Generating the Pfaffian closure with total Pfaffian functions

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Abstract: Given an o-minimal expansion \( \mathcal{R} \) of the real field, we show that the structure obtained from \( \mathcal{R} \) by iterating the operation of adding all total Pfaffian functions over \( \mathcal{R} \) defines the same sets as the Pfaffian closure of \( \mathcal{R} \).

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There are various possibilities for adding Pfaffian objects to o-minimal expansions of the real field and preserving o-minimality. One example is the Pfaffian closure of an o-minimal expansion of the real field, which was shown to be o-minimal by the second author \([7]\). The purpose of this note is to present a somewhat simpler construction of the Pfaffian closure. Although not as simple as the description in terms of nested leaves obtained by Lion and the second author \([5]\), our construction has the novelty of only using total Pfaffian functions and is reminiscent of the original Pfaffian expansion of the real field constructed by Wilkie \([9]\).

In order to state our result, we need to introduce some terminology. Suppose that \( \mathcal{R} \) is an o-minimal expansion of the real field, and that \( U \subseteq \mathbb{R}^n \) is an \( \mathcal{R} \)-definable open subset of \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). We say that a \( C^1 \) function \( f : U \to \mathbb{R} \) is Pfaffian over \( \mathcal{R} \) if there exist \( \mathcal{R} \)-definable \( C^1 \) functions \( P_i : U \times \mathbb{R} \to \mathbb{R} \), for \( i = 1, \ldots, n \) such that

\[
\frac{\partial f}{\partial x_i}(x) = P_i(x, f(x))
\]

for all \( x \in U \).

Given \( n, l \in \mathbb{N} \) such that \( l \leq n \), we let \( G^l_n \) be the Grassmannian of all linear subspaces of \( \mathbb{R}^n \) of dimension \( l \). This is an analytic manifold and is naturally definable in the real field (see \([1, 3.4.2]\)). We also set \( G_n = \bigcup_{l=0}^n G^l_n \). Now fix an embedded \( C^1 \) submanifold \( M \) of \( \mathbb{R}^n \) and let \( l \leq n \). A \( C^1 \) map \( d : M \to G_n \) is said to be a distribution on \( M \) if \( d(x) \subseteq T_x M \) for all \( x \in M \), where \( T_x M \) is the tangent space of \( M \) at \( x \). A distribution \( d \) is an \( l \)-distribution if \( d(M) \subseteq G^l_n \). Given an \( l \)-distribution \( d \) on \( M \) and an immersed \( C^1 \) submanifold \( V \) of \( M \), we say that \( V \) is an integral manifold of \( d \) if \( T_x V = d(x) \) for all
$x \in V$. A maximal connected integral manifold is called a leaf of the distribution. Now suppose that $d$ has codimension one. A leaf $L$ of $d$ is said to be a Rolle leaf of $d$ if it is a closed embedded submanifold of $M$ and is such that for all $C^1$ curves $\gamma : [0, 1] \to M$ satisfying $\gamma(0), \gamma(1) \in L$, we have $\gamma'(t) \in d(\gamma(t))$ for some $t \in [0, 1]$. A Rolle leaf over $\mathcal{R}$ is a Rolle leaf of an $\mathcal{R}$-definable codimension one distribution defined on $\mathbb{R}^n$ for some $n \in \mathbb{N}$. For example, a result due to Khovanskii (see [8, 1.6]) implies that if $f : \mathbb{R}^n \to \mathbb{R}$ is Pfaffian over $\mathbb{R}$, then the graph of $f$ is a Rolle leaf over $\mathbb{R}$.

We can now define the Pfaffian structures involved in our result. Given any o-minimal expansion of the real field $\mathcal{R}$, let $L(\mathcal{R})$ be the collection of all Rolle leaves over $\mathcal{R}$. Now let $\mathcal{R}_0 = \mathcal{R}$ and, for $i \geq 0$, let $\mathcal{R}_{i+1}$ be the expansion of $\mathcal{R}_i$ by all leaves in $L(\mathcal{R}_i)$. Let $\mathcal{L}$ be the union of all the $L(\mathcal{R}_i)$ and let $\mathcal{P}(\mathcal{R})$ be the expansion of $\mathcal{R}$ by all the leaves in $\mathcal{L}$. This structure is called the Pfaffian closure of $\mathcal{R}$. The second author showed that it is o-minimal [7].

Similarly, we let $L'(\mathcal{R})$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$, for all $n \in \mathbb{N}$ that are Pfaffian over $\mathcal{R}$. We define $\mathcal{R}_i'$ and then $\mathcal{P}'(\mathcal{R})$ by mimicking the previous paragraph. The structure $\mathcal{P}'(\mathcal{R})$ is a reduct of $\mathcal{P}(\mathcal{R})$ (by the example above) and it is the purpose of this note to show that they are in fact the same from the point of view of definability.

**Theorem 1** A set $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{P}(\mathcal{R})$ if and only if it is definable in $\mathcal{P}'(\mathcal{R})$.

If $\mathcal{R}$ admits analytic cell decomposition, then so too does $\mathcal{P}'(\mathcal{R})$ (see [8]) and it follows that in this case, the reduct of $\mathcal{P}'(\mathcal{R})$ in which only analytic functions are added also defines the same sets as $\mathcal{P}(\mathcal{R})$.

Before proving the theorem, we first recall a result from Khovanskii theory that we will repeatedly use in the proof. This originates with Khovanskii’s work on pfaffian functions (see [3]) and was adapted to the o-minimal setting by the second author [7] following work of Moussu and Roche [6] and Lion and Rolin [4]. In our proof we shall only ever need to work with a single distribution so we restrict ourselves to Khovanskii theory in this simple setting. We follow the presentation in [8]. Suppose that $d$ is a $C^2$ distribution on a $C^2$ manifold $M \subseteq \mathbb{R}^n$ and that $N \subseteq M$ is a $C^2$ submanifold of $M$. The pull-back of $d$ to $N$ is the distribution $d^N$ on $N$ defined by

$$d^N(x) = T_xN \cap d(x).$$

We say that $N$ is compatible with $d$ if $d^N$ is an $l$-distribution on $N$, for some $l$. From now on, we use the word definable to mean $\mathcal{P}'(\mathcal{R})$-definable. In particular, cell means $\mathcal{P}'(\mathcal{R})$-definable cell. The result we need is as follows (see [8, 3.6]).
We now need to show that $C$ is a partition of $M$ into $C^2$ cells compatible with $d$. Then there is a $k \in \mathbb{N}$ such that whenever $C \in C$ and $L$ is a Rolle leaf of $d$ the set $C \cap L$ is a union of at most $k$ Rolle leaves of $d^C$.

Given the definition of $P'(R)$, in order to prove the theorem it suffices to show that if $L$ is a Rolle leaf over $P'(R)$ then $L$ is definable. For the proof of this, we assume that the reader is familiar with o-minimality (as presented in [2]). First, an easy observation.

Lemma 3 Suppose that $C \subseteq \mathbb{R}^n$ is an open $C^2$ cell and that $f : C \rightarrow \mathbb{R}$ is Pfaffian over $P'(R)$. Then $f$ is definable.

The proof, using a definable diffeomorphism between $C$ and $\mathbb{R}^n$, is left to the reader.

Now suppose that $C \subseteq \mathbb{R}^n$ is a bounded open $C^2$ cell, and that $\alpha, \beta, \gamma, \delta : C \rightarrow \mathbb{R}$ are definable bounded $C^2$ functions such that

$$\gamma(x) < \alpha(x) < \beta(x) < \delta(x)$$

for all $x \in C$. Let $D = (\alpha, \beta)_C$ and $D' = (\gamma, \delta)_C$ and suppose that $d'$ is a definable integrable $n$-distribution on $D'$ (for a discussion of integrability in this context, see [8, Section 1]). Suppose that we are given a Rolle leaf $L'$ of $d'$. Assume that both the graph of $\alpha$ and the graph of $\beta$ are compatible with $d'$ and let $d^\alpha$ and $d^\beta$ be the pullbacks of $d'$ to the graphs of $\alpha$ and $\beta$ respectively. Let $d$ be the restriction of $d'$ to $C$. By Fact 2, $L' \cap D, L' \cap \text{graph} \alpha$ and $L' \cap \text{graph} \beta$ are finite unions of Rolle leaves of $d, d^\alpha$ and $d^\beta$ respectively.

Lemma 4 Suppose that $L$ is a connected component of $L' \cap D$ and suppose that $\text{graph} \alpha$ is transverse to $d'$. Then $\text{fr}L \cap \text{graph} \alpha$ is a clopen subset of $L' \cap \text{graph} \alpha$.

Proof Since $L'$ is a Rolle leaf in $D'$, it is closed in $D'$ and so $L$ is closed in $D$. So, $\text{fr}L \cap \text{graph} \alpha = \text{cl}L \cap \text{graph} \alpha$ is closed in the graph of $\alpha$. Using the fact that $L'$ is closed in $D'$ again, we have $\text{cl}L \cap \text{graph} \alpha \subseteq L' \cap \text{graph} \alpha$ and so $\text{fr}L \cap \text{graph} \alpha$ is a closed subset of $L' \cap \text{graph} \alpha$.

We now need to show that $\text{fr}L \cap \text{graph} \alpha$ is open in $L' \cap \text{graph} \alpha$, so let $p \in \text{fr}L \cap \text{graph} \alpha$. Let $L_p$ be the connected component of $L' \cap \text{graph} \alpha$ containing $p$. By the Frobenius theorem (see [8, Section 1]) there is a neighbourhood $U$ of $p$ and a diffeomorphism $\phi : \mathbb{R}^{n+1} \rightarrow U$ such that $\phi^*d' = \text{ker}dx_{n+1}$ and $\phi(0) = p$. Now, $L'$ is a leaf of $d'$ and $p \in L' \cap U$, so the hyperplane $\mathbb{R}^n \times \{0\}$ is a component of $\phi^{-1}(L' \cap U)$. Since $L' \cap \text{graph} \alpha \cap U$ is a submanifold of $L' \cap U$, we can find an open box $B$ centred at 0...
such that \( N := \phi^{-1}(L' \cap \text{graph}\alpha \cap U) \cap B \) is connected. Let \( B_0 = (\mathbb{R}^n \times \{0\}) \cap B \). Then \( N \) is a closed codimension one submanifold of \( B_0 \) and so \( B_0 \setminus N \) has exactly two components, \( B_1 \) and \( B_2 \), say. Since \( p \in \text{cl} \ L \), at least one of \( B_1 \) or \( B_2 \) must be contained in \( \phi^{-1}(L \cap U) \). Also, \( N = \text{fr}(B_i) \cap B_0 \) for each \( i \) and so \( \phi(N) \) is contained in \( \text{fr}L \cap \text{graph}\alpha \). But \( \phi(N) \) is open in \( L' \cap \text{graph}\alpha \), by our choice of \( B \), and the lemma is proved. \( \square \)

The following proposition suffices to prove the theorem.

**Proposition 5** Let \( L \subseteq \mathbb{R}^n \) be a Rolle leaf over \( \mathcal{P}'(\mathcal{R}) \). Then \( L \) is definable in \( \mathcal{P}'(\mathcal{R}) \).

**Proof** The proof is by induction on \( n \). The \( n = 1 \) case is trivial, so we assume that \( n > 1 \) and that the proposition is true for Rolle leaves over \( \mathcal{P}'(\mathcal{R}) \) contained in \( \mathbb{R}^m \) with \( m < n \). Thus if \( C \subseteq \mathbb{R}^n \) is a \( C^2 \) cell of dimension less than \( n \) and \( V \subseteq C \) is a Rolle leaf of a definable codimension one distribution on \( C \), then \( V \) is definable.

Suppose that \( L \subseteq \mathbb{R}^n \) is a Rolle leaf over \( \mathcal{P}'(\mathcal{R}) \). Then \( L \) is a closed embedded proper submanifold of \( \mathbb{R}^n \), and so there are \( p \in \mathbb{R}^n \setminus L \) and \( r > 0 \) such that \( B(p, 2r) \cap L = \emptyset \), where \( B(a, \varepsilon) \) is the open ball around \( a \) of radius \( \varepsilon \). Perhaps after translating and stretching, we may assume that \( p = 0 \) and that \( r = 1 \). Let \( \phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \) be the semialgebraic diffeomorphism \( \phi(x) = \frac{x}{\|x\|} \). Then \( \phi(L) \) is contained in \( B(0, 1/2) \) and \( \text{cl}(\phi(L)) \subseteq \phi(L) \cup \{0\} \). So, after replacing \( L \) by \( \phi(L) \), we may assume that \( L \) is a Rolle leaf of a definable integrable \((n - 1)\)-distribution \( d \) on \( B'(0, 1) := B(0, 1) \setminus \{0\} \), that \( L \subseteq B(0, 1/2) \) and that \( \text{cl}L \subseteq L \cup \{0\} \).

Let \( \Pi_{n-1} \) be the projection onto the first \( n - 1 \) coordinates. For each compatible permutation \( \sigma \) on \( \mathbb{R}^n \), the set \( B_{\sigma} = \{ x \in B'(0, 1) : \Pi_{n-1}|_{\sigma^{-1}(\sigma^{-1}(x))} \text{ has rank } n - 1 \} \) is open and together these sets cover \( B'(0, 1) \). So it suffices to show that \( L \cap B_{\sigma} \) is definable for each \( \sigma \). Fix \( \sigma \), which we may assume to be the identity. Let \( C \) be a \( C^2 \) cell decomposition of \( B'(0, 1) \) compatible with \( B_{\text{id}}, B'(0, 1/2) \) and \( d \). We show that \( C \cap L \) is definable for each cell \( C \in C \) such that \( C \subseteq B_{\text{id}} \). If \( C \in C \) is not open then \( L \cap C \) is definable, by Fact 2 and the inductive hypothesis. So, suppose that \( C \in C \) is open and that \( C \subseteq B_{\text{id}} \). Let \( N \) be a component of \( L \cap C \). Since \( N \) is a Rolle leaf of \( d|_C \) and \( C \) is a cell, \( N \) is the graph of a function \( f : \Pi_{n-1}(N) \to \mathbb{R} \). Let \( \alpha, \beta : \Pi_{n-1}(C) \to \mathbb{R} \) be the functions such that \( \text{graph}\alpha \) and \( \text{graph}\beta \) are the two cells in \( C \) forming the ‘bottom’ and ‘top’ of \( C \). Then the graph of \( \alpha \) is compatible with \( d \) and so it is either tangent to \( d \) or transverse to \( d \). Since \( \text{graph}\alpha \) is connected, if it is tangent to \( d \), then either \( \text{graph}\alpha \subseteq L \) or \( L \cap \text{graph}\alpha = \emptyset \). If the graph of \( \alpha \) is transverse to \( d \) then by Fact 2 and the inductive hypothesis, \( L \cap \text{graph}\alpha \) is definable. By
Lemma 4, frN \cap \text{graph} \alpha is a clopen subset of L \cap \text{graph} \alpha and so frN \cap \text{graph} \alpha is also definable. This all also holds with the graph of \beta in place of the graph of \alpha. Since N is bounded and the graph of a continuous function, x \in fr\Pi_{n-1}(N) if and only if there is a y such that (x, y) \in frN. So the set fr\Pi_{n-1}(N) \cap \Pi_{n-1}(C) is definable. Let \mathcal{D} be a cell decomposition of \Pi_{n-1}(C) compatible with fr\Pi_{n-1}(N) \cap \Pi_{n-1}(C). Then for each D \in \mathcal{D} we either have D \subseteq \Pi_{n-1}(N) or D \cap \Pi_{n-1}(N) = \emptyset. For each non-open cell D \in \mathcal{D} such that D \subseteq \Pi_{n-1}(N), let E_D = (\alpha|_D, \beta|_D). Take a cell decomposition of E_D compatible with d. Let E' be a cell in this decomposition such that graph f|_D \cap E' is non-empty. Then by Fact 2, graph f|_D \cap E' is either a finite union of Rolle leaves of the pullback of d to E' and so definable by the inductive hypothesis, or is equal to E' (in the case that E' is tangent to d). So the graph of f|_D is definable. Finally, for each open cell D \in \mathcal{D} such that D \subseteq \Pi_{n-1}(N), the restriction of f to D is Pfaffian over \mathcal{P}'(\mathbb{R}) and so is definable by Lemma 3. So N is definable, as required.

\[ \square \]

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References


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