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Inverse problem for cuts

Renling Jin

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Abstract Let U be an initial segment of ${}^*\mathbb{N}$ closed under addition (such U is called a cut) with uncountable cofinality and A be a subset of U, which is the intersection of U and an internal subset of ${}^*\mathbb{N}$. Suppose A has lower U-density α strictly between 0 and $\frac{3}{5}$. We show that either there exists a standard real $\epsilon > 0$ and there are sufficiently large x in A such that $|(A+A)\cap[0,2x]| > \left(\frac{10}{3}+\epsilon\right)|A\cap[0,x]|$ or A is a large subset of an arithmetic progression of difference greater than 1 or A is a large subset of the union of two arithmetic progressions with the same difference greater than 2 or A is a large subset of the union of three arithmetic progressions with the same difference greater than 4.

Keywords Inverse problem · Nonstandard analysis · Cut · Additive number theory

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1 Introduction

The main purpose of this article is to improve an important lemma [12, Lemma 2.12] in nonstandard analysis so that it becomes powerful enough to be used in the forthcoming efforts on settling Conjecture 1.1. The applications of nonstandard methods to problems on additive/combinatorial number theory have been fruitful. See, for example, [15–18] and [1,9–14].

R. Jin (✉)
Department of Mathematics, College of Charleston, Charleston, SC 29424, USA
e-mail: jinr@cofc.edu

Let A and B be two subsets of an abelian semigroup. We denote $A \pm B$ for the set $\{a \pm b : a \in A \text{ and } b \in B\}$. If a is an element, we write $A \pm a$ for the set $A \pm \{a\}$. For a positive integer g we denote gA for the set $\{ga : a \in A\}$.[1]

Inverse problems concern the structure of A when the sumset $A + A$ is relatively small. During the late 1950s and early 1960s, G. Freiman obtained a series of results indicating that for a finite set A of integers, if $A + A$ is "small", then A must have some arithmetic structure (cf. [3,22]). He characterized, with precision, the structure of A when $|A+A| \leqslant 3|A|-2$. For $|A+A| > 3|A|-2$, the following Conjecture 1.1 was made in [4] and in [2].

A set $X = \{a+id : i = 0, 1, \ldots, k-1\}$ or $X = \{a+id : i \in \mathbb{N}\}$ for some integer a and positive integers d, k is called an *arithmetic progression*, or *a.p.* for abbreviation, of difference d. If X is finite, we call $|X| = k$ the length of X. A set X is called a *bi-arithmetic progression*, or *b.p.* for abbreviation, of difference d if $X = I_0 \cup I_1$ for two *a.p.*'s I_0 and I_1 with a common difference d such that $I_0 + I_0$, $I_0 + I_1$, $I_1 + I_1$ are pairwise disjoint. If a *b.p.* X is finite, we call $|X|$ the length of X. When we say that a set A is a subset of a *b.p.* $X = I_0 \cup I_1$, we always assume $A \cap I_i \neq \emptyset$ for $i = 0, 1$.

Conjecture 1.1 There exists a natural number K such that for any finite set A of integers with $|A| = k > K$ and $|A + A| = 3k - 3 + b$ for $0 \leqslant b < \frac{1}{3}k - 2$, A is either a subset of an *a.p.* of length at most $2k - 1 + 2b$ or a subset of a *b.p.* of length at most $k + b$.

Conjecture 1.1 is false if b is allowed to be $\frac{1}{3}k - 2$. Simply let A be the union of three intervals of the same size $\frac{k}{3}$ so that the middle interval has an equally long distance away from the interval in each side. In the past few decades some efforts have been made to generalize Freiman's theorems mentioned above. However, these generalizations either have some restrictions on A, see [19], or do not concern the case when $|A + A| > 3|A| - 2$, see [7,20]. In [13], the author applied methods from nonstandard analysis to the problems in this direction and proved the following weak version of Conjecture 1.1.

Theorem 1.2 *There exists a positive real number ϵ and a positive integer K such that for every finite set A of integers with $|A| = k$, if $k > K$ and $|A + A| = 3k - 3 + b$ for $0 \leqslant b \leqslant \epsilon k$, then A is either a subset of an* a.p. *of length at most $2k - 1 + 2b$ or a subset of a* b.p. *of length at most $k + b$.*

Note that even for $b = 2$ Theorem 1.2 gives a new result. However, we still have a long way to go to settle Conjecture 1.1. In order to be prepared for attacking the conjecture, we need to re-examine the ideas used in [13]. One important step in the proof of Theorem 1.2 is [12, Lemma 2.12], which is the key for applying nonstandard methods to the case.

For introducing the ideas from nonstandard analysis we first need to have some notation and definitions. The reader may also consult [5,8,21] or other typical nonstandard analysis introductions for basic information on nonstandard analysis.

[1] In some literature nA is used for n-fold sum of A.

We fix a countably saturated nonstandard universe ${}^*V$ throughout this article. For each standard set $A \subseteq \mathbb{N}$ we denote ${}^*A$ for its version in ${}^*V$. All integers in ${}^*\mathbb{N} \setminus \mathbb{N}$ are called *hyperfinite* integers. For any real numbers $a \leqslant b$ let $[a, b]$ denote the set of integers $\{k : a \leqslant k \leqslant b\}$. For a set A we denote $A[a, b]$ for the set $A \cap [a, b]$ and $A(a, b)$ for the number of elements in $A[a, b]$. For any $x > 0$ the term $A(x)$ is used for $A(1, x)$. In this paper we need to handle a lot of cases of the density of a set inside an interval. In order to avoid writing fractions too often we use the notation $\mathcal{D}_x(A(a, b))$ for the density of A in an interval $[a, b]$ of length x, i.e., $\mathcal{D}_x(A(a, b)) = \frac{A(a,b)}{x}$. Theoretically, we can express any fraction $\frac{a}{b}$ as $\mathcal{D}_a(b)$. But we will use $\mathcal{D}$ primarily for the density of a set A in an interval $[a, b]$. A set $U \subseteq {}^*\mathbb{N}$ is called a *cut* if $\mathbb{N} \subseteq U$, $U + U \subseteq U$, and for any $x, y \in {}^*\mathbb{N}$, $x < y$ and $y \in U$ imply $x \in U$. Clearly, $\mathbb{N}$ itself is a cut. So a cut can be seen as a generalization of $\mathbb{N}$ as the semigroup of all nonnegative elements of a convex subgroup of the additive group ${}^*\mathbb{Z}$. Let H be a hyperfinite integer and let $U_H = \bigcap_{n\in\mathbb{N}}[0, \frac{H}{n}]$. Then U_H is the largest cut below H. By countable saturation the cut U_H has uncountable cofinality, i.e., any countable increasing sequence of integers in U_H is upper bounded in U. If U is a cut different from ${}^*\mathbb{N}$, then U is an external set because it does not have a maximal element. For a cut U and $a \in {}^*\mathbb{N}$ we often write $a < U$ for $a \in U$ and $a > U$ for $a \notin U$. For a cut U a set $A \subseteq U$ is called U-internal if there is an internal set $B \subseteq {}^*\mathbb{N}$ such that $A = U \cap B$. For a U-internal set A the *lower U-density* of A is defined by

$$\underline{d}_U(A) = \sup\{\inf\{st(\mathcal{D}_x(A(x))) : x \in U \setminus [0, m]\} : m \in U\} \tag{1}$$

where st is the standard part map, which will be defined in the next paragraph. If $U = \mathbb{N}$, then any $A \subseteq \mathbb{N}$ is $\mathbb{N}$-internal and $\underline{d}_U(A)$ coincides with $\underline{d}(A)$ where

$$\underline{d}(A) = \liminf_{n\to\infty} \mathcal{D}_n(A(n))$$

is the usual definition of lower asymptotic density of A. Hence lower U-density is a generalization of the usual lower asymptotic density. Note that if A contains some negative integers, we can still define $\underline{d}_U(A)$ simply by ignoring these negative integers. For a cut U and a set $A \subseteq U$, when we say "for all sufficiently large $x \in A$ the property $P(x)$ is true", we mean that "there is $y \in U$ such that for all $x \in A$ and $x > y$, the property $P(x)$ is true". When we say "there exist sufficiently large $x \in A$ such that the property $P(x)$ is true", we mean that "for every $y \in U$ there is $x \in A$ and $x > y$ such that the property $P(x)$ is true".

We use lower case Greek letters for standard real numbers. For any real $r \in {}^*\mathbb{R}$ such that $|r| < n$ for some $n \in \mathbb{N}$, let $st\,(r) = \alpha$ for the unique $\alpha \approx r$ where $\approx$ means "infinitesimally close to". The map st is called the standard part map. For two real numbers $r, s \in {}^*\mathbb{R}$, we write $r \ll s$ $(r \gg s)$ if $r < s$ $(r > s)$ but $r \not\approx s$ and write $r \lessapprox s$ $(r \gtrapprox s)$ if $r < s$ $(r > s)$ or $r \approx s$.

Let's go back to the standard world. Keep in mind that our future goal is to characterize the structure of a finite set A when $|A + A|$ is more than $3|A| - 2$ but less than $\frac{10}{3}|A| - 5$. For a finite set A we call a number c the doubling constant of A if $|A + A| = c|A|$. From Theorem 1.2 we can characterize the structure of A when $|A|$ is large enough and the doubling constant of A is less than $3 + \epsilon$ for a small positive constant ϵ independent of $|A|$. For A with doubling constant between $3+\epsilon$ and $\frac{10}{3}$, the structure of A is unsettled. Interestingly, there is a result of Kneser for an infinite subset A of $\mathbb{N}$, which gives us desirable arithmetic structure of A when, roughly speaking, the "doubling constant" of A is between 3 and $\frac{10}{3}$. This will be made clear in the comments after Corollary 1.4. Corollary 1.4 follows from Theorem 1.3 and the proof of Theorem 1.3 can be found in [6].

Theorem 1.3 (M. Kneser) *Let $A, B \subseteq \mathbb{N}$. If $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$, then there are $g > 0$ and $G \subseteq [0, g-1]$ such that*

(a) $\underline{d}(A+B) \geqslant \underline{d}(A) + \underline{d}(B) - \frac{1}{g}$,
(b) $A + B \subseteq G + g\mathbb{N}$, *and*
(c) $(G + g\mathbb{N}) \setminus (A+B)$ *is finite.*

Kneser's Theorem actually deals with the sum of multiple sets. We state only the version for the sum of two sets here for simplicity. We can easily derive a corollary of Kneser's Theorem, which appears closer to the style of Freiman's inverse problems.

Corollary 1.4 *Suppose $A, B \subseteq \mathbb{N}$ and $\underline{d}(A+B) < \underline{d}(A) + \underline{d}(B)$. Then there are $g > 0$ and $F, F' \subseteq [0, g-1]$ such that $A \subseteq F + g\mathbb{N}$, $B \subseteq F' + g\mathbb{N}$, and $\underline{d}(A) + \underline{d}(B) > \frac{|F|+|F'|}{g} - \frac{1}{g}$.*

Proof Let g be the least positive integer satisfying Theorem 1.3. Let $F, F' \subseteq [0, g-1]$ be the minimal sets such that $A \subseteq F + g\mathbb{N}$ and $B \subseteq F' + g\mathbb{N}$. Let G be the set in Theorem 1.3. Clearly $F + F' = G$ in $\mathbb{Z}/g\mathbb{Z}$. If $|G| \geqslant |F| + |F'|$, then $\underline{d}(A+A) = \frac{|G|}{g} \geqslant \frac{|F|+|F'|}{g} \geqslant \underline{d}(A) + \underline{d}(B)$. So we can assume $|G| < |F| + |F'|$. If $|G| < |F| + |F'| - 1$, then by Lemma 2.1 we can find a positive $g' < g$ such that G is the union of some cosets of the nontrivial subgroup generated by g' in $\mathbb{Z}/g\mathbb{Z}$. This contradicts the minimality of g. Hence we have $|G| = |F| + |F'| - 1$, which implies the conclusion of the corollary. □

One can also prove Theorem 1.3 easily by assuming Corollary 1.4. The inequality $\underline{d}(A) + \underline{d}(B) > \frac{|F|+|F'|}{g} - \frac{1}{g}$ implies that $A+B$ is essentially the union of $|F|+|F'|-1$ arithmetic progressions with the same difference g. This is true because for any $a \in F$ and $b \in F'$ we have $\underline{d}(A \cap (a + g\mathbb{N})) + \underline{d}(B \cap (b + g\mathbb{N})) > \frac{1}{g}$.

Suppose $A \subseteq \mathbb{N}$ such that $0 < \underline{d}(A) = \alpha \leqslant \frac{1}{2}$. If A is neither a subset of an *a.p.* of difference > 1 nor a subset of a *b.p.* of the form $F + g\mathbb{N}$ with $F \subseteq [0, g-1]$, $|F| = 2$, and $\alpha > \frac{3}{2g}$, then by Corollary 1.4 with $A = B$ we have that either (a) $\underline{d}(A+A) \geqslant 2\underline{d}(A)$ or (b) $A \subseteq F + g\mathbb{N}$ with $F \subseteq [0, g-1]$, $|F| \geqslant 3$, and $\alpha > \frac{|F|}{g} - \frac{1}{2g}$. Note that (a) implies that there are infinitely many $x \in A$ such that $(A+A)(0, 2x) > \frac{10}{3}A(0, x)$. If (b) is true, then by Corollary 1.4 and $|F| \geqslant 3$ we have $\underline{d}(A+A) = \frac{2|F|-1}{g} \geqslant \frac{5|F|}{3g} \geqslant \frac{5}{3}\alpha$, which implies that there are infinitely many $x \in \mathbb{N}$ such that $(A+A)(0, 2x) \geqslant (\frac{10}{3} - \delta)A(0, x)$ for any small positive real δ. This is what we mentioned above that for an infinite set $A \subseteq \mathbb{N}$, if the "doubling constant" of A is less than $\frac{10}{3} - \delta$ for an arbitrarily small fixed $\delta > 0$, then A must have the desired arithmetic structure.

How can this result for an infinite set be used for a finite set? In a nonstandard model, arbitrarily large finite sets can be interpreted to a hyperfinite set and $\mathbb{N}$ as a subset of a hyperfinite interval $[0, H]$. So the behavior of $A \subseteq \mathbb{N}$ can be spilled over to a hyperfinite set $^*A[0, N]$ for some hyperfinite integer N.

Suppose we want to prove Theorem 1.2 by a contrapositive argument and assume that the theorem is false. Then we can find a counterexample A' of the theorem with $|A'|$ greater than any $K \in \mathbb{N}$ and $|A' + A'|$ less than $(3 + \epsilon)|A'|$ for any positive standard real ϵ. Hence $|A'|$ is hyperfinite and $\frac{|A'+A'|}{|A'|} \lessapprox 3$. Without loss of generality we can assume $A' \subseteq [0, H]$, $0, H \in A'$, and $0 \ll \mathcal{D}_H(|A'|) \lessapprox \frac{1}{2}$. Let $A = A' \cap \mathbb{N}$. Suppose also that $0 < \underline{d}(A) = \alpha \leqslant \frac{1}{2}$. If $\underline{d}(A + A) < \frac{5}{3}\alpha$, then A is either a subset of an *a.p.* with difference > 1 or a subset of a *b.p.* by Corollary 1.4. Hence there is a hyperfinite integer N such that $A'[0, N]$ is either a subset of an *a.p.* with difference > 1 or a subset of a *b.p.* If $\underline{d}(A + A) \geqslant \frac{5}{3}\alpha$, then there is a hyperfinite integer N such that $\mathcal{D}_N((A' + A')(0, 2N)) \gtrapprox \frac{10}{3} \cdot \mathcal{D}_N(A'(0, N))$. Hence Corollary 1.4 gives us a "localized" version of Theorem 1.2 with "doubling constant" $3 + \epsilon$ for any positive standard real $\epsilon < \frac{1}{3}$.

Unfortunately, this localized version may not be useful if N is significantly smaller than H because if $\frac{N}{H} \approx 0$, then the set $A'[0, N]$ is too small to make some meaningful impact on the structure of A'.

How about dealing with $A' \cap U_H$ instead of dealing with $A' \cap \mathbb{N}$? If $N \in [0, H]$ and $N > U_H$, then $\frac{N}{H} \gg 0$. So the structural information from $A' \cap U_H$ can be spilled over to $A'[0, N]$ for some hyperfinite integer N and the structural information of $A'[0, N]$ can have a significant impact on the structure of A'. But can we have something similar to Corollary 1.4 for U_H instead of $\mathbb{N}$? Although both U_H and $\mathbb{N}$ are convex semigroups, they do have significant differences. For example, if f is a function from an initial segment of $\mathbb{N}$ to $\mathbb{N}$, then the image of f is bounded in $\mathbb{N}$. But there are functions f, which map a proper initial segment of U_H cofinally to U_H. This is the main reason why we are unable to imitate the proof of Kneser's Theorem to derive a result parallel to Kneser's Theorem for U_H. Fortunately, the proof of Theorem 1.2 needs only a weak version of Kneser's Theorem for U_H with "doubling constant" slightly greater than 3, see [12, Lemma 2.12].

If we want to solve Conjecture 1.1, we have to improve [12, Lemma 2.12] so that the "doubling constant" there can be raised to $\frac{10}{3}$. The main purpose of this article is to prove such an improved lemma. Since the improved lemma is the main result here, we should call it a theorem instead. In fact the main theorem stated below is actually more general than what we want. The "doubling constant" involved there can be slightly greater than $\frac{10}{3}$. Before stating the theorem, we want to introduce another new notation. A set $X = I_0 \cup I_1 \cup I_2$ is called a tri-arithmetic progression, or *t.p.* for abbreviation, of difference d if I_i is an *a.p.* of difference d for $i = 0, 1, 2$ and exactly five of the six sets $I_i + I_j$ for $i \leqslant j$ in $\{0, 1, 2\}$ are pairwise disjoint. For example, $\{0, 1, 2\} + 7U$ is a *t.p.* of difference 7. When we say a set A is a subset of a *t.p.* $X = I_0 \cup I_1 \cup I_2$, we always assume $A \cap I_i \neq \emptyset$ for $i = 0, 1, 2$.

Theorem 1.5 *Let U be a cut with uncountable cofinality and $A_0 \subseteq U$ be U-internal. Suppose $0 \in A_0$ and $0 < \underline{d}_U(A_0) = \alpha < \frac{3}{5}$. Then one of the following is true:*

(a) *A_0 is a subset of an* a.p. *of difference $g > 1$.*
(b) *A_0 is a subset of a* b.p. *of the form $F + gU$ where $g > 2$ and $F = \{0, a\} \subseteq [0, g-1]$.*
(c) *A_0 is a subset of a* t.p. *of the form $F + gU$ where $g > 4$ and $F = \{0, a_1, a_2\} \subseteq [0, g-1]$.*
(d) *There is $\epsilon > 0$ and there are sufficiently large $x \in A_0$ such that $(A_0+A_0)(2x) > \left(\frac{10}{3} + \epsilon\right) A_0(x)$.*

In (a), (b), and (c) we do not explain why we call the set A_0 a *large* subset of these arithmetic progressions as mentioned in the abstract. We postpone the explanation to Theorem 3.1 in the third section in order to split the technical difficulties. In the next section we present the proof of Theorem 1.5.

2 Proof of Theorem 1.5

We first list and prove some lemmas and introduce some tools. We will frequently refer to the following lemma by Kneser.

Lemma 2.1 *Let G be an abelian group and A, B be finite subsets of G. Let $S = \{g \in G : g + A + B = A + B\}$ be the stabilizer of $A + B$. If $|A + B| < |A| + |B|$, then $|A + B| = |A + S| + |B + S| - |S|$.*

The proof of Lemma 2.1 can be found in [22, p. 115]. The next lemma is due to Freiman. The proof can be found in [3] or in [22].

Lemma 2.2 *Let A be a finite set of integers and $|A| = k$. If $|A + A| = 2k - 1 + b < 3k - 3$, then A is a subset of an* a.p. *of length at most $k + b$.*

The next lemma is a part of a theorem in page 28 of [3].

Lemma 2.3 *Let A be a finite subset of a* b.p. *and $|A| = k > 10$. If $|A+A| = 3k-3+b$ for $0 \leqslant b < k - 3$, then A is a subset of a* b.p. *of length at most $k + b$.*

In page 28 of [3] b in the theorem is restricted to be less than $\frac{1}{3}k - 2$. However, the restriction is only used for the first half of the theorem. It is not hard to see that the restriction can be weakened to $b < k - 3$ here when Lemma 2.2 is applied in the proof.

The next lemma is due to Lev and Smeliansky (see [20] and [22, p.118]).

Lemma 2.4 *Let A and B be two finite sets of nonnegative integers. Suppose $0 \in A \cap B$, both A and B contain more than one element, $\gcd(A) = 1$, $m = \max A$, and $n = \max B \leqslant m$. If $m = n$, then $|A+B| \geqslant \min\{m + |B|, |A| + 2|B| - 3\}$. If $m > n$, then $|A + B| \geqslant \min\{m + |B|, |A| + 2|B| - 2\}$.*

One can easily derive various versions of Lemma 2.4 when, for example, A and B are subsets of arithmetic progressions of a common difference and the smallest elements of A and B are not 0.

The next lemma is a trivial consequence of the definition of the lower U-density. We list it so that we can conveniently refer to it later.

Lemma 2.5 *Let U be a cut and A be U-internal. Suppose $\gamma \in \mathbb{R}$. If $\underline{d}_U(A) > \gamma$, then there is $\epsilon > 0$ such that for all sufficiently large $x \in U$ we have $\mathcal{D}_x(A(x)) > \gamma + \epsilon$.*

The next lemma is a consequence of the fact that the cofinality of U is uncountable.

Lemma 2.6 *Let U be a cut and A be U-internal. If $\underline{d}_U(A) \geqslant \gamma$, then for all sufficiently large $x \in U$ we have $\mathcal{D}_x(A(x)) \gtrapprox \gamma$.*

Proof For each $n \in \mathbb{N}$ let

$$X_n = \left\{ x \in U : \mathcal{D}_x(A(x)) < \gamma - \frac{1}{n} \right\}.$$

Since $\underline{d}_U(A) \geqslant \gamma$, then X_n is upper bounded in U. Since U has an uncountable cofinality, there is $y \in U$ such that y is an upper bound of X_n for every $n \in \mathbb{N}$. It is now easy to see that for all $x \in U$, if $x > y$, then $\mathcal{D}_x(A(x)) \gtrapprox \gamma$. □

An important tool for additive/combinatorial number theory is called e-transform (cf. [22, p. 42]).[2] Let $A, B \subseteq U$ and $c \in {}^*\mathbb{Z}$ with $|c| \in U$. An e_c-transform of (A, B) is the pair $(A', B') = e_c(A, B)$ such that

$$A' = (A \cup (B + c)) \cap U \quad \text{and} \quad B' = (B \cap (A - c)) \cap U.$$

Remark 2.7 If A and B are finite or hyperfinite and $(A', B') = e_c(A, B)$, then $|A| + |B| = |A'| + |B'|$ because $A' \smallsetminus A = c + (B \smallsetminus B')$.

Remark 2.8 The following are important properties of the e_c-transform. Let $A, B \subseteq U$ be U-internal and $(A', B') = e_c(A, B)$. Then A' and B' are U-internal and

(a) $A' \supseteq A$ and $B' \subseteq B$,
(b) if $B' \neq \emptyset$, then $A' + B' \subseteq A + B$,
(c) if $x \in U$ and $\frac{c}{x} \approx 0$, then for every y with $x < y < U$ we have

$$\mathcal{D}_y(A(y) + B(y)) \approx \mathcal{D}_y(A'(y) + B'(y)).$$

This is a simple consequence of Remark 2.7.

An e_c-transform of (A, B) is *legitimate* if $c = a - b$ for some $a \in A$ and $b \in B$. Note that if e_c is legitimate for $c = a - b$ and $(A', B') = e_c(A, B)$, then $b \in B'$. In particular, $B' \neq \emptyset$. From now on, all e-transforms used in this article are legitimate e-transforms. So we can omit the word "legitimate". We will exclusively write $\mathcal{E}$ for a *finite* sequence of e-transforms. When we write $(A', B') = \mathcal{E}(A, B)$ we mean that (A', B') is obtained by applying a finite sequence $\mathcal{E}$ of e-transforms $e_{c_1}, e_{c_2}, \ldots, e_{c_n}$ successively to (A, B) such that each step of the application is legitimate. We say that e_c occurs in $\mathcal{E}$ if e_c is in the sequence. If $(A', B') = \mathcal{E}(A, B)$, then (a) and (b) of Remark 2.8 are still true and (c) of Remark 2.8 is true for all $x \in U$ satisfying $\frac{c}{x} \approx 0$ for all e_c occurring in $\mathcal{E}$.

The following are some remarks on lower U-density.

[2] The e-transform is also called τ-transformation in [6].

Remark 2.9 Let U be a cut and A be U-internal. Then

(a) For any $a \in U$, $\underline{d}_U(A \setminus [0,a]) = \underline{d}_U(A)$.
(b) For any $a \in U$, $\underline{d}_U(A \pm a) = \underline{d}_U(A)$.

Lemma 2.10 *Let A and B be two U-internal sets, $\underline{d}_U(A) = \alpha$, and $\underline{d}_U(B) = \beta$. Suppose $\alpha + \beta \leqslant 1$. If for every $k \in \mathbb{N}$ there exists a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and A' contains k consecutive integers, then $\underline{d}_U(A+B) \geqslant \alpha+\beta$.*

The proof of Lemma 2.10 is very similar to the proof of [6, Theorem 20]. The reader can also find the proof in [12, Subclaim 2.12.4.2.2] with $A + A$ and 2α being replaced by $A + B$ and $\alpha + \beta$.

Lemma 2.11 *Let U be a cut with uncountable cofinality and A be a U-internal set. If there are sufficiently large $x \in A$ such that*

$$\mathcal{D}_{2x}((A + A)(2x)) \gg \tfrac{5}{3}\mathcal{D}_x(A(x)),$$

then there is $\epsilon > 0$ and there are sufficiently large $x \in A$ such that

$$\mathcal{D}_{2x}((A + A)(2x)) > \left(\tfrac{5}{3} + \epsilon\right)\mathcal{D}_x(A(x)).$$

Proof Let B be internal and $A = B \cap U$. For each $n \in \mathbb{N}$ let

$$X_n = \left\{x \in B : \mathcal{D}_{2x}((B + B)(2x)) > \left(\frac{5}{3} + \frac{1}{n}\right)\mathcal{D}_x(B(x))\right\}.$$

Then X_n is internal and $U \cap \bigcup\{X_n : n \in \mathbb{N}\}$ is upper unbounded in U. Since the cofinality of U is uncountable, there is $n \in \mathbb{N}$ such that $U \cap X_n$ is upper unbounded in U. So the lemma is true with $\epsilon = \frac{1}{n}$. □

Proof of Theorem 1.5 We assume that (a), (b), (c) of Theorem 1.5 are not true and show that (d) of Theorem 1.5 must be true.

Suppose $\frac{1}{2} < \alpha < \frac{3}{5}$. Then $\underline{d}(A_0 + A_0) = 1 > \frac{5}{3}\alpha$. Hence for each $x \in A_0$ with $\mathcal{D}_x(A_0(x)) \approx \alpha$ and $\mathcal{D}_{2x}((A_0 + A_0)(2x)) \approx 1$ we have $\mathcal{D}_{2x}((A_0 + A_0)(2x)) \gg \frac{5}{3}\mathcal{D}_x(A_0(x))$. By Lemma 2.11, (d) of Theorem 1.5 is true. So for the rest of the proof we can assume $\alpha \leqslant \frac{1}{2}$.

Let $C \subseteq U$ be U-internal. If there are $a, a' \in C$ such that $a < a'$ and $a' - a \in \mathbb{N}$, then define $f(C) = \min\left\{a' - a : a < a' \text{ and } a, a' \in C\right\}$. Otherwise, define $f(C) = \infty$. Let $m_C = \min C$. If $\gcd(C - m_C) \in \mathbb{N}$, define $g(C) = \gcd(C - m_C)$. Otherwise, define $g(C) = \infty$. Clearly, $g(C) \leqslant f(C)$ and if $C \subseteq C'$, then $f(C) \geqslant f(C')$ and $g(C) \geqslant g(C')$.

Let $\mathcal{E}$ be a sequence of e-transforms and $(A, B) = \mathcal{E}(A_0, A_0)$. If $f(B)$ is greater than $\frac{3}{\alpha}$, then by Remark 2.8 we have $\underline{d}_U(A_0 + A_0) \geqslant \underline{d}_U(A) \gtrapprox 2\alpha - \frac{1}{f(B)} > \frac{5}{3}\alpha$. This implies (d) of Theorem 1.5. Hence we can assume that $f(B) \leqslant \frac{3}{\alpha}$. Note that $g(B) \leqslant f(B)$. Without loss of generality we can assume that $(A, B) = \mathcal{E}(A_0, A_0)$ satisfies the following. For any sequence $\mathcal{E}'$ of e-transforms and $(A', B') = \mathcal{E}'(A, B)$ we have $f(B') = f(B)$, $g(B') = g(B)$, and $f(A') = f(A)$. Let $f = f(B)$, $g = g(B)$, and $t = f(A)$ be fixed.

For each $C \subseteq U$ let

$$F_C = \{y \in [0, g-1] : \exists x \in C\ (x \equiv y \pmod g)\} \quad \text{and} \quad C^y = C \cap (y + g\,{}^*\mathbb{N})$$

for each $y \in F_C$. Note that $|F_B| = 1$.

Let $y \in F_A$ and $a_y = \min A^y$, and let $m_B = \min B$. With few more e-transforms we can also assume that $a_y - m_b + B \subseteq A$. Note that since $A + B \subseteq A_0 + A_0$, then we have $F_A + F_B \subseteq F_{A_0} + F_{A_0}$ in $\mathbb{Z}/g\mathbb{Z}$.

Suppose $(A', B') = \mathcal{E}(A_0, A_0)$. Let

$$\Gamma(A', B') = \left\{x \in B' : \exists z \in U\ \ (\mathcal{D}_z(A_0(z)) \approx \alpha) \quad \text{and} \quad x = \min(B' \smallsetminus [0, z-1])\right\}.$$

Note that $\mathcal{D}_x(A(x)+B(x)) \approx 2\mathcal{D}_x(A_0(x)) \lessapprox 1$ for all sufficiently large $x \in \Gamma(A, B)$.

Let $k = |F_A|$ and $k_0 = |F_{A_0}|$.

Claim 1.5.1 *Let $\mathcal{E}$ be a finite sequence of e-transforms and $(A', B') = \mathcal{E}(A, B)$. If there are sufficiently large $x \in U$ such that*

$$\mathcal{D}_x(B'(x)) \ll \tfrac{1}{3}\mathcal{D}_x(A_0(x)),$$

then (d) of Theorem 1.5 is true. □

Proof of Claim 1.5.1 Suppose B' is bounded in U. For any sufficiently large $x \in A_0$ such that $\mathcal{D}_x(A_0(x)) \approx \alpha$ we have

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\quad \gtrapprox \mathcal{D}_{2x}((A' + B')(2x)) \gtrapprox \mathcal{D}_{2x}(A'(2x)) \\
&\quad \approx 2\mathcal{D}_{2x}(A_0(2x)) \gtrapprox 2\alpha \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}$$

Suppose B' is unbounded in U. Again let $m_{B'} = \min B'$. Given any sufficiently large $x \in U$ such that $\mathcal{D}_x(B'(x)) \ll \frac{1}{3}\mathcal{D}_x(A_0(x))$, we can assume $x \in B'$. Then we have

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\quad \gtrapprox \mathcal{D}_{2x}((A' + B')(2x)) \gtrapprox \mathcal{D}_{2x}((A' + \{m_{B'}, x\})(2x)) \\
&\quad \gtrapprox \mathcal{D}_x(A'(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}$$

Hence (d) of Theorem 1.5 is true by Lemma 2.11. □

By Claim 1.5.1 we can assume that for any sequence $\mathcal{E}$ of e-transforms with $(A', B') = \mathcal{E}(A, B)$ and for all sufficiently large $x \in B'$,

$$D_x(B'(x)) \gtrapprox \tfrac{1}{3}\mathcal{D}_x(A_0(x)) \quad \text{and} \quad \mathcal{D}_x(A'(x)) \lessapprox \tfrac{5}{3}\mathcal{D}_x(A_0(x)). \tag{2}$$

Claim 1.5.2 *For all sufficiently large x,*

$$\mathcal{D}_x(B(x)) \lessapprox \frac{2}{k+1}\mathcal{D}_x(A_0(x)) \quad \text{and} \quad \mathcal{D}_x(A(x)) \gtrapprox \frac{2k}{k+1}\mathcal{D}_x(A_0(x)). \tag{3}$$

Proof of Claim of 1.5.2 Since for each $y \in F_A$, we assumed that $a_y - m_b + B \subseteq A$. Hence by the definition of k, we have that $\mathcal{D}_x(A(x)) \gtrapprox k\mathcal{D}_x(B(x))$ for all sufficiently large x. This implies that

$$\mathcal{D}_x(B(x)) \approx 2\mathcal{D}_x(A_0(x)) - \mathcal{D}_x(A(x)) \lessapprox 2\mathcal{D}_x(A_0(x)) - k\mathcal{D}_x(B(x)),$$

which clearly implies $\mathcal{D}_x(B(x)) \lessapprox \frac{2}{k+1}\mathcal{D}_x(A_0(x))$. We also have

$$\begin{aligned}\mathcal{D}_x(A(x)) &\approx 2\mathcal{D}_x(A_0(x)) - \mathcal{D}_x(B(x))\\ &\gtrapprox 2\mathcal{D}_x(A_0(x)) - \frac{2}{k+1}\mathcal{D}_x(A_0(x))\\ &= \frac{2k}{k+1}\mathcal{D}_x(A_0(x)).\end{aligned}$$

□

Claim 1.5.3 *Suppose $k > 1$. If there are two distinct $y_1, y_2 \in F_A$ and sufficiently large $x \in B$ such that $\mathcal{D}_x(A^{y_i}(x) + B(x)) \lessapprox \frac{1}{g}$, then (d) of Theorem 1.5 is true.*

Proof of Claim 1.5.3 By Lemma 2.4 we have that there are sufficiently large $x \in B$ such that

$$\begin{aligned}&\mathcal{D}_{2x}((A_0 + A_0)(2x))\\ &\quad \gtrapprox \sum_{y\in F_A} \mathcal{D}_{2x}(|A^y[0,x] + B[0,x]|)\\ &\quad \gtrapprox \sum_{i=1}^{2} \mathcal{D}_{2x}(2A^{y_i}(x) + B(x)) + \sum_{y\neq y_1,y_2} \mathcal{D}_{2x}(2A^y(x))\\ &\quad \gtrapprox \mathcal{D}_x(A(x)) + \mathcal{D}_x(B(x))\\ &\quad \gtrapprox 2\mathcal{D}_x(A_0(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)).\end{aligned}$$

Hence (d) of Theorem 1.5 is true. □

Claim 1.5.4 *If $k = g > 2$, then (d) of Theorem 1.5 is true.*

Proof of Claim 1.5.4 By Claim 1.5.3 we can assume that either there is only one $y_0 \in F_A$ such that $\mathcal{D}_x(A^{y_0}(x)+B(x)) \lessapprox \frac{1}{g}$ or for every $y \in F_A$, we have $\mathcal{D}_x(A^{y_0}(x)+ B(x)) \gg \frac{1}{g}$ for all sufficiently large $x \in B$. Let $x \in \Gamma(A, B)$ be sufficiently large.

Suppose there is only one $y_0 \in F_A$ such that $\mathcal{D}_x(A^{y_0}(x) + B(x)) \lessapprox \frac{1}{g}$. Since $\frac{k-1}{g} = \frac{k-1}{k} \gg \frac{1}{2} \gtrapprox \mathcal{D}_x(A_0(x))$, then by Lemma 2.4 we have

$$
\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\gtrapprox \sum_{y \in F_A \smallsetminus \{y_0\}} \mathcal{D}_{2x}(A^y(x) + x/g) + \mathcal{D}_{2x}(2A^{y_0}(x) + B(x)) \\
&\gtrapprox \mathcal{D}_{2x}(A(x) + B(x)) + \mathcal{D}_{2x}(A^{y_0}(x)) + \frac{k-1}{2g} \\
&\gg \mathcal{D}_x(A_0(x)) + \tfrac{1}{6}\mathcal{D}_x(A_0(x)) + \tfrac{1}{2}\mathcal{D}_x(A_0(x)) = \tfrac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}
$$

Suppose $\mathcal{D}_x(A^{y_0}(x) + B(x)) \gg \frac{1}{g}$ for every $y \in F_A$. Recall that $\mathcal{D}_x(A(x)) \gtrapprox \frac{2k}{k+1}\mathcal{D}_x(A_0(x))$. Then

$$
\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\gtrapprox \sum_{y \in F_A} \mathcal{D}_{2x}(A^y(x) + x/g) \gtrapprox \mathcal{D}_{2x}(A(x)) + \frac{1}{2} \\
&\gtrapprox \frac{k}{k+1}\mathcal{D}_x(A_0(x)) + \mathcal{D}_x(A_0(x)) \gg \frac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}
$$

$\square$

Claim 1.5.5 *If $g = k = 2$, then (d) of Theorem 1.5 is true.*

Proof of Claim 1.5.5 Let $F_A = \{y_1, y_2\}$. Again we can assume that either there is only one $y \in F_A$ such that $\mathcal{D}_x(A^y(x) + B(x)) \lessapprox \frac{1}{2}$ or $\mathcal{D}_x(A^{y_i}(x) + B(x)) \gg \frac{1}{2}$ for $i = 1, 2$ and for all sufficiently large $x \in B$. Let $x \in \Gamma(A, B)$ be sufficiently large.

Suppose, say, $\mathcal{D}_x(A^{y_1}(x) + B(x)) \lessapprox \frac{1}{2}$. Then by the first displayed inequalities in the proof of Claim 1.5.4 we can assume $\mathcal{D}_x(A^{y_1}(x)) \approx \mathcal{D}_x(B(x)) \approx \frac{1}{3}\mathcal{D}_x(A_0(x))$. This implies $\mathcal{D}_x(A(x)) \approx \frac{5}{3}\mathcal{D}_x(A_0(x))$. Note that $\mathcal{D}_x(A^{y_2}(x)) \lessapprox \frac{1}{2}$. Hence

$$
\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\gtrapprox \mathcal{D}_{2x}(2A^{y_1}(x) + B(x)) + \mathcal{D}_{2x}(A^{y_2}(x) + x/2) \\
&\gtrapprox \mathcal{D}_x(A(x)) + \tfrac{1}{2}\mathcal{D}_x(B(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}
$$

Suppose $\mathcal{D}_x(A^{y_i}(x) + B(x)) \gg \frac{1}{2}$ for $i = 1, 2$. Then by Claim 1.5.2

$$
\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\gtrapprox \sum_{i=1,2} \mathcal{D}_{2x}(A^{y_i}(x) + x/2) \\
&\gtrapprox \tfrac{1}{2}\mathcal{D}_x(A(x)) + \tfrac{1}{2} \gtrapprox \tfrac{2}{3}\mathcal{D}_x(A_0(x)) + \mathcal{D}_x(A_0(x)).
\end{aligned}
$$

Hence we can assume that for all sufficiently large $x \in \Gamma(A, B)$, $\mathcal{D}_x(A_0(x)) \approx \frac{1}{2}$ and $\mathcal{D}_x(A(x)) \approx \frac{4}{3}\mathcal{D}_x(A_0(x)) \approx \frac{2}{3}$ because otherwise some $\gtrapprox$ in the inequality above can be replaced by $\gg$, which implies (d) of Theorem 1.5. We also have $\mathcal{D}_x(B(x)) \approx \frac{2}{3}\mathcal{D}_x(A_0(x)) \approx \frac{1}{3}$. Hence $t = 1$ and $f = 2$.

Now we show that for each $n \in \mathbb{N}$, there is a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and A' contains n consecutive integers. If this is done, then $\underline{d}_U(A_0 + A_0) \geqslant 2\alpha > \frac{5}{3}\alpha$ by Lemma 2.10. Hence (d) of Theorem 1.5 is true by Lemmas 2.5 and 2.11.

Since $t = 1$, then A contains two consecutive numbers. Suppose we have found a sequence $\mathcal{E}'$ of e-transforms such that $(A', B') = \mathcal{E}'(A, B)$ and A' contains $n - 1 \geqslant 2$ consecutive integers $\{a, a + 1, \dots, a + n - 2\}$. Since $f(B') = 2$, then B' contains two integers $b, b + 2$. Let $(A'', B'') = e_{a+n-3-b}(A', B')$. Then $a + n - 1 \in A''$. This shows A'' contains n consecutive integers $\{a, a + 1, \dots, a + n - 1\}$. $\square$

We now prove the theorem case by case according to the values of k. The case for $k = 1$ is the most difficult part of the proof.

Case 1.5.1 $k \geqslant 6$.

Since for all sufficiently large $x \in B$ we have $\mathcal{D}_x(B(x)) \gtrapprox \frac{1}{3}\mathcal{D}_x(A_0(x))$ and $\mathcal{D}_x(A(x)) \gtrapprox 6\mathcal{D}_x(B(x))$, then $\mathcal{D}_x(A(x)) \gtrapprox 2\mathcal{D}_x(A_0(x))$, which contradicts (2).

Case 1.5.2 $k = 5$.

By the same argument above we can assume that for all sufficiently large $x \in B$ and any $y \in F_A$ we have

$$\mathcal{D}_x(A^y(x)) \approx \mathcal{D}_x(B(x)) \approx \frac{1}{3}\mathcal{D}_x(A_0(x)).$$

Suppose there are sufficiently large $x \in B$ such that $\mathcal{D}_x(B(x)) \ll \frac{1}{g}$. Then for these x we have

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\quad \gtrapprox \mathcal{D}_{2x}((A + B)(2x)) \gtrapprox \mathcal{D}_{2x}(|A[0, x] + B[0, x]|) \\
&\quad \gg \mathcal{D}_x(A(x)) \gtrapprox \frac{5}{3}\mathcal{D}_x(A_0(x))
\end{aligned}$$

by Lemma 2.4. Hence, by Lemma 2.11, (d) of Theorem 1.5 is true. So we can assume $\mathcal{D}_x(B(x)) \approx \frac{1}{g} \approx \frac{1}{3}\mathcal{D}_x(A_0(x))$ and $\mathcal{D}_x(A(x)) \approx \frac{5}{3}\mathcal{D}_x(A_0(x))$ for all sufficiently large $x \in B$.

Note that the possible values of k_0 are 3, 4, and 5. Suppose $k_0 = 3$.

Because $k_0 = 3$, A_0 is a subset of the union of three *a.p.*'s. If $|F_{A_0} + F_{A_0}| = 5$ in $\mathbb{Z}/g\mathbb{Z}$, then A_0 is a subset of a *t.p.* of the form $F_{A_0} + gU$. Hence (c) of Theorem 1.5 is true. So we can assume $|F_{A_0} + F_{A_0}| = 6$ in $\mathbb{Z}/g\mathbb{Z}$ and let $y_1, y_2 \in F_{A_0}$ be such that $y_1 + y_2 \notin F_A + F_B$ in $\mathbb{Z}/g\mathbb{Z}$. Then either (d) of Theorem 1.5 is true or $\mathcal{D}_x(A_0^{y_1}(x)) \approx 0$ for all sufficiently large $x \in B$ because $A + B \subseteq A_0 + A_0$ and $a_{y_2} + A_0^{y_1} \subseteq (A_0 + A_0) \setminus (A + B)$. But we have $\mathcal{D}_x(B(x)) \approx \frac{1}{g}$ for all sufficiently large $x \in B$. So $\mathcal{D}_x(A_0(x)) \approx 3\mathcal{D}_x(B(x)) \approx \frac{3}{g}$, which implies $\mathcal{D}_x(A_0^{y_1}(x)) \approx \frac{1}{g}$. Hence we have a contradiction.

Suppose $k_0 = 4$. If $|F_{A_0} + F_{A_0}| = 5$ in $\mathbb{Z}/g\mathbb{Z}$. Then by Lemma 2.1 the stabilizer S of $F_{A_0} + F_{A_0}$ in $\mathbb{Z}/g\mathbb{Z}$ satisfies $|S| = 5$ and $|F_{A_0} + F_{A_0}| = 2|F_{A_0} + S| - |S|$ in $\mathbb{Z}/g\mathbb{Z}$. Note that $F_{A_0} = S$. Let S be generated by d for some $d|g$. If $d > 1$, then (a) of Theorem 1.5 is true. If $d = 1$, then $g = 5 = k$. Hence (d) of Theorem 1.5 is true by Claim 1.5.4.

If $|F_{A_0} + F_{A_0}| = 6$ in $\mathbb{Z}/g\mathbb{Z}$, then there are $y_1, y_2 \in F_{A_0}$ such that $z = y_1 + y_2 \notin F_A + F_B$. Hence for sufficiently large $x \in B$ we have $\mathcal{D}_x(A_0^{y_i}(x)) \approx 0$ for $i = 1, 2$. If $y_1 \neq y_2$, then there is $y_3 \in F_{A_0}$ such that $\mathcal{D}_x(A_0^{y_3}(x)) \gtrapprox \frac{1}{2}\mathcal{D}_x(A_0(x)) \approx \frac{3}{2g}$, which is absurd because $A_0^{y_3} \subseteq y_3 + g\,{}^*\mathbb{N}$. Suppose $y_1 = y_2$ and let $F' = F_{A_0} \setminus \{y_1\}$. Then $|F_{A_0} + F'| = 6$ because otherwise $|F_{A_0} + F'| = 5 < |F_{A_0}| + |F'| - 1$ implies $F_{A_0} + F' = S$ where S is the stabilizer of $F_{A_0} + F'$. But if S is generated by $d > 1$, then (a) of Theorem 1.5 is true. If S is generated by 1, then $g = 5 = k$, which implies (d) of Theorem 1.5 by Claim 1.5.4. Since $|F_{A_0} + F'| = 6$, then there is $y_3 \in F'$ such that for all sufficiently large $x \in B$ we have $\mathcal{D}_x(A_0^{y_3}(x)) \approx 0$. So there is $y \in F' \setminus \{y_3\}$ such that $\mathcal{D}_x(A_0^{y}(x)) \gg \frac{1}{g}$, which is again absurd.

If $|F_{A_0} + F_{A_0}| > 6$ in $\mathbb{Z}/g\mathbb{Z}$, then there are $y_1, y_2 \in F_{A_0}$ such that $y_1 \neq y_2$ and $\mathcal{D}_x(A_0^{y_i}(x)) \approx 0$ for $i = 1, 2$ and for all sufficiently large $x \in B$. This again leads to a contradiction to $\mathcal{D}_x(A_0(x)) \approx \frac{3}{g}$.

Suppose $k_0 = 5$. So we have $F_{A_0} = F_A$. Let $y_0 \in F_B$, $y \in F_{A_0}$, $y \neq y_0$, and $F' = \{y_0, y\}$. If $|F_{A_0} + F'| = 5$ in $\mathbb{Z}/g\mathbb{Z}$, then $F_{A_0} + F' = S$ where S is the stabilizer of $F_{A_0} + F'$. If (a) of Theorem 1.5 is not true, then $g = 5 = k$, which implies (d) of Theorem 1.5.

If $|F_{A_0} + F'| > 5$ in $\mathbb{Z}/g\mathbb{Z}$, then $\mathcal{D}_x(A_0^{y}(x)) \approx 0$ for all sufficiently large $x \in B$, which implies $\mathcal{D}_x(A_0(x)) \approx \mathcal{D}_x(A_0^{0}(x))$ for all sufficiently large $x \in B$. But this contradicts $\mathcal{D}_x(A_0^{0}(x)) \approx \frac{1}{g}$.

Case 1.5.3 $k = 4$.

In this case the possible values for k_0 are 3 and 4. Suppose $k_0 = 3$.

If $|F_{A_0} + F_{A_0}| = 4$ in $\mathbb{Z}/g\mathbb{Z}$, then $F_{A_0} + F_{A_0}$ is the union of S cosets, where S is the stabilizer of $F_{A_0} + F_{A_0}$ in $\mathbb{Z}/g\mathbb{Z}$ with $|S| > 1$. If $|S| = 2$, then $4 = |F_{A_0} + F_{A_0}| = 2|F_{A_0} + S| - |S| = 2(|F_{A_0} + S| - 1)$. But $|F_{A_0} + S|$ is even. So we have a contradiction. If $|S| = 4$, then (a) of Theorem 1.5 is true unless $g = 4$. But $g = 4 = k$ implies (d) of Theorem 1.5 by Claim 1.5.4.

If $|F_{A_0} + F_{A_0}| = 5$ in $\mathbb{Z}/g\mathbb{Z}$, then (c) of Theorem 1.5 is true. So we can assume $|F_{A_0} + F_{A_0}| > 5$. Since $k = 4$, there are distinct $y_1, y_2 \in F_{A_0}$ and $y_1', y_2' \in F_{A_0}$ such that $y_1 + y_1' \neq y_2 + y_2'$, $y_1 + y_1' \notin F_A + F_B$ and $y_2 + y_2' \notin F_A + F_B$. Since $\mathcal{D}_x(A(x)) \gtrapprox \frac{8}{5}\mathcal{D}_x(A_0(x))$ by (3), we can assume $\mathcal{D}_x(A_0^{y_1}(x) + A_0^{y_2}(x)) \lessapprox \frac{2}{15}\mathcal{D}_x(A_0(x))$ for all sufficiently large $x \in B$. This implies $\mathcal{D}_x(A_0^{y_3}(x)) \gtrapprox \frac{13}{15}\mathcal{D}_x(A_0(x))$ for $y_3 \in F_{A_0} \setminus \{y_1, y_2\}$, which implies $\frac{1}{g} \gtrapprox \frac{13}{15}\mathcal{D}_x(A_0(x))$. Since for each $y \in F_A$ we have, by Lemma 2.4,

$$
\begin{aligned}
&\mathcal{D}_{2x}((A^y + B)(2x)) \\
&\quad \gtrapprox \min\left\{\mathcal{D}_{2x}(2A^y(x) + B(x)), \mathcal{D}_{2x}(A^y(x) + x/g)\right\} \\
&\quad \gtrapprox \min\left\{\tfrac{3}{2}\mathcal{D}_x(B(x)), \mathcal{D}_{2x}(B(x) + 13A_0(x)/15)\right\},
\end{aligned}
$$

then we have

$$\mathcal{D}_{2x}((A^y + B)(2x)) \gtrapprox \min\left\{\tfrac{1}{2}\mathcal{D}_x(A_0(x)), \tfrac{18}{30}\mathcal{D}_x(A_0(x))\right\} \gtrapprox \tfrac{1}{2}\mathcal{D}_x(A_0(x)).$$

Hence

$$\mathcal{D}_{2x}((A_0 + A_0)(2x)) \gtrapprox \mathcal{D}_{2x}((A + B)(2x)) \gtrapprox 2\mathcal{D}_x(A_0(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)),$$

which implies (d) of Theorem 1.5.

Now let $k_0 = 4$. So $F_{A_0} = F_A$. Let $y_0 \in F_B$ and $y \in F_{A_0}$, $y \neq y_0$. Let $F' = \{y_0, y\}$. If $|F_{A_0} + F'| = 4$, then, by Lemma 2.1, we have $|S| = 2$ or $|S| = 4$. If $|S| = 2$, then $F_{A_0} + \{y_0\} = F_{A_0} + F'$ is the union of two S cosets. Let $S = \{0, \frac{g}{2}\}$ and $F_{A_0} = \{0, y', \frac{g}{2}.\frac{g}{2} + y'\}$. If $2y' \neq \frac{g}{2}$, then (b) of Theorem 1.5 is true. If $2y' = \frac{g}{2}$, then (a) of Theorem 1.5 is true unless $g = 4$. But $g = 4 = k$ implies (d) of Theorem 1.5. If $|S| = 4$, then we have that either (a) of Theorem 1.5 is true or (d) of Theorem 1.5 is true by Claim 1.5.4.

Case 1.5.4 $k = 3$.

In this case the possible values for k_0 are 2 and 3. If $k_0 = 2$, then (b) of Theorem 1.5 is true. So we can assume $k_0 = 3$.

Suppose $|F_{A_0} + F_{A_0}| = 3$ in $\mathbb{Z}/g\mathbb{Z}$. By Lemma 2.1 $S = F_{A_0} + F_{A_0} = F_A = F_{A_0}$ where S is the stabilizer of $F_{A_0} + F_{A_0}$. Let S be generated by d. If $d > 1$, then (a) of Theorem 1.5 is true. So we can assume $d = 1$. Then $g = 3 = k$, which implies (d) of Theorem 1.5 by Claim 1.5.4.

Suppose $|F_{A_0} + F_{A_0}| = 4$ in $\mathbb{Z}/g\mathbb{Z}$. By Lemma 2.1 the stabilizer S of $F_{A_0} + F_{A_0}$ is nontrivial. Note that $|F_{A_0} + S| \geqslant 4$. So $|S| = 2$ is impossible. Hence $|S| = 4$. Let $F_A = F_{A_0} = \{0 = y_0 < y_1 < y_2\}$ and, without loss of generality, $F_B = \{0\}$. Note that by (3) we have $\mathcal{D}_x(B(x)) \lessapprox \frac{1}{2}\mathcal{D}_x(A_0(x))$ and $\mathcal{D}_x(A(x)) \gtrapprox \frac{3}{2}\mathcal{D}_x(A_0(x))$ for all sufficiently large $x \in U$. Let $x \in U$ be sufficiently large. Without loss of generality, we can assume $\mathcal{D}_x(A_0^{y_1}(x)) \lessapprox \mathcal{D}_x(A_0^{y_2}(x))$. Let $F' = \{0, y_2\}$. If $|F_{A_0} + F'| = 3$ in $\mathbb{Z}/g\mathbb{Z}$, then, by Lemma 2.1, the stabilizer S' of $F_{A_0} + F'$ in $\mathbb{Z}/g\mathbb{Z}$ has three elements, which implies that $F_{A_0} = S'$. But this contradicts $|F_{A_0} + F_{A_0}| = 4$. So we can assume $|F_{A_0} + F'| = 4$. Hence there is $y \in \{y_1, y_2\}$ such that $y + y_2 \notin F_A + F_B$. This implies

$$(A_0^y[0, x] + A_0^{y_2}[0, x]) \cap A[0, x] = \emptyset.$$

So if (d) of Theorem 1.5 is not true, then we have

$$\tfrac{1}{6}\mathcal{D}_x(A_0(x)) \gtrapprox \mathcal{D}_{2x}(|A_0^y[0, x] + A_0^{y_2}[0, x]|) \gtrapprox \tfrac{1}{2}\mathcal{D}_x(A_0^{y_1}(x) + A_0^{y_2}(x)).$$

Hence

$$\frac{1}{g} \gtrapprox \mathcal{D}_x(A_0^{y_0}(x)) \gtrapprox \frac{2}{3}\mathcal{D}_x(A_0(x)). \tag{4}$$

Let $x \in B$ be sufficiently large. By Claim 1.5.3 we can assume that either there is exactly one $y' \in F_A$ such that $\mathcal{D}_x(A^{y'}(x) + B(x)) \lessapprox \frac{1}{g}$ or for all $y \in F_A$ we have $\mathcal{D}_x(A^y(x) + B(x)) \gg \frac{1}{g}$ for all sufficiently large $x \in B$.

If for all $y \in F_A$ we have $\mathcal{D}_x(A^y(x) + B(x)) \gg \frac{1}{g}$, then

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\quad \gtrapprox \mathcal{D}_{2x}((A + B)(2x)) \gtrapprox \sum_{y \in F_A} \mathcal{D}_{2x}(A^y(x) + x/g) \\
&\quad = \frac{1}{2}\mathcal{D}_x(A(x)) + \frac{3}{2g} \gtrapprox \frac{3}{4}\mathcal{D}_x(A_0(x)) + \mathcal{D}_x(A_0(x)) \\
&\quad = \tfrac{7}{4}\mathcal{D}_x(A_0(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)),
\end{aligned}$$

which implies (d) of Theorem 1.5.

If there is exactly one $y' \in F_A$ such that $\mathcal{D}_x(A^{y'}(x) + B(x)) \lessapprox \frac{1}{g}$, then

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x)) \\
&\quad \gtrapprox \mathcal{D}_{2x}((A + B)(2x)) \\
&\quad \gtrapprox \sum_{y \neq y'} \mathcal{D}_{2x}(A^y(x) + x/g) + \mathcal{D}_{2x}(2A^{y'}(x) + B(x)) \\
&\quad = \mathcal{D}_{2x}(A(x) + B(x)) + \frac{1}{g} + \mathcal{D}_{2x}(A^{y'}(x)) \\
&\quad \gtrapprox \mathcal{D}_x(A_0(x)) + \tfrac{2}{3}\mathcal{D}_x(A_0(x)) + \tfrac{1}{6}\mathcal{D}_x(A_0(x)) \\
&\quad = \tfrac{11}{6}\mathcal{D}_x(A_0(x)) \gg \tfrac{5}{3}\mathcal{D}_x(A_0(x)),
\end{aligned}$$

which again implies (d) of Theorem 1.5.

Suppose $|F_{A_0} + F_{A_0}| > 4$ in $\mathbb{Z}/g\mathbb{Z}$. Then there are $y_1, y_2 \in F_{A_0} \setminus \{y_0\}$ such that $\mathcal{D}_x(A_0^{y_1}(x) + A_0^{y_2}(x)) \lessapprox \frac{1}{3}\mathcal{D}_x(A_0(x))$ by (3). Hence $\frac{1}{g} \gtrapprox \mathcal{D}_x(A_0^{y_0}(x)) \gtrapprox \frac{2}{3}\mathcal{D}_x(A_0(x))$. Now the same argument after (4) can be used to derive (d) of Theorem 1.5.

Case 1.5.5 $k = 2$.

In this case k_0 must be 2. If $|F_{A_0} + F_{A_0}| = 3$, then (b) of Theorem 1.5 is true. So we can assume $|F_{A_0} + F_{A_0}| = 2$. Hence by Lemma 2.1 we have $g = 2 = k$. Now (d) of Theorem 1.5 follows from Claim 1.5.5 or (a) of Theorem 1.5 is true.

Case 1.5.6 $k = 1$.

If $g > 1$, then (a) of Theorem 1.5 is true. So we can assume $g = 1$, i.e., $\gcd(B - m_B) = 1$. We first prove a few claims and then further divide this case into two subcases. Although in the following claims we only deal with (A, B), the claims are still true if (A, B) is replaced by (A', B') where $(A', B') = \mathcal{E}(A, B)$ for some sequence $\mathcal{E}$ of e-transforms.

Claim 1.5.5.1 *We can assume that* $\mathcal{D}_x(A(x)) \lessapprox \frac{4}{3}\mathcal{D}_x(A_0(x))$ *and, consequently,* $\mathcal{D}_x(B(x)) \gtrapprox \frac{2}{3}\mathcal{D}_x(A_0(x))$ *for all sufficiently large* $x \in B$.

Proof of Claim 1.5.5.1 Suppose there are sufficiently large $x \in U$ such that $\mathcal{D}_x(A(x) + B(x)) \gg 1$ we want to show that for any $n \in \mathbb{N}$ there is a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and A' contains n consecutive integers, which implies (d) of Theorem 1.5 by Lemma 2.10.

Suppose m is the maximum integer such that there is a sequence $\mathcal{E}$ and $(A', B') = \mathcal{E}(A, B)$ such that A' contains m consecutive integers. Since $B \subseteq A$, then $\mathcal{D}_x(A(x)) \gg \frac{1}{2}$ for some sufficiently large $x \in B$. This implies that A contains two consecutive integers. So $m \geqslant 2$. Since A' does not contain $m + 1$ consecutive integers, we have that $\mathcal{D}_x(A'(x)) \lessapprox \frac{m}{m+1}$ for all sufficiently large $x \in U$. This implies that $\mathcal{D}_x(B'(x)) \gg \frac{1}{m+1}$ for some sufficiently large $x \in U$. This implies that B contain two numbers $b_1 < b_2$ such that $b_2 - b_1 < m + 1$. Let $a, a + 1, \ldots, a + m - 1 \in A'$ and let $c = a + m - (b_2 - b_1) - b_1$. Since $b_1 \in B'$ and $0 \leqslant m - (b_2 - b_1) \leqslant m - 1$ implies $a + m - (b_2 - b_1) \in A'$, the e_c is a legitimate e-transform to (A', B'). Let $(A'', B'') = e_c(A', B')$. Then $a + m \in A''$. Hence A'' contains $m + 1$ consecutive integers $\{a, a + 1, \ldots, a + m\}$, which contradicts the maximality of m.

So we can assume that for all sufficiently large $x \in U$, $\mathcal{D}_x(A(x) + B(x)) \lessapprox 1$. For all sufficiently large $x \in B$

$$\mathcal{D}_{2x}((A + B)(2x)) \gtrapprox \mathcal{D}_{2x}(2A(x) + B(x)) \approx \mathcal{D}_x(A_0(x)) + \tfrac{1}{2}\mathcal{D}_x(A(x))$$

by Lemma 2.4. If there are sufficiently large $x \in B$ such that $\mathcal{D}_x(A(x)) \gg \frac{4}{3}\mathcal{D}_x(A_0(x))$, then (d) of Theorem 1.5 is true. So we can assume that for all sufficiently large $x \in B$, $\mathcal{D}_x(A(x)) \lessapprox \frac{4}{3}\mathcal{D}_x(A_0(x))$. This clearly implies $\mathcal{D}_x(B(x)) \gtrapprox \frac{2}{3}\mathcal{D}_x(A_0(x))$. □

Claim 1.5.5.2 *If there are $a_1, a_2, a_3 \in A$ such that*

$$\mathcal{D}_x((a_i + B) \cap (a_j + B))(x)) \ll \tfrac{2}{9}\mathcal{D}_x(A_0(x))$$

for all $i < j$ in $\{1, 2, 3\}$ and for all sufficiently large $x \in U$, then (d) of Theorem 1.5 is true.

Proof of Claim 1.5.5.2 Let $a_1, a_2, a_3 \in A$ be such that

$$\mathcal{D}_x((a_i + B) \cap (a_j + B))(x)) \ll \tfrac{2}{9}\mathcal{D}_x(A_0(x))$$

for all $i < j$ in $\{1, 2, 3\}$ and for all sufficiently large $x \in U$. Let $b \in B$. Then we can obtain (A', B') by applying three e-transforms $e_{a_i - b}$ to (A, B) so that $a_i - b + B' \subseteq A'$ for $i = 1, 2, 3$. By the inclusion–exclusion principle we have $\mathcal{D}_x(A'(x)) \gg 3\mathcal{D}_x(B'(x)) - \frac{2}{3}\mathcal{D}_x(A_0(x)) \gtrapprox \frac{4}{3}\mathcal{D}_x(A_0(x))$ for all sufficiently large $x \in B'$, which contradicts Claim 1.5.5.1. □

Claim 1.5.5.3 *We can assume that* $\gcd((B - b) \cap U) = 1$ *for any* $b \in B$.

Proof of Claim 1.5.5.3 Suppose $b \in B$ such that $d = \gcd((B - b) \cap U) > 1$. Without loss of generality we can assume that for any $\mathcal{E}$ with $(A', B') = \mathcal{E}(A, B)$ and any $b \in B'$ we have $\gcd((B' - b) \cap U) = d$. Let $\bar{B} = B \smallsetminus [0, b - 1]$. Since $\gcd(A) = 1$, then there are $a_1, a_2 \in A$ such that $(a_1 + \bar{B}) \cap (a_2 + \bar{B}) = \emptyset$. If A is not a subset of $\{a_1, a_2\} + d \cdot {}^*\mathbb{Z}$, then (d) of Theorem 1.5 is true by Claim 1.5.5.2. Hence we can assume $A \subseteq \{a_1, a_2\} + d \cdot {}^*\mathbb{Z}$.

Since $g = 1$, there is $b' \in B$ such that $b' \not\equiv b \pmod{d}$. If the set $\{a_1, a_2\} + \{b, b'\}$ mod d contains more than two elements, then we can apply one e-transform to (A, B) to obtain (A', B') such that there are three elements a'_1, a'_2, a'_3 in A' such that $(a'_i + B') \cap (a'_j + B')$ for $i < j$ are all bounded in U. Hence (d) of Theorem 1.5 is true by Claim 1.5.5.2. So we can assume that the set $\{a_1, a_2\} + \{b, b'\}$ mod d contains exactly two elements. By Lemma 2.1 we have that A is a subset of an *a.p.* of difference $\frac{d}{2}$. Since (a) of Theorem 1.5 is not true, then $d = 2$. Now we can prove (d) of Theorem 1.5 by the same reason as in the proof of Claim 1.5.5 with g replaced by d and B replaced by $\bar{B}$. □

Claim 1.5.5.4 *Suppose U' is an initial segment of $^*\mathbb{N}$ and $\{x_i \in U : i \in U'\}$ is a strictly increasing U-internal sequence unbounded in U. Note that U' is external. If for each $i \in U'$ either*

$$\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1} - 1) + B(x_i, x_{i+1} - 1)) \approx 0 \tag{5}$$

or

$$\mathcal{D}_{x_{i+1}-x_i}((A + B)(2x_i, 2x_{i+1}-1)) \gg \tfrac{5}{3}\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)), \tag{6}$$

then (d) of Theorem 1.5 is true.

Proof of Claim 1.5.5.4 Fix a sufficiently large x_j for some $j \in U'$. For each $n \in \mathbb{N}$ let

$$\begin{aligned}
I_n &= \left\{ i < j : \mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1} - 1) + B(x_i, x_{i+1} - 1)) < \frac{1}{n} \right\},\\
J_n &= \left\{ i < j : \mathcal{D}_{x_{i+1}-x_i}((A + B)(2x_i, 2x_{i+1} - 1)) \right.\\
&\qquad \left. > \left(\frac{5}{3} + \frac{1}{n}\right) \mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1} - 1) + B(x_i, x_{i+1} - 1)) \right\},\\
X_n &= \bigcup \{[x_i, x_{i+1} - 1] : i \in I_n\}, \text{ and}\\
Y_n &= \bigcup \{[x_i, x_{i+1} - 1] : i \in J_n\}.
\end{aligned}$$

Note that $X_n \supseteq X_{n+1}$ and $Y_n \subseteq Y_{n+1}$ for every $n \in \mathbb{N}$.

It is easy to see by the assumption of the claim that $[0, x_j - 1] = (\bigcap_{n\in\mathbb{N}} X_n) \cup (\bigcup_{n\in\mathbb{N}} Y_n)$. Since $\mathcal{D}_{x_j}(A(x_j) + B(x_j)) \gtrapprox 2\alpha > 0$ then there is $n \in \mathbb{N}$ such that $\mathcal{D}_{x_j}(|X_n|) < 1 - \epsilon$ for some $\epsilon > 0$. This implies that there is $m \in \mathbb{N}$ such that $X_n \cup Y_m = [0, x_j - 1]$, which implies $\mathcal{D}_{x_j}(|Y_m \setminus X_n|) \gg 0$ and $J_m \cup I_n = [0, j - 1]$. Let $k \in \mathbb{N}$ be large enough so that $k > n$ and

$$\frac{1}{k} \ll \frac{3}{5mn}\mathcal{D}_{x_j}(|Y_m \setminus X_n|).$$

For each $i \in J_m \setminus I_n$ we have

$$\begin{aligned}&\mathcal{D}_{x_{i+1}-x_i}((A+B)(2x_i, 2x_{i+1}-1))\\&\quad > \left(\frac{5}{3}+\frac{1}{m}\right)\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1))\end{aligned}$$

and

$$\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)) \geqslant \frac{1}{n}.$$

For each $i \in I_n \setminus I_k$ we have

$$\begin{aligned}&\mathcal{D}_{x_{i+1}-x_i}((A+B)(2x_i, 2x_{i+1}-1))\\&\quad > \tfrac{5}{3}\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)).\end{aligned}$$

And for each $i \in I_k$ we have

$$\mathcal{D}_{x_{i+1}-x_i}(A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)) < \frac{1}{k}.$$

Then

$$\begin{aligned}&\mathcal{D}_{2x_j}((A_0+A_0)(2x_j)) \gtrapprox \mathcal{D}_{2x_j}((A+B)(2x_j))\\&\gtrapprox \frac{1}{2x_j}\left(\sum_{i\in J_m\setminus I_n}(A+B)(2x_i, 2x_{i+1}-1)+\sum_{i\in I_n\setminus I_k}(A+B)(2x_i, 2x_{i+1}-1)\right)\\&\gtrapprox \frac{5}{3}\mathcal{D}_{2x_j}(A(x_j)+B(x_j))+\frac{1}{m}\cdot\mathcal{D}_{2x_j}\left(\sum_{i\in J_m\setminus I_n}A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)\right)\\&\quad -\frac{5}{3}\mathcal{D}_{2x_j}\left(\sum_{i\in I_k}A(x_i, x_{i+1}-1)+B(x_i, x_{i+1}-1)\right)\\&\gtrapprox \frac{5}{3}\mathcal{D}_{2x_j}(A(x_j)+B(x_j))+\frac{1}{2mn}\mathcal{D}_{x_j}(|Y_m\setminus X_n|)-\frac{5}{6k}\\&\gg \tfrac{5}{6}\mathcal{D}_{x_j}(A(x_j)+B(x_j)) \approx \tfrac{5}{3}\mathcal{D}_{x_j}(A_0(x_j)),\end{aligned}$$

by the choice of k. Hence (d) of Theorem 1.5 is true.

Note that $g = 1$. If $f = 1$, then we can find $\mathcal{E}$ and $(A', B') = \mathcal{E}(A, B)$ such that A' contains n consecutive integers for any given $n \in \mathbb{N}$, which implies (d) of Theorem 1.5. So we can assume $f > 1$.

For each $C \subseteq U$ let

$$h(C) = \max\{|C \cap [x, x+f-1]| : x \in U\}.$$

For any sequence $\mathcal{E}$ of e-transforms and $(A', B') = \mathcal{E}(A, B)$, if $h(A') \geqslant 3$, then (d) of Theorem 1.5 is true by Claim 1.5.5.2. So we can always assume $h(A') = 1$ or 2.

Subcase 1.5.5.1 There is a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and $h(A') = 2$.

Without loss of generality we can assume $h(A) = 2$. We can also assume $0, a, f, f + a \in A$ for some $a \in [1, f-1]$, $0, f \in B$, and $B \cup (a+B) \cup (f+B) \cup (f+a+B) \subseteq A$ after applying a few e-transforms and replacing B by $B - b$ for some b such that $b, b + f \in B$. Since $B \cap (a + B) = \emptyset$, we have that $\mathcal{D}_x(A(x)) \gtrapprox 2\mathcal{D}_x(B(x))$ for all sufficiently large $x \in U$, which implies $\mathcal{D}_x(A(x)) \approx 2\mathcal{D}_x(B(x))$ by Claim 1.5.5.1. So for simplicity we can assume that $A = B \cup (a + B) \cup (f + B) \cup (f + a + B)$.

If $f = 2$, then $a = 1$. Hence A contains two consecutive integers and B' contains two integers $b, b+2$ for any B' such that (A', B') is obtained by applying a sequence of e-transforms to (A, B). This implies that for any $n \in \mathbb{N}$ we can find $\mathcal{E}$ such that $(A', B') = \mathcal{E}(A, B)$ and A' contains n consecutive integers. Hence (d) of Theorem 1.5 is true by Lemma 2.10. So we can assume $f > 2$.

It suffices to construct a strictly increasing U-internal sequence $S = \{x_i \in B : i \in U'\}$ unbounded in U satisfying the conditions (5) or (6) for each $i \in U'$ by Claim 1.5.5.4.

Let $x_0 = 0$. Suppose $\{i \in B : i \in [0, j]\}$ has been found so that for each $i < j$ either (5) or (6) is true.

If $2a = f$, then let

$$x_{j+1} = \min\{x \in B : x > x_j \text{ and } x \not\equiv x_j \pmod{a}\}. \tag{7}$$

Since $a > 1$, x_{j+1} is well defined by Claim 1.5.5.3. If $x_{j+1} - x_j \in \mathbb{N}$, then we can find $\mathcal{E}$ and $(A', B') = \mathcal{E}(A, B)$ such that $h(A') \geqslant 3$ by the following process. Note that $x_j, x_j + a \in A$. Choose $b, b + f \in B$. Let $c = x_j - b$ and $(A_1, B_1) = e_c(A, B)$. Then $x_j + a, x_j + f \in A_1$. Choose $b', b' + f \in B_1$. Let $c' = x_j + a - b'$ and $(A_2, B_2) = e_{c'}(A_1, B_1)$, then $x_j + f, x_j + f + a \in A_2$. If we continue this n many times, we can obtain (A_{2n}, B_{2n}) such that A_{2n} contains $x_j + nf, x_j + nf + a$. Since $x_{j+1} - x_j \in \mathbb{N}$, we can find $n \in \mathbb{N}$ such that $x_{j+1} \in [x_j + nf, x_j + (n+1)f - 1]$. Hence $h(A_{2n}) \geqslant 3$ by (7). So we can assume that $x_{j+1} - x_j$ is hyperfinite.

Suppose (5) is not true for $i = j$. Then

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}((A + B)(2x_j, 2x_j - 1)) \\
&\quad \gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1} - 1] + B[x_j, x_{j+1} - 1]| + |A[x_j, x_{j+1} - 1] + x_{j+1}|) \\
&\quad \gtrapprox 2\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1))
\end{aligned}$$

by (7). Hence (6) is true for $i = j$.

If $2a \neq f$, then let

$$x_{j+1} = \min\{x \in B : x > x_j \text{ and } x \not\equiv x_j \pmod{f}\}. \tag{8}$$

Let $y = x_{j+1}$ if $x_{j+1} \not\equiv x_j + a \pmod f$ and $y = x_{j+1} + a$ otherwise. Note that $y \in A$ and $y \not\equiv x_j, x_j + a \pmod f$. If $x_{j+1} - x_j \in \mathbb{N}$, then we can find $\mathcal{E}$ and $(A', B') = \mathcal{E}(A, B)$ such that $h(A') \geqslant 3$. So we can assume that $x_{j+1} - x_j$ is hyperfinite.

Suppose (5) is not true. Let $z = \max(B[x_j, x_{j+1} - 1])$. Note that $\gcd(B[x_j, z] - x_j) = mf$ for some positive integer $m \in \mathbb{N}$. If $m > 1$, then $B[x_j, z]$, $a + B[x_j, z]$, $f + B[x_j, z]$, and $f + a + B[x_j, z]$ are pairwise disjoint subsets of $A[x_j, z + f + a]$. So $\mathcal{D}_{x_{j+1}-x_j}(A(x_j, z)) \gtrapprox 4\mathcal{D}_{x_{j+1}-x_j}(B(x_j, z))$. Hence

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}((A + B)(2x_j, 2x_{j+1} - 1)) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A[x_j, z] + x_j| + |A[x_j, z] + z| + |y + B[x_j, z]|) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}((11A(x_j, x_{j+1} - 1)/6) + (5B(x_j, x_{j+1} - 1)/3)) \\
&\quad\gg \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).
\end{aligned}$$

By the argument above we can assume $m = 1$. Let $A_1 = A[x_j, x_{j+1} - 1] \cap (x_j + fU)$, $A_2 = A[x_j, x_{j+1} - 1] \cap (x_j + a + fU)$, and $B_0 = B[x_j, x_{j+1} - 1]$. Clearly, we have

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1}] + B[x_j, x_{j+1}]|) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A_1 + B_0| + |A_2 + B_0| + |y + B_0|) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2|A_1| + 2|A_2| + |B_0|) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(5A(x_j, x_{j+1} - 1)/3 + 5B(x_j, x_{j+1} - 1)/3) \\
&\quad= \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).
\end{aligned}$$

If (6) for $i = j$ is not true, then we have

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1}] + B[x_j, x_{j+1}]|) \\
&\quad\lessapprox \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)),
\end{aligned}$$

which implies $\mathcal{D}_{x_{j+1}-x_j}(|A_1 + B_0|) \approx 2\mathcal{D}_{x_{j+1}-x_j}(|A_1|)$ and $\mathcal{D}_{x_{j+1}-x_j}(|A_2 + B_0|) \approx 2\mathcal{D}_{x_{j+1}-x_j}(|A_2|)$. By Lemma 2.4 and $m = 1$ we have that $\mathcal{D}_{x_{j+1}-x_j}(|A_i|) \approx \frac{1}{f}$ for $i = 1, 2$ and $\mathcal{D}_{x_{j+1}-x_j}(|B_0|) \approx \frac{1}{f}$.

Suppose $\frac{x_{j+1}-z}{x_{j+1}-x_j} \gg 0$. If $y = x_{j+1}$, then

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1}] + B[x_j, x_{j+1}]|) \\
&\quad\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2|A_1| + 2|A_2| + |A_2 + x_{j+1}| + |A_1[2z - x_{j+1}, z] + x_{j+1}|) \\
&\quad\gg \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).
\end{aligned}$$

If $y = x_{j+1} + a$, then

$$\begin{aligned}&\mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1}] + B[x_j, x_{j+1}]|)\\&\quad \gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2|A_1| + 2|A_2| + |y + B_0| + |A_1[2z - x_{j+1}, z] + x_{j+1}|)\\&\quad \gg \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).\end{aligned}$$

So we can assume $\frac{x_{j+1}-z}{x_{j+1}-x_j} \approx 0$.

Let $\bar{x} = \max\{x < y : x \equiv x_j \pmod f\}$. Note that $\bar{x} < y < \bar{x} + f$ and $y \not\equiv x_j + a \pmod f$. We want to find a sequence $\mathcal{E}$ of e-transforms with $(A'', B'') = \mathcal{E}(A, B)$ such that $\bar{x}, \bar{x} + a, y \in A''$, which implies $h(A'') \geqslant 3$.

Let $u = \min\{x \in A_1 :$ there is $v \in B_0$ such that $u + v - x_j = \bar{x}\}$. Then $\frac{u-x_j}{x_{j+1}-x_j} \approx 0$ due to the fact that $\mathcal{D}_{x_{j+1}-x_j}(|A_1|) \approx \frac{1}{f} \approx \mathcal{D}_{x_{j+1}-x_j}(|B_0|)$. Let $(A', B') = e_{u-x_j}(A, B)$. Then $\bar{x} \in A'$ and $x_j \in B'$. Since $B' = B \cap (A - u + x_j)$, $B \subseteq (x_j + fU)$, and $\mathcal{D}_{x_{j+1}-x_j}(|A_1|) \approx \frac{1}{f}$, then $\mathcal{D}_{x_{j+1}-x_j}(B'(x_j, x_{j+1} - 1)) \approx \mathcal{D}_{x_{j+1}-x_j}(B(x_j, x_{j+1} - 1))$. So now we can find $u' \in A_2$ and $v' \in B'$ such that $\bar{x} + a = u' + v' - x_j$. Let $(A'', B'') = e_{u'-x_j}(A', B')$. Then $\bar{x}, \bar{x} + a, y \in A''$. Hence $h(A'') \geqslant 3$. This ends the proof of Subcase 1.5.5.1.

Subcase 1.5.5.2 For any sequence $\mathcal{E}$ of e-transforms with $(A', B') = \mathcal{E}(A, B)$ we have $h(A') = 1$.

Note that if $a_1 < a_2$ in A such that $a_2 - a_1 \in \mathbb{N}$ and $a_2 - a_1 \not\equiv 0 \pmod f$, then there is $\mathcal{E}$ such that $(A', B') = \mathcal{E}(A, B)$ and $h(A') \geqslant 2$. This is true by the following. Let $b, b+f \in B$ and $(A_1, B_1) = e_{a_1-b}(A, B)$. Then $a_1 + f, a_2 \in A_1$ and $f(B_1) = f$. After a few e-transforms we can have $(A_n, B_n) = \mathcal{E}(A, B)$ such that $a_1 + nf, a_2 \in A_n$ and $a_1 + nf < a_2 < a_1 + (n + 1)f$. So $h(A_n) \geqslant 2$. Before going further we need to prove a claim first.

Claim 1.5.5.2.1 *If there are $u < v < w$ in A and $m \in \mathbb{N}\setminus\{0\}$ such that $u, v \in B$, $w - u$ is hyperfinite, $w - u \not\equiv 0 \pmod f$, $\frac{v-u}{w-u} \gg 0$, $A_1 = A[u, v] \cap (u + mfU)$, $\gcd(A_1 - u) = \gcd(B[u, v] - u) = mf$, and $\mathcal{D}_{v-u}(|A_1| + B(u, v)) \gg \frac{1}{mf}$, then there is a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and $h(A') \geqslant 2$.*

Proof of Claim 1.5.5.2.1 Suppose $\frac{w-v}{w-u} \approx 0$. Let $\bar{x} = \max\{x < w : x \equiv u \pmod{mf}\}$. Then $\bar{x} < w < \bar{x} + mf$. Since $(\bar{x} - A_1) \cap (B[u, \bar{x}] - u) \neq \emptyset$, there are $a \in A_1$ and $b \in B[u, \bar{x}]$ such that $\bar{x} = a + b - u$. Let $(A', B') = e_{a-u}(A, B)$. Then $\bar{x} = a - u + b \in A'$. This implies $h(A'') \geqslant 2$ for some A'' obtained by applying a sequence e-transforms to (A', B'). So we can assume $\frac{w-v}{w-u} \gg 0$.

Let $\epsilon > 0$ be such that $\mathcal{D}_{v-u}(|A_1| + B(u, v)) \gg \frac{1}{mf} + \epsilon$. We want to find $b_1 < b_2 < b_3$ in $B[u, v]$ such that $\min\{\frac{b_2-b_1}{v-u}, \frac{b_3-b_2}{v-u}\} \gg 0$, $\frac{b_3-b_1}{u-v} < \frac{\epsilon}{2}$, $\mathcal{D}_{b_2-b_1}(A_1(b_1, b_2) + B(b_1, b_2)) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$, and $\mathcal{D}_{b_3-b_2}(A_1(b_2, b_3) + B(b_2, b_3)) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$ by the following.

Let $u_1, v_1 \in [u, v]$ be such that $0 \ll \frac{v_1-u_1}{v-u} < \frac{\epsilon}{2}$ and $\mathcal{D}_{v_1-u_1}(A_1(u_1, v_1) + B(u_1, v_1)) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$. By increasing u_1 and decreasing v_1 if necessary we can assume $u_1, v_1 \in B[u, v]$ because $A_1 \subseteq (u + mfU)$. Let $r = \left[\frac{u_1+v_1}{2}\right]$. If $\mathcal{D}_{r-u_1}(A_1(u_1, r) +$

$B(u_1, r)) < \frac{1}{mf} + \frac{\epsilon}{2}$, then let $u_2 = \min B[r, v_1]$ and $v_2 = v_1$. If $\mathcal{D}_{r-u_1}(A_1(u_1, r) + B(u_1, r)) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$, let $s = \max B[u_1, r]$. If $\mathcal{D}_{v_1-s}(A_1(s, v_1) + B(s, v_1)) < \frac{1}{mf} + \frac{\epsilon}{2}$, then let $u_2 = u_1$ and $v_2 = s$. If $\mathcal{D}_{v_1-s}(A_1(s, v_1) + B(s, v_1)) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$, then we stop and let $b_1 = u_1$, $b_2 = s$, and $b_3 = v_1$.

Repeat the steps above with u_1, v_1 being replaced by u_2, v_2 if the steps do not stop. Then we can have $[u_1, v_1] \supseteq [u_2, v_2] \supseteq \cdots$. This sequence of intervals will eventually stop because otherwise we would have $\mathcal{D}_{v_1-u_1}(A_1(u_1, v_1) + B(u_1, v_1)) \ll \frac{1}{mf} + \epsilon$, a contradiction to the choice of u_1, v_1.

By applying three e-transforms we can assume that $b_i - u + B \subseteq A$ for $i = 1, 2, 3$. By Claim 1.5.5.2 we can find $i < j$ in $\{1, 2, 3\}$ and sufficiently large x such that

$$\mathcal{D}_x(((b_i - u + B) \cap (b_j - u + B))(x)) \gtrapprox \frac{2}{9}\mathcal{D}_x(A_0(x)) \gtrapprox \frac{2}{9}\alpha.$$

Without loss of generality let $i = 1$ and $j = 2$. The above argument shows that for any $\mathcal{E}$ and $(A', B') = \mathcal{E}(A, B)$ there are always $b, b' \in B'$ such that $b' - b = b_2 - b_1$.

Let $k = b_2 - b_1$. Note that k is a multiple of mf. Since $\frac{k}{v-u} \gg 0$, we have $\frac{k}{w-u} = \frac{k}{v-u} \cdot \frac{v-u}{w-u} \gg 0$. Hence there is $n \in \mathbb{N}$ such that $b_1 + nk < w < b_1 + (n+1)k$. Let

$$\bar{A} = \bigcup_{i=0}^{n-1}(ik + A_1[b_1, b_2 - 1]),$$
$$\bar{B} = \bigcup_{i=0}^{n-1}(ik + B_1[b_1, b_2 - 1]),$$

and $\bar{x} = \max\{x < w : x \equiv u \pmod{mf}\}$. Since $\mathcal{D}_{nk}(|\bar{A}| + |\bar{B}|) \geqslant \frac{1}{mf} + \frac{\epsilon}{2}$, then there are $i_1, i_2 \in [0, n-1]$, $a \in A_1[b_1, b_2 - 1]$, and $b \in B[b_1, b_2 - 1]$ such that

$$\bar{x} = i_1 k + a + i_2 k + b - b_1.$$

Now we need to find a sequence $\mathcal{E}$ of e-transforms such that $(A', B') = \mathcal{E}(A, B)$ and $\bar{x} \in A'$.

Let $(A_1, B_1) = e_{a-b_1}(A, B)$. Then $a + b - b_1 \in A_1$. Since there are $b', b' + k \in B_1$ we can let $(A_2, B_2) = e_{a+b-b_1-b'}(A_1, B_1)$. Then $a + b - b_1 + k \in A_2$. Now by doing this $i_1 + i_2 - 1$ more times we can get $A_{i_1+i_2+1}$ with $\bar{x} = a + b - b_1 + (i_1 + i_2)k \in A_{i_1+i_2+1}$. Since $w - \bar{x} \in \mathbb{N}$ and $w \not\equiv \bar{x} \pmod{f}$, we can find $(A', B') = \mathcal{E}(A_{i_1+i_2+1}, B_{i_1+i_2+1})$ such that $h(A') \geqslant 2$. □

We are now ready to construct a strictly increasing U-internal sequence $S = \{x_i \in B : i \in U'\}$ unbounded in U satisfying (5) or (6) for each $i \in U'$. Without loss of generality we can assume $0, f \in B$ and $B \cup (f + B) \subseteq A$.

Let $x_0 = 0$. Suppose $\{i \in B : i \in [0, j]\}$ has been found so that for each $i < j$ either (5) or (6) is true.

Let $y = \min\{x \in A : x > x_j \text{ and } x \not\equiv x_j \pmod f\}$. If $y - x_j \in \mathbb{N}$, then there is $\mathcal{E}$ such that $(A', B') = \mathcal{E}(A, B)$ and $h(A') \geqslant 2$. So we can assume $y - x_j$ is hyperfinite. Let $z = \max B[0, y]$ and

$$x_{j+1} = \min\{x \in B : x \geqslant y\}.$$

Note that $B[z+1, x_{j+1} - 1] = \emptyset$. Without loss of generality we can assume that (5) for $i = j$ is not true. We need to show that (6) is true for $i = j$.

If $\mathcal{D}_{x_{j+1}-x_j}(B(x_j, z)) \approx 0$, then

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}(|A[x_j, x_{j+1}] + B[x_j, x_{j+1}]|) \\
&\quad \gtrapprox 2\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1)) \\
&\quad \gtrapprox \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).
\end{aligned}$$

So we can assume $\mathcal{D}_{x_{j+1}-x_j}(B(x, z)) \gg 0$.

Clearly, we have that there is a finite positive integer m such that $\gcd(B[x_j, z] - x_j) = mf$. Suppose $m > 1$. For each $k = 0, 1, \ldots, m-1$ let $A_k = A[x_j, z] \cap (x_j + kf + mfU)$ and $B_0 = B[x_j, z]$. Note that $x_j, z \in A_0$ and $x_j + f, z + f \in A_1$. If there are distinct $k_1, k_2 \in [0, m-1]$ such that $\mathcal{D}_{x_{j+1}-x_j}(|A_{k_i}| + |B_0|) \lessapprox \frac{1}{mf}$, then by Lemma 2.4

$$\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}((A + B)(2x_j, 2x_{j+1} - 1)) \\
&\quad \gtrapprox \mathcal{D}_{x_{j+1}-x_j}(2A(x_j, z) + 2B(x_j, z) + |A[z, x_{j+1} - 1] + z| \\
&\qquad + |A[z, x_{j+1} - 1] + x_{j+1}|) \\
&\quad \approx \mathcal{D}_{x_{j+1}-x_j}(2A(x_j, x_{j+1} - 1) + 2B(x_j, x_{j+1} - 1)) \\
&\quad \gg \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j, x_{j+1} - 1) + B(x_j, x_{j+1} - 1)).
\end{aligned}$$

So we can assume that there is $k \in [0, m-1]$ such that

$$\mathcal{D}_{z-x_j}(|A_k| + |B_0|) \gg \frac{1}{mf}.$$

But this contradicts that $h(A') = 1$ for any $\mathcal{E}$ with $(A', B') = \mathcal{E}(A, B)$ by Claim 1.5.5.2.1.

Suppose $m = 1$. Let $A_1 = A[x_j, x_{j+1} - 1] \cap (x_j + fU)$ and $A_2 = A[x_j, x_{j+1} - 1] \smallsetminus A_1$. If $\mathcal{D}_{x_{j+1}-x_j}(|A_1| + B(x_j, z)) \lessapprox \frac{1}{f}$, then

$$\mathcal{D}_{x_{j+1}-x_j}(|A_1 + B[x_j, z]|) \gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A_1| + 2B(x_j, z)).$$

Hence

$$
\begin{aligned}
&\mathcal{D}_{x_{j+1}-x_j}((A+B)(2x_j, 2x_{j+1}-1))\\
&\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A_1+B[x_j,z]|+|A_2+x_j|+|A[x_j,x_{j+1}-1]+x_{j+1}|)\\
&\gtrapprox \mathcal{D}_{x_{j+1}-x_j}(|A_1|+2B(x_j,z)+|A_2|+A(x_j,x_{j+1}-1))\\
&\approx 2\mathcal{D}_{x_{j+1}-x_j}(A(x_j,x_{j+1}-1)+B(x_j,x_{j+1}-1))\\
&\gg \tfrac{5}{3}\mathcal{D}_{x_{j+1}-x_j}(A(x_j,x_{j+1}-1)+B(x_j,x_{j+1}-1)).
\end{aligned}
$$

So we can assume $\mathcal{D}_{x_{j+1}-x_j}(|A_1|+B(x_j,z)) \gg \frac{1}{f}$. But this implies $\mathcal{D}_{z-x_j}(A_1(x_j,z)+B(x_j,z)) \gg \frac{1}{f}$, which contradicts $h(A')=1$ for any sequence $\mathcal{E}$ of e-transforms with $(A', B') = \mathcal{E}(A, B)$ by Claim 1.5.5.2.1. This ends the proof of Theorem 1.5. □

3 Further improvement

In this section we prove a better version of Theorem 1.5 by specifying explicitly the meaning of the word "large" for the set A_0 mentioned in the abstract when (a), (b), or (c) of Theorem 1.5 is true.

Theorem 3.1 *Let U be a cut with uncountable cofinality and $A_0 \subseteq U$ be U-internal. Suppose $0 \in A_0$ and $0 < \underline{d}_U(A_0) = \alpha < \frac{3}{5}$. If for all sufficiently large $x \in A_0$ we have*

$$\mathcal{D}_{2x}((A_0+A_0)(2x)) \lessapprox \tfrac{5}{3}\mathcal{D}_x(A_0(x)), \tag{9}$$

then one of the following must be true:

(a) *A_0 is a subset of an* a.p. *of difference $g > 1$ and $\alpha \geqslant \frac{3}{5g}$.*

(b) *A_0 is a subset of a* b.p. *of the form $\{0, a\}+gU$ with some $g > 2$ and $a \in [1, g-1]$ such that $\alpha \geqslant \frac{2}{(1+\beta)g}$ where*

$$
\begin{aligned}
\beta = \inf\{\gamma :\ &\text{for all sufficiently large } x \in A_0\\
&\mathcal{D}_x((A_0+A_0)(2x)) \lessapprox (3+\gamma)\mathcal{D}_x(A_0(x))\}.
\end{aligned}
$$

(c) *A_0 is a subset of a* t.p. *of the form $\{0, a_1, a_2\}+gU$ with some $g > 4$ and $a_1 < a_2$ in $[1, g-1]$ such that $\alpha = \frac{3}{g}$.*

Note that $0 \leqslant \beta \leqslant \frac{1}{3}$ in (b) of Theorem 3.1. So if (b) of Theorem 3.1 is true, then both $A_0 \cap gU$ and $A_0 \cap (a+gU)$ must be unbounded in U. If (c) of Theorem 3.1 is true, then for all sufficiently large $x \in U$, $\mathcal{D}_x((A_0 \cap gU)(x)) \approx \frac{1}{g}$, $\mathcal{D}_x((A \cap (a_1+gU))(x)) \approx \frac{1}{g}$, and $\mathcal{D}_x((A_0 \cap (a_2+gU))(x)) \approx \frac{1}{g}$.

Proof of Theorem 3.1 Suppose A_0 satisfies (9). Hence (d) of Theorem 1.5 is not true.

Case 3.1.1 A_0 satisfies (a) but does not satisfy (b) or (c) of Theorem 1.5.

Assume $g = \gcd(A_0)$. Let $A' = \{\frac{a}{g} : a \in A_0\}$. Note that $\gcd(A') = 1$. Suppose $0 < \underline{d}_U(A') < \frac{3}{5}$. Then A' does not satisfy any of (a), (b), (c), or (d) of Theorem 1.5. Hence $\underline{d}_U(A') \geqslant \frac{3}{5}$, which implies $\underline{d}_U(A_0) \geqslant \frac{3}{5g}$.

Case 3.1.2 A_0 satisfies (b) but does not satisfy (c) of Theorem 1.5.

Suppose $A_0 \subseteq \{0, a\} + gU$ for some $a \in [1, g-1]$ with $2a \not\equiv g \pmod g$. Let $A_1 = A_0 \cap gU$ and $A_2 = A_0 \cap (a + gU)$. Without loss of generality we can assume that $g = \gcd(A_1 \cup (A_2 - \min A_2))$, $A_1 \neq \emptyset$, and $A_2 \neq \emptyset$.

Suppose there are sufficiently large $x \in U$ and $i \in \{1, 2\}$ such that $\mathcal{D}_x(A_i(x)) \ll \frac{1}{3}\mathcal{D}_x(A_0(x))$. Without loss of generality we can assume $i = 1$ and $x \in A_1$. Let $a_1 = \min A_1$, $a_2 = \min A_2$, and $(A, B) = e_{a_1 - a_2}(A_0, A_0)$. Then $\mathcal{D}_x(B(x)) \ll \frac{1}{3}\mathcal{D}_x(A_0(x))$, which implies (d) of Theorem 1.5. So we can assume $\mathcal{D}_x(A_i(x)) \gtrapprox \frac{1}{3}(A_0(x))$ for $i = 1, 2$ and for all sufficiently large $x \in U$.

Let $g_1 = \gcd(A_1)$ and $g_2 = \gcd(A_2 - a_2)$. Suppose $g_1 > g$. Then we have $\mathcal{D}_x(|A_1[0, x] + A_2[0, x]|) \gtrapprox \mathcal{D}_x(2A_1(x) + A_2(x))$ for all sufficiently large $x \in U$. Hence

$$\begin{aligned}
&\mathcal{D}_{2x}((A_0 + A_0)(2x))\\
&\quad \gtrapprox \mathcal{D}_{2x}(|A_1[0, x] + A_1[0, x]| + |A_1[0, x] + A_2[0, x]| + |A_2[0, x] + A_2[0, x]|)\\
&\quad \gtrapprox \mathcal{D}_{2x}(4A_1(x) + 3A_2(x)) = \mathcal{D}_{2x}(3A_0(x) + A_1(x)) \gtrapprox \tfrac{5}{3}\mathcal{D}_x(A_0(x)).
\end{aligned}$$

By (9) we have $\mathcal{D}_x(|A_1[0, x] + A_1[0, x]|) \approx 2\mathcal{D}_x(A_1(x))$ and $\mathcal{D}_x(|A_2[0, x] + A_2[0, x]|) \approx 2\mathcal{D}_x(A_2(x))$. Let $x_i = \max A_i[0, x]$ for $i = 1, 2$. By Lemma 2.2 we can conclude that $\mathcal{D}_{x_i}(A_i(x_i)) \approx \frac{1}{g_i}$ for $i = 1, 2$. Suppose $x_1 < x_2$. Let $x_1' = \min\{z \in A_1 : z > x\}$. By the argument above with x replaced by x_1', we have $\mathcal{D}_{x_1'}(A_1(x_1')) \approx \frac{1}{g_1}$. Since $A_1[x_1 + 1, x_1' - 1] = \emptyset$, we have that $\frac{x - x_1}{x} \approx 0$. This also implies $\frac{x - x_2}{x} \approx 0$. If $x_2 < x_1$, the same conclusion can be derived by the same argument. So for all sufficiently large $x \in U$ and $x_i = \max A_i[0, x]$ we always have $\frac{x - x_i}{x} \approx 0$. By Lemma 2.3 we have that

$$\frac{2}{g} \lessapprox (1 + \beta)\mathcal{D}_x(A_0(x))$$

for all sufficiently large $x \in U$. Let $x \in U$ be sufficiently large such that $\mathcal{D}_x(A_0(x)) = \alpha$. Then we have $\alpha \geqslant \frac{2}{(1+\beta)g}$. Note that the same argument can be applied if $g_2 > g$.

We now assume $g_1 = g_2 = g$. Let $x \in \Gamma(A_0, A_0)$ be sufficiently large. Without loss of generality we assume $x \in A_1$. Let $x_2 = \max A_2[0, x]$ and $x_2' = \min\{z \in A_2 : z > x\}$. If $\frac{x_2' - x}{x_2'} \approx 0$, then by Lemma 2.3 we have $\frac{2}{g} \lessapprox (1 + \beta)\mathcal{D}_x(A_0(x))$, which implies (b) of Theorem 3.1. So we can assume $\frac{x_2' - x}{x_2'} \gg 0$.

Suppose $\frac{x_2}{x} \gg \frac{1}{3}$. By Lemma 2.3 we have $\frac{1}{g} + \frac{x_2}{xg} \lessapprox (1+\beta)\mathcal{D}_x(A_0(x))$. Hence $\alpha > \frac{1}{g}$ by the fact that $\beta \leqslant \frac{1}{3}$. Since $\mathcal{D}_{x_2' - x}(A_1(x, x_2')) \lessapprox \frac{1}{g} \ll \alpha$ and $\mathcal{D}_{x_2'}(A_2(x, x_2')) \approx 0$, then $\mathcal{D}_{x_2'}(A_0(x_2')) \ll \mathcal{D}_x(A_0(x)) \approx \alpha$. This contradicts the definition of α. So we can assume $\frac{x_2}{x} \lessapprox \frac{1}{3}$. This implies that $\mathcal{D}_{x_2'}(|A_1[0, x_2'] + A_1[0, x_2']|) \gtrapprox 2\mathcal{D}_{x_2'}(A_1(x_2'))$, $\mathcal{D}_{x_2'}(|A_2[0, x_2'] + A_2[0, x_2']|) \gtrapprox 3\mathcal{D}_x(A_2(x_2'))$, and

$$\begin{aligned}&\mathcal{D}_{x_2'}(|A_1[0,x_2']+A_2[0,x_2']|)\\&\quad\gtrapprox \min\left\{\mathcal{D}_{x_2'}(2A_1(x_2')+A_2(x_2')),\mathcal{D}_{x_2'}(A_1(x_2')+x_2'/g)\right\}.\end{aligned}$$

If $\mathcal{D}_{x_2'}(|A_1[0,x_2']+A_2[0,x_2']|)\gtrapprox \mathcal{D}_{x_2'}(2A_1(x_2')+A_2(x_2'))$, then

$$\mathcal{D}_{2x_2'}((A_0+A_0)(2x_2'))\gtrapprox 2\mathcal{D}_{x_2'}(A_0(x_2'))\gg \tfrac{5}{3}\mathcal{D}_{x_2'}(A_0(x_2')).$$

If $\mathcal{D}_{x_2'}(|A_1[0,x_2']+A_2[0,x_2']|)\gtrapprox \mathcal{D}_{x_2'}(A_1(x_2')+x_2'/g)$, then

$$\mathcal{D}_{2x_2'}((A_0+A_0)(2x_2'))\gtrapprox \mathcal{D}_{2x_2'}(3A_0(x_2')+x_2'/g).$$

By (9) we have $\frac{1}{g}\lessapprox \frac{1}{3}\mathcal{D}_{x_2'}(A_0(x_2'))$. This contradicts $\mathcal{D}_{x_2'}(A_0(x_2'))\lessapprox \frac{2}{g}$ because $A_0=A_1\cup A_2$.

Case 3.1.3 A_0 satisfies (c) of Theorem 1.5.

Let $A_0\subseteq F=\{a_1,a_2,a_3\}+gU$ where $0=a_1<a_2<a_3<g$ and $|F+F|=5$ in $\mathbb{Z}/g\mathbb{Z}$. Let $A_i=A_0\cap(a_i+gU)$ for $i=1,2,3$. Without loss of generality we can assume $A_i\neq\emptyset$ for $i=1,2,3$ and $g=\gcd(A_1\cap(A_2-a_2)\cup(A_3-a_3))$. We want to prove that $\alpha=\frac{3}{g}$.

Suppose A_1 and A_2 are bounded in U. Then

$$\underline{d}_U(A_0+A_0)=3\underline{d}_U(A_3)=3\underline{d}_U(A_0)>\tfrac{5}{3}\underline{d}_U(A_0),$$

which implies (d) of Theorem 1.5 because for any sufficiently large $x\in\Gamma(A_0,A_0)$ we have $\mathcal{D}_{2x}((A_0+A_0)(2x))\gtrapprox \underline{d}_U(A_0+A_0)\gg\frac{5}{3}\underline{d}_U(A_0)\approx\frac{5}{3}\mathcal{D}_x(A_0(x))$. Suppose A_1 is bounded in U but A_2, A_3 are unbounded in U. Let x be sufficiently large in A_0. Suppose, without loss of generality, $\mathcal{D}_x(A_2(x))\lessapprox\mathcal{D}_x(A_3(x))$. If $x\in A_2$, then let $x'=x$. Otherwise let $x'=\min\{z\in A_2:z>x\}$. Note that $\mathcal{D}_{x'}(|A_2[0,x']+A_3[0,x']|)\gtrapprox 2\mathcal{D}_{x'}(A_3(x'))$ because $x'\in A_2$. Hence

$$\begin{aligned}&\mathcal{D}_{2x'}((A_0+A_0)(2x'))\\&\quad\gtrapprox\mathcal{D}_{2x'}(3A_2(x')+4A_3(x'))\\&\quad\gtrapprox\tfrac{3.5}{2}\mathcal{D}_{x'}(A_0(x'))\gg\tfrac{5}{3}\mathcal{D}_{x'}(A_0(x')),\end{aligned}$$

which contradicts (9). So we can assume that A_i is unbounded in U for $i=1,2,3$.

Clearly $\alpha\leqslant\frac{3}{g}$. Suppose $\alpha<\frac{3}{g}$. Given a sufficiently large $x\in\Gamma(A_0,A_0)$, there is $i\in\{1,2,3\}$ such that $\mathcal{D}_x(A_i(x))\ll\frac{1}{g}$. Without loss of generality let $i=1$ and $\mathcal{D}_x(A_1(x))\lessapprox\mathcal{D}_x(A_i(x))$ for $i=2,3$.

Let $x_1=\min\{z\in A_1:z\geqslant x\}$ and assume, without loss of generality, that $\mathcal{D}_{x_1}(A_2(x_1))\lessapprox\mathcal{D}_{x_1}(A_3(x_1))$. Let $G=\{a_1+a_i:i=1,2,3\}$ in $\mathbb{Z}/g\mathbb{Z}$.

If $2a_3 \equiv a_1 + a_2 \pmod g$, then

$$\begin{aligned}
&\mathcal{D}_{2x_1}((A_0 + A_0)(2x_1)) \\
&\gtrapprox \mathcal{D}_{2x_1}(|A_1[0, x_1] + A_1[0, x_1]| + |A_3[0, x_1] + A_3[0, x_1]| \\
&\quad +|A_1[0, x_1] + A_3[0, x_1]| + |A_2[0, x_1] + A_2[0, x_1]| \\
&\quad +|A_2[0, x_1] + A_3[0, x_1]|) \\
&\gtrapprox \mathcal{D}_{2x_1}(2A_1(x_1) + 5A_3(x_1) + 3A_2(x_1)) \\
&\gtrapprox \mathcal{D}_{2x_1}(3A_0(x_1) + A_3(x_1)) \gtrapprox \tfrac{5}{3}\mathcal{D}_{x_1}(A_0(x_1)).
\end{aligned}$$

By (9) we can assume $\mathcal{D}_{x_1}(A_3(x_1)) \approx \frac{1}{3}\mathcal{D}_{x_1}(A_0(x_1))$. Hence

$$\mathcal{D}_{x_1}(A_3(x_1)) \approx \mathcal{D}_{x_1}(A_2(x_1)) \approx \mathcal{D}_{x_1}(A_1(x_1)).$$

Also we can assume $\mathcal{D}_{x_1}(|A_1[0, x_1] + A_i[0, x_1]|) \approx \mathcal{D}_{x_1}(A_1(x_1) + A_i(x_1))$, which implies $\gcd(A_1 - a_1) = g$. Hence $\mathcal{D}_{x_1}(A_1(x_1)) \approx \frac{1}{g}$, which implies $\frac{x_1 - x}{x_1} \approx 0$. So $\alpha \approx \mathcal{D}_x(A_0(x)) \approx \frac{3}{g}$.

If $2a_3 = 2a_1 \pmod g$, then again we have

$$\begin{aligned}
\mathcal{D}_{2x_1}((A_0 + A_0)(2x_1)) &\gtrapprox \mathcal{D}_{2x_1}(2A_1(x_1) + 4A_3(x_1) + 4A_2(x_1)) \\
&\gtrapprox \mathcal{D}_{2x_1}(3A_0(x_1) + A_3(x_1)).
\end{aligned}$$

By the same argument above we must have $\alpha = \frac{3}{g}$.

If $2a_3 \notin G$ in $\mathbb{Z}/g\mathbb{Z}$, then

$$\begin{aligned}
&\mathcal{D}_{2x_1}((A_0 + A_0)(2x_1)) \\
&\gtrapprox \mathcal{D}_{2x_1}(2A_1(x_1) + 4A_2(x_1) + 4A_3(x_1)) \\
&\gtrapprox \mathcal{D}_{2x_1}(3A_0(x_1) + A_3(x_1)) \gtrapprox \tfrac{5}{3}\mathcal{D}_{x_1}(A_0(x_1)),
\end{aligned}$$

which again implies $\alpha = \frac{3}{g}$ by the same argument above. This ends the proof of Theorem 3.1. □

Remark 3.2 Similar to the style of Theorem 1.3, we can describe the structure of $A_0 + A_0$ instead of the structure of A_0.

(a) If (a) of Theorem 3.1 is true, then there is $a \in U$ such that $(A_0 + A_0)\setminus[0, a]$ is an *a.p.* of difference g.

(b) If (b) of Theorem 3.1 is true but (c) of Theorem 3.1 is not, then there is $a \in U$ such that $(A_0 + A_0)\setminus[0, a]$ is the union of three *a.p.*'s with the common difference g by the following reason.
Suppose $\beta < \frac{1}{3}$. Then both the lower U-density of $A_1 = A_0 \cap (gU)$ and the lower U-density of $A_2 = A_0 \cap (a + gU)$ are greater than $\frac{1}{2g}$. Hence there is $a \in U$ such that $(A_0 + A_0)\setminus[0, a]$ is the union of three *a.p.*'s with the common difference g. Suppose $\beta = \frac{1}{3}$. Then $\alpha \geqslant \frac{3}{2g}$. Suppose there are sufficiently large $x \in U$, such that $\mathcal{D}_x(A_i(x)) \lessapprox \frac{1}{2g}$ for $i = 1$ or 2. Without loss of generality we

can assume $i = 1$ and $x \in A_1$. Then $\mathcal{D}_x(A_2(x)) \approx \frac{1}{g}$ and $\mathcal{D}_x(A_1(x)) \approx \frac{1}{2g}$. Hence

$$\begin{aligned}&\mathcal{D}_{2x}((A_0 + A_0)(2x))\\&\quad \gtrapprox \mathcal{D}_{2x}(4A_2(x) + 2A_1(x)) \gtrapprox \frac{5}{2g} \approx \frac{5}{3}\mathcal{D}_x(A_0(x)).\end{aligned}$$

By (9) we have $\mathcal{D}_x(|A_1[0, x] + A_1[0, x]|) \approx 2\mathcal{D}_x(A_1(x))$. By Lemma 2.4 we can conclude that $A_1[0, x]$ is a subset of an *a.p.* of difference $2g$. This implies that A_1 is a subset of an *a.p.* of difference $2g$. This implies that A_0 is a subset of a *t.p.* of difference $2g$, which contradicts the assumption that (c) of Theorem 3.1 is not true. So we can assume $\mathcal{D}_x(A_i(x)) \gg \frac{1}{2g}$ for $i = 1, 2$. Hence there is $a \in U$ such that $(A_0 + A_0)\smallsetminus[0, a]$ is the union of three *a.p.*'s with the common difference g.

(c) Suppose (c) of Theorem 3.1 is true and let $A_0^0 = A_0 \cap (gU)$, $A_0^1 = A_0 \cap (a_1 + gU)$, and $A_0^2 = A_0 \cap (a_2 + gU)$. Then $\mathcal{D}_x(A_0^i(x)) \approx \frac{1}{g}$ for $i = 0, 1, 2$ and for all sufficiently large $x \in U$. So there is $a \in U$ such that $(A_0 + A_0)\smallsetminus[0, a]$ is the union of five *a.p.*'s with the common difference g.

4 A conjecture

We do not have any reason to believe that a full version of Kneser's Theorem for cut should not be true. So we would like to make the following conjecture.

Conjecture 4.1 Let U be a cut with an uncountable cofinality and let $A, B \subseteq U$ be U-internal. If $\underline{d}_U(A + B) < \underline{d}_U(A) + \underline{d}_U(B)$, then there are positive $g \in \mathbb{N}$ and $G \subseteq [0, g - 1]$ such that

(a) $\underline{d}_U(A + B) \geqslant \underline{d}_U(A) + \underline{d}_U(B) - \frac{1}{g}$,
(b) $A + B \subseteq G + gU$, and
(c) $(G + gU)\smallsetminus(A + B)$ is bounded in U.

Theorem 3.1 is a few steps away from confirming Conjecture 4.1. For example, we still do not know whether this conjecture is true when A and B have the same lower U-density. Taking one more step back, we do not know whether Conjecture 4.1 is true when A and B are the same set. The least progress we could make is to replace the condition (9) by the condition $\underline{d}_U(A_0 + A_0) \leqslant \frac{5}{3}\underline{d}_U(A_0)$ in Theorem 3.1. But even that may need some nontrivial effort.

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