $\|(S_n(A), \tilde{d}_{\mathfrak{p}''_A})\| = \aleph_0$, even though T need not be stable.

Lemma 4.13 *Let $\mathfrak{p}$ be a perturbation system for T, $A \subseteq B \subseteq M \vDash T$. Then the projection map $\pi \colon (S_n(B), \tilde{d}_{\mathfrak{p}}) \to (S_n(A), \tilde{d}_{\mathfrak{p}})$ is precise (and in particular a quotient of topometric spaces).*

Proof Exercise. □

It follows that:

Proposition 4.14 *Let T be a countable theory, $\mathfrak{p}$ a perturbation system for T. Then the collection $\{(S_n(M'), \tilde{d}_{\mathfrak{p}}) \colon M' \preceq M\}$ is a sufficient family of quotients of $(S_n(M), \tilde{d}_{\mathfrak{p}})$.*

Theorem 4.15 *Let T be a countable theory, $\mathfrak{p}$ a perturbation system for T. Then the following are equivalent*:

    (i)   *T is $\mathfrak{p}$-λ-stable for all λ.*

(ii)   *T is $\mathfrak{p}$-$\aleph_0$-stable.*
(iii) *For every separable model M: $\|(S_n(M), \tilde{d}_{\mathfrak{p}})\| = \aleph_0$.*
(iv)   *For every model M, the space $\|(S_n(M), \tilde{d}_{\mathfrak{p}})\|$ is CB-analysable.*

Proof (i) $\Longrightarrow$ (ii). Immediate.
(ii) $\Longrightarrow$ (iii). If M is separable and $A \subseteq M$ is countable and dense, then $S_n(A) = S_n(M)$.
(iii) $\Longrightarrow$ (iv). By Corollary 3.21.
(iv) $\Longrightarrow$ (i). By Löwenheim-Skolem and Proposition 3.19.

$\square$

Corollary 4.16 *Let T be a complete countable theory, $\bar{M}$ a monster model for T. For a non-empty type-definable set $X \subseteq \bar{M}^n$ and $\varepsilon > 0$, define the ε-Morley rank of X up to $\mathfrak{p}$ by: $\mathrm{RM}_{\mathfrak{p},\varepsilon}(X) = \mathrm{CB}_{f,\varepsilon}^{(S_n(\bar{M}), \tilde{d}_{\mathfrak{p}})}([X])$. Then T is $\mathfrak{p}$-$\aleph_0$-stable if and only if $\mathrm{RM}_{\mathfrak{p},\varepsilon}(X)$ is an ordinal for every type-definable set X and every $\varepsilon > 0$.*

In particular:

Corollary 4.17 *A theory T is $\aleph_0$-stable if and only if $\mathrm{RM}_\varepsilon(X)$ is an ordinal for every type-definable set X and every $\varepsilon > 0$.*

References

1. Ben Yaacov I, Berenstein A (2008) On perturbations of hilbert spaces and probability algebras with a generic automorphism (in preparation)
2. Ben Yaacov I, Berenstein A, Henson CW, Usvyatsov A (2008) Model theory for metric structures. In: Model theory with applications to algebra and analysis, vol 2, Chatzidakis Z, Macpherson D, Pillay A, Wilkie A (eds) London Math Society Lecture Note Series, vol 350, pp 315–427
3. Ben Yaacov I (2005) Uncountable dense categoricity in cats. J Symbolic Log 70(3):829–860
4. Ben Yaacov I Modular functionals and perturbations of Nakano spaces (submitted)
5. Ben Yaacov I (2008) On perturbations of continuous structures (submitted)
6. Ben Yaacov I, Usvyatsov A (2007) On d-finiteness in continuous structures. Fundam Math 194:67–88
7. Ben Yaacov I, Usvyatsov A (2008) Continuous first order logic and local stability. Trans Am Math Soc (in press)
8. Iovino J (1999) Stable Banach spaces and Banach space structures, I and II. Models, algebras, and proofs (Bogotá, 1995). Lecture Notes in Pure and Applied Mathematics, vol 203. Dekker, New York, pp 77–117
9. Keisler HJ (1987) Choosing elements in a saturated model. In: Classification theory (Chicago, IL, 1985). Lecture Notes in Mathematics, vol 1292. Springer, Berlin, pp 165–181
10. Keisler HJ (1987) Measures and forking. Ann Pure Appl Log 34(2):119–169
11. Newelski L (2003) The diameter of a Lascar strong type. Fundam Math 176(2):157–170